CONTROL ANALYSIS AND SYNTHESIS FOR GENERAL CONTROL-AFFINE SYSTEMS*

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Abstract. We study controllability and constructive synthesis for control–affine systems. We introduce trajectory–dependent Gramian maps that extend the linear time–varying Gramian and yield explicit fixed–point synthesis maps. On feasible coercivity classes (uniform eigenvalue lower bounds), the Gramian map is Lipschitz, and under a comparison estimate criterion, synthesis iterates exhibit decay, and the Banach fixed-point theorem gives a unique fixed point that steers the system and satisfies an energy identity. When, in addition, an orthogonality condition holds, this fixed point coincides with the unique global minimum–energy control on the feasible set; if the coercivity bound holds uniformly for all bounded controls, the same conclusion holds on the full bounded–control space. We provide structural conditions on the input matrix that ensure the nonemptiness of the feasible class (and, in fully actuated regimes, equality with the full space) and sufficient conditions for underactuated systems via bounded–amplitude reference controls. Case studies on Hopfield network dynamics illustrate refined estimates that enlarge reachable targets. A trajectory–freezing and compactness step extends the synthesis to general nonlinear control-affine systems. The results yield verifiable controllability criteria with explicit, numerically implementable controllers.

Key words. Control-affine systems; nonlinear controllability; fixed-point methods; controllability Gramians; reachability; minimal-energy control.

MSC codes. 93B05, 93C10, 93C15, 49K15, 49N35.

1. Introduction. We study control analysis and implementable synthesis for the control–affine class

$$\dot{x}(t) = A(t,x(t))\,N_t(x(t)) + B(t,x(t))\,u(t), \qquad x(t_0)=x^0, \quad t\in[t_0,T], \tag{1.1}$$

where at time t, the system's state is $x(t)=(x_1(t),\dots,x_d(t))^\top\in\mathbb{R}^d$, $x^0=(x_1^0,\dots,x_d^0)^\top\in\mathbb{R}^d$ is the initial state, $u(t)=(u_1(t),\dots,u_k(t))^\top\in\mathbb{R}^k$ is the control input, N_t is a nonautonomous, vector field, $A(t,x)$ and $B(t,x)$ are time- and state-dependent matrices. Models of this form cover, among others, recurrent neural dynamics [15, 24] and standard engineering systems [6], and hence their study remains important.

Systems of the form (1.1) have been extensively studied in control theory, with a wide range of mathematical techniques developed over the past decades. Classical approaches include linearization-based analysis [25, 10], homotopy continuation [8, 17, 16, 29], and the return method [9, 10], which remains one of the most powerful constructive tools for global controllability. Geometric and algebraic methods based on Lie algebras provide structural controllability criteria [14, 20, 21, 27, 28, 10], while topological arguments offer complementary perspectives [12, 24]. Other analytic techniques include power series expansions [10] and fixed-point approaches [33, 34, 10, 20]. These contributions have led to a rich theory of local and global controllability; yet, implementable minimum-energy controls remain scarce beyond the linear time-invariant (LTI) and linear time-varying (LTV) settings [18], where the controllability Gramian yields closed-form minimum-energy controls.

Beyond the LTI/LTV setting, explicit Gramian–based formulas for minimum–energy controls are rarely available, and many nonlinear controllability results are qualitative or rely on continuation along a state path. Our stance is to recover "Gramian calculus" for system (1.1) by introducing *trajectory–dependent* controllability Gramian maps and casting synthesis as a fixed–point problem on an *feasible coercivity class*. This yields (i) a constructive control law via an implicit Gramian formula; (ii) a built–in *energy certificate*; and (iii) a convergence mechanism (via explicit Volterra rank-one comparison estimates for the iterates of the synthesis map), which is directly implementable. In contrast, the homotopy continuation method (HCM) [8, 17, 29] advances by integrating the path lifting equation and therefore requires surjectivity of the differential of the endpoint map along the entire path and typically offers no intrinsic energy certificate. Note, however, that a regularized continuation method has been proposed in [22] to deal with the case where this differential is not surjective. Still, the approach requires some conditions for the solvability: the drift dynamics must be well-posed over

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Funding: This work is partially supported by grant R21MH132240 from the US National Institutes of Health to SC.

