Exact Kerr–Schild spacetimes from linearized kinetic gravity braiding

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Abstract

We generalize our recent work on k-essence sourcing Kerr–Schild spacetimes to kinetic gravity braiding scalar field. For k-essence, in order a perturbative Kerr–Schild type solution to become exact, the k-essence Lagrangian was either linear in the kinetic term (with the Kerr–Schild congruence autoparallel) or unrestricted, provided the scalar gradient along the congruence vanishes. A similar reasoning for the pure kinetic braiding contribution leads to either a vanishing Lagrangian or a scalar which is constant along the congruence. From the scalar dynamics we also derive an accompanying constraint. Finally, we discuss pp-waves, an example of Kerr–Schild spacetime generated by a constant k-essence along the Kerr–Schild congruence with vanishing Lagrangian. This allows for the construction of a Fock-type space, consisting of a tower of Kerr–Schild maps first yielding a vacuum pp-wave from flat spacetime; next a k-essence generated pp-wave from the vacuum pp-wave; and finally an arbitrary number of k-essence pp-waves with different, retarded time dependent metric functions.

I. INTRODUCTION

Kerr–Schild maps $\tilde{g}_{ab} = g_{ab} + \lambda l_a l_b$ with λ an arbitrary parameter and $l_a l^a = 0$ often appear as recipes to generate exact solutions from known ones through a null congruence in such a way that only the generator of the light cone along the congruence stays unaffected [1]. Plane-fronted waves with parallel propagation (pp-waves) are generated from a flat seed metric $g_{ab} = \eta_{ab}$ through a Kerr–Schild map with a null congruence with all optical scalars (expansion, twist, shear) vanishing. Schwarzschild black hole metrics emerge in a similar fashion, this time the null congruence having expansion. The Kerr–Schild map on a flat seed spacetime with a particular null congruence, which has both expansion and twist, generates Kerr black hole metrics. In all these cases the null congruence stays shearfree. Shearing Kerr–Schild congruences generate one of the Kóta–Perjés metrics or its nontwisting limit, the Kasner metric from a type N vacuum Kundt metric or a vacuum pp-wave, respectively [2–4]. Recently, Kerr–Schild spacetimes generated attention through attempts to relate gauge and gravitational theories [5].

Kerr–Schild vacuum spacetimes emerging perturbatively for small λ have the remarkable property that they become exact solutions of the Einstein equations, holding for arbitrary values of the Kerr–Schild parameter, as proved by Xanthopoulos [6]. As an extension of this result for the nonvacuum case, the pair $\left(\tilde{g}_{ab}, T_{ab} + \lambda T^{(1)}_{ab}\right)$ solving the linearized equation generates the exact solution $\left(\tilde{g}_{ab}, T_{ab} + \lambda T^{(1)}_{ab} + \lambda^2 l_{(a} T^{(1)}_{b)c} l^c\right)$, provided the null congruence is autoparallel (otherwise, a similar, more involved result holds) [7].

Introducing at least one additional degree of freedom, complementing the tensorial ones of general relativity could provide geometric explanations for dark matter or dark energy, also to model inflation or low energy modifications due to quantum gravity. The kinetic gravity braiding class of modified gravity theories [8] incorporates a scalar field into the gravitational sector, but still allows for the propagation of the tensorial modes at all frequencies with the speed of light in vacuum and is consistent with all available observations. The dependence of the Lagrangian on the scalar ϕ of such theories is only through ϕ, $\Box\phi$ and the kinetic term $X = -\left(g^{ab}\nabla_a \phi \nabla_b \phi\right)/2$ (with g^{ab} the inverse metric, ∇_a the Levi-Civita connection and $\Box = \nabla_a \nabla^a$). We further restrict this class by assuming minimal coupling of the kinetic gravity braiding scalar field, supressing any time evolution of the gravitational constant. This will render the equation of motion into the form of an Einstein equation, with the left-hand side the Einstein tensor and the right-hand side containing the contributions of the scalar.

This is a subclass of the Horndeski theories [9, 10], which have the convenient property that both the metric and the scalar evolve through second order dynamics (thus no Ostrogradsky instabilities occur). In addition, the propagation speed of the tensorial modes (the gravitational waves) is the speed of light in vacuum at all frequencies [11–15] (thus they comply with observations of high frequency gravitational waves by LIGO and Virgo [11]).

Cosmological evolutions in kinetic gravity braiding theories were discussed in Refs. [16, 17], which identified evolutions leading into one of the following cases: de Sitter state, the future Big Rip singularity occurring in finite time or diverging energy density.

In a previous paper [18] we have analyzed Kerr–Schild maps for k-essence spacetimes whose Lagrangians do not involve $\Box\phi$, identifying the Lagrangians linear in X as able to reproduce the property that perturbative Kerr–Schild solutions are also exact (for autoparallel Kerr–Schild congruence). In this paper we extend these investigations both to special cases of k-essence and to the more general case, which includes $\Box\phi$.

In Section 2 we present the kinetic gravity braiding dynamics and the existing results on perturbative Kerr–Schild spacetimes, which generate exact solutions in the presence of matter sources, specifying for the case of k-essence. We then turn to pure kinetic gravity scalar fields in Section 3, discussing the requirements for the exactness of perturbative solutions and we derive a scalar constraint in Section 4. In Section 5 we investigate pp-waves generated by k-essence. We summarize our results in the Section 6.