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$[t_0, T]$, and the family of regularized path lifting equations must converge to a solution of the original problem. See also [16, 23]. Our framework trades this global rank condition for verifiable, local requirements: the coercivity of a trajectory–dependent Gramian, a self–mapping inequality on the feasible class, and an explicit iterate-contractivity condition. In practice, this matters when complete controllability is not known a priori: the method still furnishes steering controls (and their cost) whenever the feasible class is nonempty, and it delivers a quantitative subset of the reachable set. When HCM does apply, the two approaches are complementary—the nonlinear Gramian recovers the classical solution in the linear case and can serve as a preconditioner or initializer for continuation—while in regimes where the rank fluctuates, the Banach fixed–point route remains robust by construction.

Our main contributions are the following: *(i) Baseline synthesis on $\dot{x} = N_t(x) + B(t,x)u$:* We work on a feasible coercivity class $\mathcal{F}(C)$ (uniform lower bound on the trajectory–dependent Gramian). On $\mathcal{F}(C)$, the synthesis operator is well defined, and—on a self–mapped ball—a Volterra rank-one comparison estimate gives an iterate-contractivity criterion, which yields a unique fixed point with the Picard convergence and an energy certificate. *(ii) From relative to global minimality:* If an orthogonality condition holds at the fixed point, then this fixed point is the *unique global* L^2–minimizer over the feasible set. *(iii) Structural guarantees:* An integral nondegeneracy of the input matrix B implies uniform coercivity ($\mathcal{F}(C) = L^\infty$). More generally, a reference control $u^\flat$ with invertible Gramian and amplitude small relative to a prescribed model-driven constant ensures the self–map property for explicit target radii; model structure (e.g., Hopfield networks) sharpens constants and enlarges admissible targets. *(iv) Reachability:* Self–mapping on $\mathcal{F}(C)$ together with the iterate-contractivity condition provides constructive steering with quantitative energy bounds; if $\mathcal{F}(C) = L^\infty$ one recovers complete controllability, otherwise one obtains a computable subset of the reachable set. *(v) Extension to the general system* (1.1)*:* Freezing along an auxiliary trajectory reduces (1.1) to a family of baseline problems with uniform Lipschitz control of the Gramian map; Schaefer's theorem then yields steering controls for admissible targets. For $N_t(x) = x$, the construction recovers [10, Theorem. 3.40].

The remainder of the paper is structured as follows. In Notation 1.1 and Assumptions 1.2, we introduce the notation and assumptions used throughout. Section 3 develops the synthesis framework for the baseline system $\dot{x} = N_t(x) + B(t,x)u$: we introduce the nonlinear Gramian representations, construct the synthesis maps, define the feasible coercivity class, and use an iterate-contractivity condition to obtain steering controls. Structurally sufficient conditions are then established, and refined estimates are illustrated on Hopfield-type networks. Section 4 extends the framework to the general control-affine system (1.1) by freezing along auxiliary trajectories and invoking a compactness argument to recover reachability for admissible targets. Section 5 summarizes the results and discusses perspectives for future work, and main proofs are collected in Section A.

Notation 1.1. In the following, we denote by $\mathbb{R}^{d\times k}$ the set of $d \times k$ matrices with real coefficients. $\mathbb{R}^d$ denotes the vector of d real columns in dimension and $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^d)$ the space of linear maps from $\mathbb{R}^k$ to $\mathbb{R}^d$ that we identify, in the usual way, with $\mathbb{R}^{d\times k}$. For all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, we denote by $|x|$ the Euclidean norm of x and by $\langle x, y\rangle$ the scalar product of x and y. For a symmetric positive definite matrix $M \in \mathbb{R}^{d\times d}$, the norm of $x \in \mathbb{R}^d$ with respect to M is defined by $|x|_M^2 = x^\top M x$. We denote the identity matrix of any size by Id, and for every $M \in \mathbb{R}^{d\times k}$, $\|M\| = \sup\{|Mx| : x \in \mathbb{R}^k,\ |x| = 1\}$ denotes the spectral norm of M. For a symmetric matrix $M \in \mathbb{R}^{d\times d}$, $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote respectively its smaller and largest eigenvalues. Finally, we use $\mathscr{L}(X, Y)$ to denote the space of linear bounded operators from two normed vector spaces X and Y.

Assumption 1.2. Throughout the following, unless otherwise stated, the nonautonomous vector field $N_t : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following assumptions

1. The map $t \mapsto N_t(x)$ belongs to $L^\infty(t_0, T)$ for every fixed $x \in \mathbb{R}^d$,
2. The map $x \mapsto N_t(x)$ is C^2 for every fixed $t \in [t_0, T]$.