II. PRELIMINARIES

A. Kinetic gravity braiding

The action for the minimally coupled kinetic gravity is a sum of the Einstein–Hilbert action with the scalar contribution

$$\begin{aligned} S_\phi &= \int \mathrm{d}^4 x \mathfrak{L}_\phi \ , \\ \mathfrak{L}_\phi &= \sqrt{-\mathfrak{g}}\left[L_2\left(\phi, X\right) + G_3(\phi, X)\Box\phi\right] \ , \end{aligned} \tag{1}$$

where L_2 is the Lagrangian of k-essence and $G_3(\phi, X)\Box\phi$ is the pure kinetic gravity braiding contribution. Both L_2 and G_3 are arbitrary functions of ϕ and the kinetic term X. It is often convenient to rewrite the scalar action through a partial integration into the form containing the Lagrangian density

$$\mathfrak{L}_{\phi}^{\mathrm{KGB}} = \sqrt{-\mathfrak{g}}\left[F\left(\phi, X\right) + H\left(\phi, X\right)\nabla_c X \nabla^c \phi\right] \ , \tag{2}$$

with $F = L_2 + 2G_{3\phi}X$ and $H = -G_{3X}$.

Variation with respect to the inverse metric g^{ab} through the prescription

$$T_{ab} = \frac{-2}{\sqrt{-\mathfrak{g}}}\frac{\delta S}{\delta g^{ab}} \tag{3}$$

gives the energy-momentum tensor

$$T_{ab}^{\phi} = T_{ab}^{F} + T_{ab}^{H} \ , \tag{4}$$

$$T_{ab}^{F} = F_X \nabla_a \phi \nabla_b \phi + g_{ab} F \ , \tag{5}$$

$$\begin{aligned} T_{ab}^{H} = \ & H\left(g_{ab}\nabla_c X \nabla^c \phi - 2\nabla_{(a} X \nabla_{b)} \phi\right) \\ & + \left(2H_{\phi} X - H \Box \phi\right) \nabla_a \phi \nabla_b \phi \ . \end{aligned} \tag{6}$$

(Derivatives with respect to ϕ and X are denoted by the respective subscripts.) Variation with respect to the scalar gives the scalar equation of motion $(-\mathfrak{g})^{-1/2}\,\delta S_{\phi}/\delta\phi = 0$ as [19]:

$$\begin{aligned} & F_{\phi} - 2F_{\phi X} X - F_{XX} \nabla_a \phi \nabla_b \phi \left(\nabla^a \nabla^b \phi\right) + F_X \Box \phi \\ & + H\left[R_{ab}\nabla^a \phi \nabla^b \phi + \left(\nabla_a \nabla_b \phi\right)\left(\nabla^a \nabla^b \phi\right) - \left(\Box \phi\right)^2\right] \\ & - 2H_{\phi}\left[\nabla_a \phi \nabla_b \phi \left(\nabla^a \nabla^b \phi\right) - 2X \Box \phi\right] \\ & + H_X \nabla_a \phi \nabla_b \phi \left[\left(\nabla^a \nabla^b \phi\right) \Box \phi - \left(\nabla^i \nabla^b \phi\right)\left(\nabla_i \nabla^a \phi\right)\right] \\ & - 4H_{\phi\phi} X^2 - 2H_{\phi X} X \nabla_a \phi \nabla_b \phi \left(\nabla^a \nabla^b \phi\right) = 0 \ . \end{aligned} \tag{7}$$

In deriving this result, the derivative of X was replaced by

$$\nabla_a X = -\nabla^c \phi \nabla_a \nabla_c \phi \ . \tag{8}$$

Variation of the Einstein–Hilbert action with respect to the inverse metric complements the set of dynamical equations with the Einstein equation

$$G_{ab} = T_{ab} \ . \tag{9}$$

Here the choice of units $8\pi G = c = 1$ has been implemented. From here, the Ricci tensor appearing in the scalar equation (7) of motion can be expressed in terms of the scalar energy-momentum tensor (4) and its trace

$$T^{\phi} = T^{F} + T^{H} \ , \tag{10}$$

with

$$T^F = -2XF_X + 4F \ , \tag{11}$$

$$T^H = 2H\nabla_c X \nabla^c \phi - 4H_\phi X^2 + 2XH\Box\phi \tag{12}$$

as

$$R_{ab} = T^\phi_{ab} - \frac{1}{2} g_{ab} T^\phi \ . \tag{13}$$

With this the curvature term can be eliminated from the equation of motion of the scalar field. By also employing Eq. (8) it exhibits the structure

$$Sc\,[F] + Sc\,[H] + Sc\left[H^2\right] + Sc\,[FH] = 0 \ , \tag{14}$$

with the contributions

$$Sc\,[F] = F_\phi - 2F_{\phi X} X - F_{XX} \nabla_a\phi\nabla_b\phi \left(\nabla^a\nabla^b\phi\right) + F_X\Box\phi \ , \tag{15}$$