Additionaly, for every $t \in [t_0, T]$, N_t is globally Λ_1-Lipschitz function for some $\Lambda_1 > 0$, viz.

$$\|DN_t(w)\| \le \Lambda_1 \qquad \forall (t, w) \in [t_0, T] \times \mathbb{R}^d, \tag{1.2}$$

where $DN_t(w) \in \mathbb{R}^{d\times d}$ is the differential of N_t at any $w \in \mathbb{R}^d$. Finally, we also assume that its second derivative

$D^2 N_t$ satisfy for some $\Lambda_2 > 0$

$$\|D^2 N_t(x)\| \le \Lambda_2 \qquad \forall (t,x) \in [t_0,T] \times \mathbb{R}^d. \tag{1.3}$$

Remark 1.3. For every fixed $t \in [t_0,T]$, the map $x \mapsto N_t(x)$ does not need to be bounded. Throughout the manuscript, we let $\Delta t_0 := T - t_0$.

2. Prerequisites and flow representation. It follows from Assumption 1.2 that the nonautonomous flow $t \in [t_0,T] \mapsto \Phi_{t_0,t}$ associated with N_t is globally Lipschitz and solves the following equation [4, Chapter 2]

$$\frac{d}{dt}\Phi_{t_0,t}(x^0) = N_t(\Phi_{t_0,t}(x^0)), \qquad \Phi_{t_0,t_0}(x^0) = x^0 \in \mathbb{R}^d. \tag{2.1}$$

Moreover, $\{\Phi_{s,t} \mid (s,t) \in [t_0,T]^2\}$ forms a two-parameter family of diffeomorphisms which satisfies the following algebraic identities

$$\Phi_{t,t} = \mathrm{Id} \qquad \forall t \in [t_0,T], \tag{2.2}$$

$$\Phi_{t_2,t_3} \circ \Phi_{t_1,t_2} = \Phi_{t_1,t_3} \qquad \forall (t_1,t_2,t_3) \in [t_0,T]^3, \tag{2.3}$$

$$(\Phi_{t_1,t_2})^{-1} = \Phi_{t_2,t_1} \qquad \forall (t_1,t_2) \in [t_0,T]^2. \tag{2.4}$$

Furthermore, for fixed $(s,t) \in [t_0,T]^2$, the maps $\Phi_{s,t}$ and $\Phi_{t,s}$ are differentiable at every x^0 and $\Phi_{s,t}(x^0)$, respectively. Denoting these differential by $D\Phi_{s,t}(x^0)$, and $D\Phi_{t,s}(\Phi_{s,t}(x^0))$, they solve

$$\begin{cases} \dfrac{d}{dt}D\Phi_{s,t}(x^0) & = DN_t(\Phi_{s,t}(x^0))\, D\Phi_{s,t}(x^0), \qquad D\Phi_{s,s}(x^0) = \mathrm{Id}, \\ \\ \dfrac{d}{dt}D\Phi_{t,s}(\Phi_{s,t}(x^0)) & = -D\Phi_{t,s}(\Phi_{s,t}(x^0))\, DN_t(\Phi_{s,t}(x^0)), \qquad D\Phi_{s,s}(\Phi_{s,t}(x^0)) = \mathrm{Id}. \end{cases} \tag{2.5}$$

In particular, the following holds for every $x^0 \in \mathbb{R}^d$,

$$[D\Phi_{s,t}(x^0)]^{-1} = D\Phi_{t,s}(\Phi_{s,t}(x^0)) \qquad \forall (s,t) \in [t_0,T]^2. \tag{2.6}$$

We collect useful estimates related to $\Phi_{s,t}$ and $\Phi_{t,s}$ in the following lemma. The proof uses the standard Gronwall lemma.