$$\begin{aligned} Sc\,[H] &= H\left[\left(\nabla_a\nabla_b\phi\right)\left(\nabla^a\nabla^b\phi\right) - \left(\Box\phi\right)^2\right] \\ &- 2H_\phi\left[\nabla_a\phi\nabla_b\phi\left(\nabla^a\nabla^b\phi\right) - 2X\Box\phi\right] \\ &+ H_X\nabla_a\phi\nabla_b\phi\left[\left(\nabla^a\nabla^b\phi\right)\Box\phi - \left(\nabla^i\nabla^b\phi\right)\left(\nabla_i\nabla^a\phi\right)\right] \\ &- 4H_{\phi\phi}X^2 - 2H_{\phi X}X\nabla_a\phi\nabla_b\phi\left(\nabla^a\nabla^b\phi\right) \ , \end{aligned} \tag{16}$$

$$Sc\left[H^2\right] = 2H^2\left(-2\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi - X\Box\phi\right)X + 4HH_\phi X^3 \ , \tag{17}$$

$$Sc\,[FH] = 2H\left(F + F_X X\right)X \ . \tag{18}$$

The first of these characterizes pure k-essence (only F-terms), the second and third pure kinetic gravity braiding scalar (with both linear and nonlinear contributions in H and its derivatives), while the last one is an interaction term emerging from the combination of both types of contributions to the action.

B. Kerr–Schild maps

In Ref. [7], it was shown that the Kerr–Schild transformed Ricci tensor

$$\tilde{R}_{ab} = R_{ab} + \lambda R^{(1)}_{ab} + \lambda^2 R^{(2)}_{ab} + \lambda^3 R^{(3)}_{ab} \ , \tag{19}$$

is a third order polynomial in λ, with the contributions

$$R_{ab}^{(1)} = \nabla_c \left[\nabla_{(a} \left(l_{b)} l^c \right) - \frac{1}{2} \nabla^c \left(l_a l_b \right) \right] , \tag{20}$$

$$R_{ab}^{(2)} = \nabla_c l^c l_{(a} D l_{b)} + \frac{1}{2} D l_a D l_b + l_{(a} D D l_{b)} + l_a l_b \nabla_c l_d \nabla^{[c} l^{d]} - \left(D l^c \right) \nabla_c l_{(a} l_{b)} , \tag{21}$$

$$R_{ab}^{(3)} = -\frac{1}{2} l_a l_b D l^c D l_c . \tag{22}$$

By taking the Kerr–Schild transformed energy-momentum tensor as an infinite series in λ

$$\tilde{T}_{ab} = \sum_{k=0}^{\infty} \lambda^k T_{ab}^{(k)} , \tag{23}$$

the Einstein equations restrict all $k > 3$ contributions to vanish, $T_{ab}^{(k>3)} = 0$. If such contributions were to exist in the energy-momentum tensor, it would not be able to source Kerr–Schild metrics.

With λ arbitrary, the Einstein equation decouples into the following λ, λ^2 and λ^3 contributions:

$$R_{ab}^{(1)} = T_{ab}^{(1)} + \frac{1}{2} g_{ab} \left(T_{cd} l^c l^d - T^{(1)} \right) - \frac{1}{2} l_a l_b T , \tag{24}$$

$$R_{ab}^{(2)} = T_{ab}^{(2)} + \frac{1}{2} g_{ab} \left(T_{cd}^{(1)} l^c l^d - T^{(2)} \right) + \frac{1}{2} l_a l_b \left(T_{cd} l^c l^d - T^{(1)} \right) , \tag{25}$$

$$R_{ab}^{(3)} = T_{ab}^{(3)} + \frac{1}{2} g_{ab} \left(T_{cd}^{(2)} l^c l^d - T^{(3)} \right) + \frac{1}{2} l_a l_b \left(T_{cd}^{(1)} l^c l^d - T^{(2)} \right) . \tag{26}$$

In Ref. [7] conditions on the energy-momentum tensor were identified, which assure that any solution of Eq. (24) also satisfies Eqs. (25)-(26). These were:

$$T_{ab}^{(3)} = -\frac{3}{4} l_a l_b \left(D l^c D l_c \right) , \tag{27}$$

$$\begin{aligned} 2T_{ab}^{(2)} &= 2 l_{(a} T_{b)c}^{(1)} l^c - \frac{1}{2} g_{ab} \left(D l^c D l_c \right) + D l_a D l_b - l_a l_b \left(\nabla_c D l^c \right) \\ &\quad + l_{(a} \left[D l_{b)} \left(\nabla_c l^c \right) + D D l_{b)} + \left(\nabla_{b)} l_c - 2 \nabla_{|c|} l_{b)} \right) D l^c \right] . \end{aligned} \tag{28}$$

For an autoparallel congruence l^a (thus $D l^a \propto l^a$, with $D = l^b \nabla_b$ the covariant derivative along l^a), these conditions further simplified, requiring that the third-order contribution to the Kerr–Schild-transformed energy-momentum tensor vanishes, while the second- and first-order contributions obey a constraint:

$$T_{ab}^{(3)} = 0 , \quad T_{ab}^{(2)} = l_{(a} T_{b)c}^{(1)} l^c . \tag{29}$$

Perturbative solutions emerge from Eq. (24). Whenever the above conditions hold for autoparallel Kerr–Schild congruences, these solutions also generate the exact solution.