LEMMA 2.1. *Let $\beta \in C^0([t_0,T];\mathbb{R}^d)$ and set $q := \Lambda_2/\Lambda_1$. It holds for every $(s,t) \in [t_0,T]^2$, $s \le t$,*

$$\begin{aligned} \|D\Phi_{s,t}(\beta(s))\| &\le e^{\Lambda_1(t-s)}, \qquad \|D\Phi_{t,s}(\beta(s))\| \le e^{\Lambda_1(t-s)}, \\ \|D\Phi_{s,t}(\beta(t))\| &\le e^{\Lambda_1(t-s)}, \qquad \|D\Phi_{t,s}(\beta(t))\| \le e^{\Lambda_1(t-s)}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \|D^2\Phi_{s,t}(\beta(s))\| &\le q(e^{2\Lambda_1(t-s)} - e^{\Lambda_1(t-s)}), \qquad \|D^2\Phi_{t,s}(\beta(s))\| \le q(e^{2\Lambda_1(t-s)} - e^{\Lambda_1(t-s)}), \\ \|D^2\Phi_{s,t}(\beta(t))\| &\le q(e^{2\Lambda_1(t-s)} - e^{\Lambda_1(t-s)}), \qquad \|D^2\Phi_{t,s}(\beta(t))\| \le q(e^{2\Lambda_1(t-s)} - e^{\Lambda_1(t-s)}), \end{aligned} \tag{2.8}$$

where $D^2\Phi_{s,t}$ is the second derivative (third-order tensor) of $\Phi_{s,t}$.

3. Control–affine dynamics with state–dependent and time-varying input matrix. In this section, we consider the following control–affine system with state–dependent and time-varying input matrix

$$\dot{x}(t) = N_t(x(t)) + B(t,x(t))u(t), \quad x(t_0) = x^0, \quad t \in [t_0,T], \tag{Σ}$$

which is a specific case of (1.1) where $A(t,x) = \mathrm{Id}$. We recall that $(t,x) \mapsto N_t(x)$ satisfies Assumption 1.2.

Assumption 3.1. The input matrix $B : [t_0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ is an element of $L^\infty((t_0,T) \times \mathbb{R}^d;\mathbb{R}^{d\times k})$. Furthermore, the map $x \mapsto B(t,x)$ is C^1 and globally Lipschitz for every $t \in [t_0,T]$, i.e., there exists $L_B \ge 0$ such that

$$\|D_x B(t,w)\| \le L_B \qquad \forall (t,w) \in [t_0,T] \times \mathbb{R}^d. \tag{3.1}$$

Note that if $B(t,\cdot) = B(t)$, then $L_B = 0$.

3.1. Representation of solutions. The existence and uniqueness of global solutions to (Σ) follow as a straightforward application of the Cauchy-Lipschitz theorem, given Assumptions 1.2 and 3.1, see, for instance, [4, Theorem 3.2.1]. However, the main objective of this paper is to synthesize a control function that solves the associated two-point boundary value problem. To this end, leveraging the regularity properties of the drift N_t, we consider solution representations of (Σ) that facilitate control design. These solution representations fall within the general framework of chronological calculus introduced in [2] and leveraged in [19, 21] for applications in geometric control theory. See also [1, Chapter 6] for a comprehensive presentation of this theory. In [30], solution representations of nonlinear control systems like (Σ) were established when $N_t = N$, $B(t,x) = B$, and the control was constant; see also the proof of [31, Theorem 4.4]. In contrast, the present work extends these representations to the framework of (Σ). For completeness, we replace in (Σ), $B(t,x)\,u(t)$ with $b(t,x) \in L^\infty((t_0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ where $x \mapsto b(t,x)$ is locally Lipschitz. The proof of the following theorem is given in Section A.1.

THEOREM 3.2. *For all $x^0 \in \mathbb{R}^d$, and $b \in L^\infty((t_0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ such that $x \mapsto b(t,x)$ is locally Lipschitz, the system $\dot{x}(t) = N_t(x(t)) + b(t,x(t))$, $x(t_0) = x^0$ admits a unique absolutely continuous solution $x \in C^0([t_0,T];\mathbb{R}^d)$ that can be written for $i \in \{1,2\}$,*

$$x(t) = \Phi_{\tau_i,t}\left(\Phi_{t_0,\tau_i}(x^0) + \int_{t_0}^{t} D\Phi_{s,\tau_i}\big(x(s)\big)\, b(s,x(s))\, ds\right), \qquad t \in [t_0,T], \quad \tau_1 = t_0,\ \tau_2 = T. \tag{3.2}$$

The representation corresponding to $\tau_1 = t_0$ is termed *forward representation*, while that corresponding to $\tau_2 = T$ is termed *backward representation*, consistent with their temporal directionality.

Observe that if $N_t \equiv A(t)$, where $A : [t_0,T] \to \mathbb{R}^{d\times d}$ is a matrix-valued function in $L^\infty((t_0,T);\mathbb{R}^{d\times d})$, then (3.2) reduces to the classical representation of solutions for linear time-varying (LTV) systems.