C. K-essence under Kerr–Schild maps

The Kerr–Schild transformation does not affect ϕ or $\nabla_c\phi$, however it changes g_{ab}, X, $\nabla_c X$ and $\Box\phi$. From among these variables, only g_{ab} and X appear in the k-essence part T^F_{ab}. As the inverse metric transforms according to $\tilde{g}^{ab} = g^{ab} - \lambda l^a l^b$, the kinetic variable X changes into

$$\tilde{X} = X + \frac{\lambda}{2}\left(D\phi\right)^2 \ . \tag{30}$$

The Kerr–Schild transformation in the k-essence case was analyzed in detail in our previous work [18]. As a main result, the functional form of the free function F characterizing the k-essence was restricted to be linear in X by the requirement to make perturbative solutions generating the exact ones.

Any Kerr–Schild null congruence still needs to obey the linearized equation (24), with the first-order contribution

$$T^{F(1)}_{ab} = l_a l_b F + \frac{1}{2}\left(F_{X^2}\nabla_a\phi\nabla_b\phi + g_{ab}F_X\right)\left(D\phi\right)^2 , \tag{31}$$

calculated in Ref. [18]. For the allowed

$$F = C\left(\phi\right)X - U\left(\phi\right) \ , \tag{32}$$

linear in X, with C and U two arbitrary functions of ϕ, energy momentum tensor (5), the linearized equation (24) reads

$$\nabla_c\left[\nabla_{(a}\left(l_{b)}l^c\right) - \frac{1}{2}\nabla^c\left(l_a l_b\right)\right] = l_a l_b U \ . \tag{33}$$

III. PURE KINETIC GRAVITY BRAIDING UNDER KERR–SCHILD MAPS

In this section we discuss the changes occurring under Kerr–Schild maps in the pure kinetic gravity braiding contribution T^H_{ab}. For this, employing Eq. (30) we calculate

$$\begin{aligned}\tilde{\nabla}_c\tilde{X} &= \nabla_c X + \lambda D\phi\nabla_c D\phi \\ &= \nabla_c X + \lambda\left[\left(\nabla_c l^a\right)\nabla_a\phi + l^a\nabla_c\nabla_a\phi\right]D\phi \ .\end{aligned} \tag{34}$$

Next, we need the Kerr–Schild transformation of the Christoffel symbols, emerging as

$$\tilde{\Gamma}^i_{ab} = \Gamma^i_{ab} + \lambda g^{ij}\left[\nabla_{(a}\left(l_j l_{b)}\right) - \frac{1}{2}\nabla_j\left(l_a l_b\right)\right] + \frac{\lambda^2}{2}l^i D\left(l_a l_b\right) \ , \tag{35}$$

to obtain

$$\tilde{\nabla}_a\tilde{\nabla}_b\phi = \nabla_a\nabla_b\phi - \lambda\left[\nabla_{(a}\left(l_c l_{b)}\right) - \frac{1}{2}\nabla_c\left(l_a l_b\right)\right]g^{cd}\nabla_d\phi - \frac{\lambda^2}{2}D\left(l_a l_b\right)D\phi\ , \tag{36}$$

which yields the Kerr–Schild transformation of the d'Alembertian:

$$\begin{aligned}\tilde{\Box}\phi &= \left(g^{ab} - \lambda l^a l^b\right)\tilde{\nabla}_a\tilde{\nabla}_b\phi \\ &= \Box\phi - \lambda g^{ab}g^{cd}\nabla_a\left(l_c l_b\right)\nabla_d\phi - \lambda l^a l^b\nabla_a\nabla_b\phi \\ &= \Box\phi - \lambda\left[D^2\phi + \left(\nabla_a l^a\right)D\phi\right]\ .\end{aligned} \tag{37}$$

The kinetic gravity braiding energy-momentum tensor under the Kerr–Schild map formally transforms into a second order polynomial in λ:

$$\begin{aligned}\tilde{T}^H_{ab} &= H(\phi,\tilde{X})g_{ab}\nabla_c X\nabla^c\phi - 2H(\phi,\tilde{X})\nabla_{(a}X\nabla_{b)}\phi \\ &+2H_\phi(\phi,\tilde{X})X\nabla_a\phi\nabla_b\phi - H(\phi,\tilde{X})\Box\phi\nabla_a\phi\nabla_b\phi \\ &+\lambda H(\phi,\tilde{X})l_a l_b\nabla_c X\nabla^c\phi + \lambda H(\phi,\tilde{X})g_{ab}D\phi\nabla_c D\phi\nabla^c\phi \\ &-2\lambda H(\phi,\tilde{X})D\phi\nabla_{(a}D\phi\nabla_{b)}\phi + \lambda H_\phi(\phi,\tilde{X})\left(D\phi\right)^2\nabla_a\phi\nabla_b\phi \\ &+\lambda H(\phi,\tilde{X})\left(\left[D^2\phi + \left(\nabla_c l^c\right)D\phi\right]\right)\nabla_a\phi\nabla_b\phi \\ &+\lambda^2 H(\phi,\tilde{X})l_a l_b D\phi\left(\nabla_c D\phi\right)\nabla^c\phi\ .\end{aligned} \tag{38}$$

In what follows, we will implement the additional expansion of $\tilde{X}$.