COROLLARY 3.3. *If $N_t \equiv A(t) \in L^\infty((t_0,T);\mathbb{R}^{d\times d})$ and $b(t,\cdot) \equiv b(t) \in L^\infty((t_0,T);\mathbb{R}^d)$, then (3.2) recasts as*

$$x(t) = R(t,t_0)\, x^0 + \int_{t_0}^{t} R(t,s)\, b(s)\, ds \qquad \forall t \in [t_0,T], \tag{3.3}$$

where $R(t,s) \in C^0([t_0,T]^2;\mathbb{R}^{d\times d})$ is the state-transition matrix of $\dot{y}(t) = A(t)\, y(t)$ satisfying $R(s,s) = \mathrm{Id}$.

Proof. First, by (2.1)-(2.4) and the definition of $R(t,s)$, one has $\Phi_{s,t}(x) = R(t,s)\,x$ for all $x \in \mathbb{R}^d$. It follows that $D\Phi_{s,t}(x(t)) = R(t,s)$ for all $(t,s) \in [t_0,T]^2$. Therefore, from (3.2), one deduces

$$x(t) = \Phi_{\tau_i,t}\left(\Phi_{t_0,\tau_i}(x^0) + \int_{t_0}^{t} D\Phi_{s,\tau_i}(x(s))\, b(s)\, ds\right) = R(t,t_0)\, x^0 + R(t,\tau_i)\int_{t_0}^{t} R(\tau_i,s)\, b(s)\, ds \tag{3.4}$$

which yields (3.3). ∎

We now establish results that relate the state-transition matrix of the linearized form of control system (Σ) to derivatives of the flow of the vector field N_t along the trajectory. It will be essential for our subsequent results. The proof is presented in Section A.2.

LEMMA 3.4. *Let $(x^0,u) \in \mathbb{R}^d \times L^\infty((t_0,T);\mathbb{R}^k)$, and let $x_u \in C^0([t_0,T];\mathbb{R}^d)$ be the solution of (Σ). Then, the state-transition matrix $R_u(t,s) \in C^0([t_0,T]^2;\mathbb{R}^{d\times d})$ of the linearized equation*

$$\dot{y}(t) = \big[DN_t(x_u(t)) + D_xB(t,x_u(t))u(t)\big]\, y(t), \qquad t \in [t_0,T] \tag{3.5}$$

satisfying $R_u(s,s) = \mathrm{Id}$, is given by $R_u(t,s) = P_{\tau_i,u}(t,s)$, $i \in \{1,2\}$ where $P_{\tau_i,u}(t,s)$ can be factorized as

$$P_{\tau_i,u}(t,s) := \big[D\Phi_{t,\tau_i}(x_u(t))\big]^{-1} M_{\tau_i,u}(t,s)\, D\Phi_{s,\tau_i}(x_u(s)), \qquad (t,s) \in [t_0,T]^2, \quad \tau_1 = t_0,\ \tau_2 = T. \tag{3.6}$$

Here, $M_{\tau_i,u}(t,s) \in C^0([t_0,T]^2;\mathbb{R}^{d\times d})$ satisfying $M_{\tau_i,u}(s,s) = \mathrm{Id}$ is the state-transition matrix of

$$\dot{y}(t) = \mathcal{A}_{\tau_i,u}(t)\, y(t), \qquad t \in [t_0,T] \tag{3.7}$$

where $\mathcal{A}_{\tau_i,u}(t) := \big[D^2\Phi_{t,\tau_i}(x_u(t))B(t,x_u(t))u(t) + D\Phi_{t,\tau_i}\big(x_u(t)\big)D_xB\big(t,x_u(t)\big)u(t)\big]\big[D\Phi_{t,\tau_i}(x_u(t))\big]^{-1} \in \mathbb{R}^{d\times d}$.

Remark 3.5. When $B(t,x) \equiv B(t) \in L^\infty((t_0,T);\mathbb{R}^k)$ is state-independent, it is clear from the factorization (3.6) that the state-transition matrix of $\dot{y}(t) = DN_t(x_u(t))y(t)$ is different from $D\Phi_{s,t}(x_u(s))$ in general, and that there is equality if the control $u = 0$ or if N_t is linear. In the former case, $M_{\tau_i,0}(t,s) = \mathrm{Id}$, and we use

$$S_{t_0,x^0}(t,s) := P_{t_0,0}(t,s) = D\Phi_{t_0,t}(x^0)\, D\Phi_{s,t_0}(\Phi_{t_0,s}(x^0)) = D\Phi_{s,t}(\Phi_{t_0,s}(x^0)) \qquad \forall (s,t) \in [t_0,T]^2 \tag{3.8}$$

as the representation of the state-transition matrix of the linearized equation $\dot{y}(t) = DN_t(\Phi_{t_0,t}(x^0))\, y(t)$.

3.2. Controllability results. Given $(x^0,x^1) \in \mathbb{R}^d \times \mathbb{R}^d$, our goal in this section is to identify sufficient conditions under which system (Σ) can be steered from the initial state x^0 to the target x^1 over the horizon $[t_0,T]$. Whenever these conditions are met, we provide a synthesis of two feasible controls realizing this transfer.

3.2.1. Synthesis, analysis, and optimal control results. As announced, we present our control synthesis, analysis, and optimal control results in this section. We start with recalling the following

DEFINITION 3.6. *Given $x^0 \in \mathbb{R}^d$, we say that $x^1 \in \mathbb{R}^d$ is* reachable *on $[t_0,T]$ from x^0 if there exists a control $u \in L^\infty((t_0,T);\mathbb{R}^k)$ such that the solution $x_u(\cdot)$ of (Σ) satisfies $x_u(T) = x^1$. The reachable set is denoted by*

$$\mathcal{R}_\Sigma(T,x^0) := \Big\{\, x_u(T) \;:\; x_u(\cdot) \text{ solves } (\Sigma) \text{ with } x(t_0) = x^0 \,\Big\}. \tag{3.9}$$

If $\mathcal{R}_\Sigma(T,x^0) = \mathbb{R}^d$ for every $x^0 \in \mathbb{R}^d$, one says that (Σ) is completely controllable on $[t_0,T]$.

For a fixed $x^0 \in \mathbb{R}^d$ and control $u \in L^\infty((t_0,T);\mathbb{R}^k)$, let $x_u(\cdot) := x_{u,x^0}(\cdot)$ be the corresponding trajectory of (Σ). We associate with u and x^0 the matrices ($\tau_1 = t_0$ and $\tau_2 = T$)

$$\mathcal{N}_{\tau_i}(u) := \mathcal{N}_{\tau_i}(u,x^0) = \int_{t_0}^T D\Phi_{t,\tau_i}(x_u(t))\, B(t,x_u(t))\, B(t,x_u(t))^\top D\Phi_{t,\tau_i}(x_u(t))^\top\, dt, \quad i \in \{1,2\}. \tag{3.10}$$

Remark 3.7. (1) If $N_t \equiv A(t) \in L^\infty((t_0,T);\mathbb{R}^{d\times d})$ and $B(t,\cdot) \equiv B(t) \in L^\infty((t_0,T);\mathbb{R}^{d\times k})$, then $\mathcal{N}_{\tau_1}(u)$ is congruent to the controllability Gramian W_c of the LTV system $\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$, and $\mathcal{N}_{\tau_2}(u) = W_c$.

(2) For any $y \in \mathbb{R}^d$,

$$y^\top \mathcal{N}_{\tau_i}(u)\, y = \int_{t_0}^T |y^\top D\Phi_{t,\tau_i}(x_u(t))\, B(t,x_u(t))|^2\, dt, \tag{3.11}$$

hence $\mathcal{N}_{\tau_i}(u)$ is symmetric and positive semi-definite (PSD).

(3) The Gramian map[1] $\mathcal{N}_{\tau_i} : L^\infty((t_0,T);\mathbb{R}^k) \to \mathcal{S}_d^+(\mathbb{R})$ defined by (3.10) is in general different from the notion of Gramian used in the homotopy continuation method [8, 16, 29] where the Gramian is defined as

$$M(u) := \int_{t_0}^T R_u(T,t)\, B(t,x_u(t))\, B(t,x_u(t))^\top R_u(T,t)^\top\, dt, \qquad u \in L^\infty((t_0,T);\mathbb{R}^k) \tag{3.12}$$

where $R_u(t,s)$ is defined in Lemma 3.4.

The proof of the following proposition uses standard arguments of minimization in Hilbert spaces.