A. Infinitesimal Kerr–Schild maps

The kinetic gravity braiding contribution to the energy-momentum tensor can be fully expanded in powers of a small λ. For this we insert infinite power serie expansions of the functions with argument $\tilde{X}$:

$$H(\phi,\tilde{X}) = \sum_{j=0}^{\infty}\left(\frac{\lambda}{2}\right)^j\frac{\left(D\phi\right)^{2j}}{j!}H_{X^j}(\phi,X)\ , \tag{39}$$

$$H_\phi(\phi,\tilde{X}) = \sum_{j=0}^{\infty}\left(\frac{\lambda}{2}\right)^j\frac{\left(D\phi\right)^{2j}}{j!}H_{\phi X^j}(\phi,X)\ . \tag{40}$$

The transformed energy-momentum tensor contains the contribution of the original kinetic gravity braiding field,

$$T^{(0)}_{ab} = T^H_{ab}\ , \tag{41}$$

together with the leading order correction

$$
\begin{aligned}
T_{ab}^{(1)} &= \frac{(D\phi)^2}{2}\left[g_{ab}\nabla_c X\nabla^c\phi - 2\nabla_{(a}X\nabla_{b)}\phi - \nabla_a\phi\nabla_b\phi\left(\Box\phi - 2X\partial_\phi\right)\right]H_X \\
&+ \left[l_a l_b \nabla_c X\nabla^c\phi + g_{ab}D\phi\left(\nabla_c D\phi\right)\nabla^c\phi - 2D\phi\nabla_{(a}D\phi\nabla_{b)}\phi\right]H \\
&+\nabla_a\phi\nabla_b\phi\left[D^2\phi + \left(\nabla_c l^c\right)D\phi + (D\phi)^2\partial_\phi\right]H \ ,
\end{aligned} \tag{42}
$$

and higher order ($k \geq 2$) contributions

$$
\begin{aligned}
T_{ab}^{(k)} &= \frac{(D\phi)^{2k}}{2^k k!}\left[g_{ab}\nabla_c X\nabla^c\phi - 2\nabla_{(a}X\nabla_{b)}\phi - \Box\phi\nabla_a\phi\nabla_b\phi + 2X\nabla_a\phi\nabla_b\phi\partial_\phi\right]H_{X^k} \\
&+\frac{(D\phi)^{2(k-1)}}{2^{k-1}(k-1)!}\left\{l_a l_b\nabla_c X\nabla^c\phi + g_{ab}D\phi\left(\nabla_c D\phi\right)\nabla^c\phi - 2D\phi\nabla_{(a}D\phi\nabla_{b)}\phi\right. \\
&+\left.\left[D^2\phi + \left(\nabla_c l^c\right)D\phi\right]\nabla_a\phi\nabla_b\phi + (D\phi)^2\nabla_a\phi\nabla_b\phi\partial_\phi\right\}H_{X^{k-1}} \\
&+\frac{(D\phi)^{2k-3}}{2^{k-2}(k-2)!}l_a l_b\left(\nabla_c D\phi\right)\nabla^c\phi H_{X^{k-2}} \ .
\end{aligned} \tag{43}
$$

These are the coefficients appearing in the final polynomial expansion in λ.

B. Kinetic gravity braiding with H linear in X

A condition necessary to have a Kerr–Schild type spacetime is $T_{ab}^{(k)} = 0$, $k \geq 4$, therefore we announce

Theorem 1. Kinetic gravity braiding scalar fields with H linear in X (or quadratic in X, when $\nabla_c\phi\left(\nabla^c D\phi\right) = 0$ also holds) generate at most third order energy-momentum tensors in λ under infinitesimal Kerr–Schild maps.

Proof. The fourth order energy-momentum tensor contribution reads

$$
\begin{aligned}
T_{ab}^{(4)} &= \frac{1}{16}\frac{(D\phi)^8}{4!}\left[g_{ab}\nabla_c X\nabla^c\phi - 2\nabla_{(a}X\nabla_{b)}\phi - \Box\phi\nabla_a\phi\nabla_b\phi + 2X\nabla_a\phi\nabla_b\phi\partial_\phi\right]H_{X^4} \\
&+\frac{1}{8}\frac{(D\phi)^6}{3!}\left[l_a l_b\nabla_c X\nabla^c\phi + g_{ab}D\phi\left(\nabla_c D\phi\right)\nabla^c\phi - 2D\phi\nabla_{(a}D\phi\nabla_{b)}\phi\right]H_{X^3} \\
&+\frac{1}{8}\frac{(D\phi)^6}{3!}\left[\left[D^2\phi + \left(\nabla_c l^c\right)D\phi\right]\nabla_a\phi\nabla_b\phi + (D\phi)^2\nabla_a\phi\nabla_b\phi\partial_\phi\right]H_{X^3} \\
&+\frac{1}{4}\frac{(D\phi)^5}{2!}l_a l_b\left(\nabla_c D\phi\right)\nabla^c\phi H_{X^2} \ .
\end{aligned} \tag{44}
$$

Assuming $T_{ab}^{(4)} = 0$ and contracting with l^b gives

$$
\begin{aligned}
0 &= \frac{(D\phi)^2}{8}\left[l_a\nabla_c X\nabla^c\phi - DX\nabla_a\phi - D\phi\left(\nabla_a X + \Box\phi\nabla_a\phi - 2X\nabla_a\phi\partial_\phi\right)\right]H_{X^4} \\
&+\left[l_a D\phi\left(\nabla_c D\phi\right)\nabla^c\phi - (D\phi)^2\left(\nabla_a D\phi - \left(\nabla_c l^c\right)\nabla_a\phi\right) + (D\phi)^3\nabla_a\phi\partial_\phi\right]H_{X^3} \ .
\end{aligned} \tag{45}
$$