PROPOSITION 3.8. *Let $x^0 \in \mathbb{R}^d$, $u \in L^\infty((t_0,T);\mathbb{R}^k)$, and $x_u \in C^0([t_0,T];\mathbb{R}^d)$ be the solution to (Σ). Fix $i \in \{1,2\}$ and define the bounded linear operator $L_{u,\tau_i} : L^2((t_0,T);\mathbb{R}^k) \to \mathbb{R}^d$ by*

$$L_{u,\tau_i} v = \int_{t_0}^T D\Phi_{t,\tau_i}(x_u(t))\, B(t,x_u(t))\, v(t)\, dt, \qquad \tau_1 := t_0,\ \tau_2 := T. \tag{3.13}$$

For $y_i \in \mathrm{Ran}(L_{u,\tau_i})$, the problem

$$\min\{\|v\|_2 : L_{u,\tau_i} v = y_i\} \tag{3.14}$$

[1]We use $\mathcal{S}_d^+(\mathbb{R})$ to denote the set of symmetric positive semi-definite matrices of order d with real coefficients.

has the solution $v_i(u) := v_i(u, x^0, y_i)$ *given by*

$$v_i(u)(t) = \left[L_{u,\tau_i}^* \mathcal{N}_{\tau_i}(u)^\dagger y_i\right](t) = B(t, x_u(t))^\top D\Phi_{t,\tau_i}(x_u(t))^\top \mathcal{N}_{\tau_i}(u)^\dagger y_i, \qquad t \in [t_0, T]. \tag{3.15}$$

Here L_{u,τ_i}^* *is the adjoint of* L_{u,τ_i}, *and* $\mathcal{N}_{\tau_i}(u)^\dagger$ *denotes the Moore–Penrose pseudoinverse of* $\mathcal{N}_{\tau_i}(u)$.

Proposition 3.8 ensures that $v_i(u)$ is L^2–minimal within the affine subspace defined by L_{u,τ_i}. In the nonlinear setting, these subspaces depend on u, so minimality is relative rather than global.

Using (2.7) and the singular value decomposition of symmetric PSD matrices, one finds that

$$\|v_i(u)\|_\infty \le \frac{\|B\|_\infty e^{\Lambda_1 \Delta t_0}|y_i|}{\lambda_{\min}^*(\mathcal{N}_{\tau_i}(u))} \qquad \forall u \in L^\infty((t_0,T);\mathbb{R}^k),\ i \in \{1,2\}, \tag{3.16}$$

where $\lambda_{\min}^*(\mathcal{N}_{\tau_i}(u)) > 0$ is the smallest nonzero eigenvalue of $\mathcal{N}_{\tau_i}(u)$.

Feasible coercivity class. Let $\lambda_{\min}\big(\mathcal{N}_{\tau_i}(u)\big)$ denote the smallest eigenvalue of $\mathcal{N}_{\tau_i}(u)$, and let $C_i > 0$ be a constant depending only on system data (e.g. $\|B\|_\infty$, Λ_1, Λ_2, Δt_0, and possibly x^0). Guided by (3.16), we encode the positivity of $\mathcal{N}_{\tau_i}(u)$ through the *feasible coercivity class*

$$\mathcal{F}(C_i) := \Big\{u \in L^\infty((t_0,T);\mathbb{R}^k):\ \lambda_{\min}\big(\mathcal{N}_{\tau_i}(u)\big) \ge C_i^{-1}\Big\}, \quad i \in \{1,\,2\}. \tag{3.17}$$

The following observations on $\mathcal{F}(C_i)$ are immediate.

Remark 3.9.

1. *Possibility of emptiness.* For a given $C_i > 0$, the feasible coercivity class $\mathcal{F}(C_i)$ may be empty. Non-emptiness requires the existence of at least one control u with $\lambda_{\min}(\mathcal{N}_{\tau_i}(u)) \ge C_i^{-1}$.
2. *Monotonicity in* C_i. If $C_i \le C_i'$, then $\mathcal{F}(C_i) \subseteq \mathcal{F}(C_i')$.
3. *Topological property.* By Assumptions 1.2, the map $u \mapsto \mathcal{N}_{\tau_i}(u)$ is continuous, hence $u \mapsto \lambda_{\min}(\mathcal{N}_{\tau_i}(u))$ is continuous. Therefore $\mathcal{F}(C_i)$ is closed in $L^\infty((t_0,T);\mathbb{R}^k)$. We clarify this point later in Lemma 3.21 and Corollary 3.23.
4. *Non-emptiness criterion.* If there exists a control reference $u^\flat \in L^\infty((t_0,T);\mathbb{R}^k)$ with $\lambda_{\min}(\mathcal{N}_{\tau_i}(u^\flat)) \ge C_i^{-1}$, then $\mathcal{F}(C_i) \neq \emptyset$. By the continuity of $u \mapsto \lambda_{\min}(\mathcal{N}_{\tau_i}(u))$, a neighborhood of $u^\flat$ also lies in $\mathcal{F}(C_i)$. A sufficient condition ensuring $\mathcal{F}(C_i) = L^\infty((t_0,T);\mathbb{R}^k)$ for every $x^0 \in \mathbb{R}^d$ is provided in Proposition 3.19.