Further contracting with l^a yields

$$0 = \frac{D\phi}{8}\left[2DX + D\phi\left(\Box\phi - 2X\partial_\phi\right)\right] H_{X^4} + \left[D^2\phi - D\phi\left(\nabla_c l^c\right) - \left(D\phi\right)^2\partial_\phi\right] H_{X^3} \ . \quad (46)$$

The simplest solution to this equation is provided by the quadratic expression $H = A(\phi)X^2 + B(\phi)X - V(\phi)$, with A, B, V arbitrary functions of the scalar. Reinserting this into Eq. (44), the condition $T_{ab}^{(4)} = 0$ leads to

$$0 = \nabla_c\phi\left(\nabla^c D\phi\right) H_{X^2} \ , \quad (47)$$

which vanishes either for a function

$$H = B(\phi)X - V(\phi) \ , \quad (48)$$

linear in X, or for a particular scalar field $\nabla_c\phi\left(\nabla^c D\phi\right) = 0$. Either of these conditions further implies the vanishing of all higher order contributions to $\tilde{T}_{ab}^H$.

Targeting generic classes of the scalar field, we adopt the linear function (48). While this obeys the conditions $T_{ab}^{(k\geq 4)}$ necessary to source Kerr–Schild spacetimes with infinitesimal parameter λ, it could equally lead to Kerr–Schild spacetimes with arbitrary parameter λ, as can be seen from the identical expansions $H\left(\phi, \tilde{X}\right) = BX - V + \lambda\left(D\phi\right)^2 B/2$ obtained from Eq. (30) and from Eq. (39) specified for the linear H.

C. Perturbative Kerr–Schild solutions are not exact

Next, we announce the following result holding for pure kinetic gravity braiding theories.

Theorem 2. For autoparallel Kerr–Schild null congruence and H linear in X the perturbative solutions can be exact only if the Lagrangian of the pure kinetic braiding scalar field vanishes.

Proof: For autoparallel null congruences $Dl^a \propto l^a$ the condition $T_{ab}^{(3)} = 0$ should be imposed in order to get Kerr–Schild metrics. A glance on Eq. (43) immediately gives (unless $D\phi = 0$)

$$0 = \nabla_c\phi\left(\nabla^c D\phi\right) B \ . \quad (49)$$

Unless $\nabla_c\phi\left(\nabla^c D\phi\right) = 0$, the condition becomes $B = 0$, hence the function $H = -V(\phi)$ loses any X-dependence.

The expressions

$$T^{(1)}_{bc}l^c = -\left[l_b D\phi\left(\nabla_i D\phi\right)\nabla^i\phi - \left(D\phi\right)^2\nabla_b D\phi + \left(D\phi\right)^2\nabla_b\phi\left(\nabla_i l^i + D\phi\partial_\phi\right)\right]V\ , \tag{50}$$

and

$$T^{(2)}_{ab} = -l_a l_b D\phi\left(\nabla_c D\phi\right)\nabla^c\phi V\ , \tag{51}$$

necessary for the perturbative Kerr–Schild solution to be also exact, through the second condition (28) yield (unless $D\phi = 0$)

$$\left[l_{(a}\nabla_{b)}D\phi - \left(\nabla_c l^c\right)l_{(a}\nabla_{b)}\phi\right]V = l_{(a}\nabla_{b)}\phi D\phi V_\phi\ . \tag{52}$$

Either its trace or contracting with l^b gives

$$\left[D^2\phi - \left(\nabla_c l^c\right)D\phi\right]V = \left(D\phi\right)^2 V_\phi\ . \tag{53}$$

Reinserting this in Eq. (52) results in

$$\left[l_{(a}\nabla_{b)}\phi D^2\phi - \left(l_{(a}\nabla_{b)}D\phi\right)D\phi\right]V = 0\ . \tag{54}$$

With the exception of the very special scalar field obeying $l_{(a}\left(\nabla_{b)}\phi D^2\phi - D\phi\nabla_{b)}D\phi\right) = 0$, we obtain $V = 0$. Hence Kerr–Schild type solutions of the linearized Einstein equation sourced by pure kinetic gravity braiding and generated by autoparallel null congruences belong to the

$$H = 0\ . \tag{55}$$

class.

Note that the scalar equation for $H = -V(\phi)$ simplifies to

$$\begin{aligned}0 = &-V\left[\left(\nabla_a\nabla_b\phi\right)\left(\nabla^a\nabla^b\phi\right) - \left(\Box\phi\right)^2\right]\\ &+2V_\phi\left[\nabla_a\phi\nabla_b\phi\left(\nabla^a\nabla^b\phi\right) - 2X\Box\phi\right] + 4V_{\phi\phi}X^2\\ &+2V^2\left(-2\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi - X\Box\phi\right)X + 4VV_\phi X^3\ .\end{aligned} \tag{56}$$

This is also solved for $V = 0$.

With $B = 0 = V$ any H linear in X reduces to pure vacuum, as $H_X = B = 0$ and $H_\phi = 0$, therefore the scalar equation of motion becomes trivial.

D. The $D\phi = 0$ case

In the considerations above we excluded the $D\phi = 0$ case. In this subsection we consider this special scalar field.

With $D\phi = 0$, Eqs. (34) and (37) imply

$$\tilde{\nabla}_c \tilde{X} = \nabla_c X \ , \quad \tilde{\Box}\phi = \Box\phi \ . \tag{57}$$

The perturbative contributions (42) and (43) also simplify as

$$T^{(1)}_{ab} = l_a l_b \nabla_c X \nabla^c \phi H \ , \quad T^{(k\geq 2)}_{ab} = 0 \ . \tag{58}$$

Next, we discuss the conditions for the linearized Einstein equation to generate an exact Kerr–Schild spacetime. As $T^{(3)}_{ab} = 0$, Eq. (27) implies an autoparallel Kerr–Schild congruence. Further, the second condition (29) is trivially satisfied. Therefore (as it was the case for the k-essence), no restrictions on H emerge in this case. This is consequence of neither ϕ nor X being affected by the Kerr–Schild map in the special case of a scalar constant along the congruence.

IV. SCALAR CONSTRAINT

We have applied Kerr–Schild maps on the metric, which do not change the scalar field. We equally assumed that the energy-momentum tensor of the scalar has the same functional form before and after the map. In this section we discuss, whether such assumptions could impose any constraints on the scalar field. We illustrate this for the case (32) of the k-essence linear in X, for which the scalar equation (7) becomes

$$C\Box\phi - U_\phi = C_\phi X \ . \tag{59}$$

For $C = 1$ this reproduces the Klein–Gordon type equation for quintessence with potential U. Requiring the same equation to hold for the scalar after the KS map:

$$C\tilde{\Box}\phi - U_\phi = C_\phi \tilde{X} \ , \tag{60}$$

Eqs. (30) and (37) yield the constraint

$$-2C\left[D^2\phi + \left(\nabla_a l^a\right) D\phi\right] - U_\phi = C_\phi \left(D\phi\right)^2 \ . \tag{61}$$

This is a serious constraint on the possible scalar fields. Even in the simplest case $C = 1$, $U = 0$ this gives

$$D^2\phi = -\left(\nabla_a l^a\right) D\phi \ . \tag{62}$$

Such equations should be obeyed together with the linearized Kerr–Schild maps when searching for perurbative solutions, which are also exact.

Note that in the special case $D\phi = 0 = U_\phi$, the constraint (61) is trivially obeyed.

V. PP-WAVES IN KINETIC GRAVITY BRAIDING

General relativity predicted gravitational waves, the existence of which was spectacularly confirmed by more than 200 detections of gravitational waves from coalescing compact binaries by LIGO-Virgo-KAGRA. In the geometrical optics / high frequency approximation these waves are characterized by null rays along which the waves propagate. This is similar to how electromagnetic waves are presented in the geometrical optics approximation. The first examples of gravitational waves were the family of cylindrically symmetric Einstein–Rosen waves [20], later shown to contain beyond stationary solutions also solitonic and pulse-type solutions [21]. The simplest plane waves can be generalized into the class of plane-fronted waves with parallel propagation (pp-waves) discussed below. A remarkable result by Penrose states that any spacetime has a plane wave as a limit [22]. Plane waves are identical with their Penrose limit. Some of the Penrose limit plane waves were shown to be diagonalizable [23]. The Penrose limit in general is a pp-wave [24], which, beside plane waves also include the Aichelburg–Sexl ultraboost, an impulsive pp-wave spacetime perceived by observers moving with high speed close to the speed of light in the vicinity of a black hole [25]. The spacetime about null geodesics can also be modelled through pp-waves.

A. Pp-waves

The line element of pp-waves in Brinkmann coordinates is manifestly in Kerr–Schild form, Eq. (24.40) of Ref. [26]

$$ds^2 = \underbrace{-2dudv + dx^2 + dy^2}_{\text{flat}} + L\left(u, x, y\right) du^2 \ . \tag{63}$$

Whenever $L(u,x,y)$ is quadratic in x and y, the spacetime represents a plane wave with extra planar symmetry and the coordinates

$$u = \frac{ct - z}{\sqrt{2}} \ , \quad v = \frac{ct + z}{\sqrt{2}} \tag{64}$$

are null.

The uu-component of Ricci tensor is

$$R_{uu} = -\frac{1}{2}\left(\partial_x^2 + \partial_y^2\right) L(u,x,y) \ , \tag{65}$$

while the other components vanish. As $R = g^{uu} R_{uu} = 0$, the Einstein equations read $R_{ab} = \left(c^4/8\pi G\right) T_{ab}$.

B. K-essence pp-waves

Pp-waves emerging from a k-essence source with linear dependence on X fall into the $F = 0$ class. (This is, because the $F = 0$ class is not necessarely empty, similarly to $R = 0$ containing all vacuum metrics). Indeed, the vv component of the Einstein equation leads to $\partial_v \phi = 0$, thus $\phi = \phi(u,x,y)$. Then the uv component immediately gives $F = 0$.

Inserting this into the xx, yy components yields either $C = 0 = U$ (the latter stemming then from $F = 0$), thus vanishing energy-momentum or for $C \neq 0$

$$\partial_x \phi = 0 \ , \quad \partial_y \phi = 0 \ , \tag{66}$$

thus $\phi = \phi(u)$, a harmonic scalar field. The uu component of the Einstein equation remains the only nontrivial one:

$$\left(\partial_x^2 + \partial_y^2\right) L(u,x,y) = -\frac{c^4}{4\pi G} C \left(\partial_u \phi\right)^2 \ . \tag{67}$$

In the special case of constant ϕ this reproduces the vacuum pp-waves [27], with solutions including all functions

$$L_{\text{hom}} = L_{\text{hom}}(u) \tag{68}$$

and other, x and y dependent solutions possible for particular boundary conditions. A particular solution of the inhomogeneous equation (67) emerges as follows. The solution of the two-dimensional Poisson equation $\left(\partial_x^2 + \partial_y^2\right) P(x,y) = 1$ is $P = \left(x^2 + y^2\right)/4$, therefore

$$L_{\text{inhom}}(u,x,y) = -\frac{c^4}{16\pi G} C \left(\partial_u \phi\right)^2 \left(x^2 + y^2\right) \ , \tag{69}$$

represents a plane wave (being quadratic in x and y). Thus, the general solution is

$$L = L_{\text{hom}}(u) + L_{\text{inhom}}(u, x, y) \ . \tag{70}$$

Note that for $C \neq 0$ the condition $F = 0$ could be obeyed only with

$$U = 0 = X \ . \tag{71}$$

The latter condition means either vacuum or that the gradient of the scalar field is a null vector. Canonical scalar fields with vanishing potential and null gradient were shown to represent expansionless null dust [28, 29].

C. The scalar constraint is satisfied

The Kerr–Schild vector field

$$l^a \partial_a = -\left|L(u, x, y)\right|^{1/2} \partial_v \tag{72}$$

and $\partial_v \phi = 0$ (obtained from the Einstein equations) allows to show that

$$D\phi = l^a \nabla_a \phi = l^v \partial_v \phi = 0 \ . \tag{73}$$

With (71) holding the scalar constraint (61) is identically satisfied.

D. The tower of Kerr–Schild maps

The first such map takes the flat seed metric

$$ds^2 = \underbrace{-2dudv + dx^2 + dy^2}_{\text{flat}} \tag{74}$$

into a vacuum pp-wave with L_{hom}:

$$d\tilde{s}^2 = \underbrace{-2dudv + dx^2 + dy^2}_{\text{flat}} + L_{\text{hom}}(u)\, du^2 \ , \tag{75}$$

with the null Kerr–Schild congruence $l_a dx^a = \left|L_{\text{hom}}(u)\right|^{1/2} du$ (this is in the absence of ϕ).

A subsequent Kerr–Schild map with $l_a dx^a = \left|L_{\text{inhom}}(u, x, y)\right|^{1/2} du$ then takes $d\tilde{s}^2$ into

$$d\tilde{\tilde{s}}^2 = \underbrace{-2dudv + dx^2 + dy^2 + L_{\text{hom}}(u)\, du^2}_{\text{vacuum pp-wave with } L_{\text{hom}} \text{ (in the absence of } \phi)} + L_{\text{inhom}}(u, x, y)\, du^2 \ , \tag{76}$$

another pp-wave with $L = L_{\text{hom}} + L_{\text{inhom}}$.

Application of further Kerr-Schild maps could follow, changing the retarded time dependent function $C(\phi)$ in L_{inhom}.

This construction is very similar to Kerr–Schild maps generating Schwarzschild from flat spacetime, followed by other Kerr–Schild maps merely changing the mass.

VI. CONLUDING REMARKS

Exact vacuum Kerr–Schild spacetimes can be recovered as solutions of the linearized Einstein equations. This advantageous property was shown to persist in the presence of specific matter sources. K-essence scalar fields could source exact Kerr–Schild spacetimes induced by the perturbative solution for Lagrangian either linear in the kinetic term (in this case the Kerr–Schild congruence being autoparallel) or unrestricted, when the scalar is constant along the Kerr–Schild congruence, $D\phi = 0$. In this paper we generalized the source term to include the full class of kinetic braiding scalar fields. We proved that the property withstands for the pure kinetic gravity braiding contribution only for a vanishing Lagrangian (equivalent to vacuum for this case) or in the case $D\phi = 0$ (a scalar constant along the Kerr–Schild congruence). We also showed that the requirement of an unchanged scalar field and an unchanged functional form of its energy-momentum tensor under the Kerr–Schild map induces a scalar constraint, which has to be considered in addition to the linearized Kerr–Schild equation.

Finally, we discussed pp-waves, which are manifestly of Kerr–Schild type. Beyond vacuum pp-waves we also derived those which are generated by a k-essence with vanishing Lagrangian and $D\phi = 0$ (such that they obey the scalar constraint). At the end, we identified a Fock space type construction through successive application of a tower of Kerr–Schild maps.

Gravitational waves in the geometrical optics approximation are pp-waves. With the forthcoming LISA and other prospective space detectors, able to monitor gravitational wave for weeks or even months, vacuum pp-waves (with $C = 0$) could in principle be distinguished from those sourced by k-essence (with $C \neq 0$), raising the possibility to constraint such scalar fields. Working out such methods would provide another challenging way for testing general relativity.

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