For any $u \in \mathcal{F}(C_i)$, estimate (3.16) yields the uniform bound

$$\|v_i(u)\|_\infty \ \le \zeta(y_i) = C_i\,\|B\|_\infty\, e^{\Lambda_1 \Delta t_0}\,|y_i|, \qquad i \in \{1,2\}. \tag{3.18}$$

Let $v_i(u, x^0, y_i)$ is defined as in Proposition 3.8, we define the synthesis maps

$$\mathcal{S}_i : L^\infty((t_0,T);\mathbb{R}^k) \to L^\infty((t_0,T);\mathbb{R}^k),\ u \mapsto v_i(u, x^0, y_i), \qquad y_i \in \mathbb{R}^d, \quad i \in \{1,\,2\}. \tag{3.19}$$

Motivated by (3.18), we introduce the feasibility coercivity ball

$$\mathcal{F}(y_i) := \mathcal{F}(C_i) \cap \mathcal{B}_i \quad \text{where} \quad \mathcal{B}_i := \{u \in L^\infty((t_0,T);\mathbb{R}^k) : \|u\|_\infty \le\ \zeta_i := \zeta(y_i)\}. \tag{3.20}$$

Under the non-emptiness assumption on $\mathcal{F}(C_i)$, and if $\mathcal{S}_i$ acts as a self-map on $\mathcal{F}(C_i)$, namely

$$\mathcal{S}_i(\mathcal{F}(C_i)) \subset \mathcal{F}(C_i), \tag{3.21}$$

we can prove the existence and uniqueness of fixed points for $\mathcal{S}_i$ in $\mathcal{F}(y_i)$, whenever an associated Volterra rank-one comparison estimate is contractive.

The proof of the following result is given in Section A.3.

THEOREM 3.10. *Fix* $i \in \{1,2\}$. *Assume that* $\mathcal{F}(C_i) \neq \emptyset$ *and that* (3.21) *is satisfied. Then there exists a positive sequence* $(\varrho_m)_{m\ge1}$ *with* $\varrho_m := K_i^m \gamma_m(p_i)$ *such that*

$$\|\mathcal{S}_i^m(u) - \mathcal{S}_i^m(v)\|_\infty \le \varrho_m \|u - v\|_\infty, \qquad \forall\, u, v \in \mathcal{F}(y_i),\ \forall\, m \ge 1, \tag{3.22}$$

where $K_i > 0$ depends only on system data and C_i, and for some $p_i \in (0,1)$ and $\gamma_m(p_i) \to 0$ as $m \to \infty$. Furthermore, if there exists $m_i \geq 1$ such that

$$\varrho_{m_i} < 1, \tag{3.23}$$

then $\mathcal{S}_i$ admits a unique fixed point $u_i \in \mathcal{F}(y_i)$, and the Picard iteration $u^{(n+1)} = \mathcal{S}_i(u^{(n)})$ converges to u_i for any $u^{(0)} \in \mathcal{F}(y_i)$.

Remark 3.11. If $\mathcal{F}(C_i) \neq \emptyset$ and (3.21) is satisfied, then $\mathcal{F}(y_i) \neq \emptyset$. In fact, under these assumptions, $\mathcal{S}_i(u) \in \mathcal{F}(y_i)$ for all $u \in \mathcal{F}(C_i)$. Furthermore, if $B(t,x) \equiv B(t)$ is not state-dependent, the decay estimate (3.22) holds for all $(u,v) \in \mathcal{F}(C_i)^2$ since the Gramian map $\mathcal{N}_{\tau_i}$ is Lipschitz continuous on the whole $L^\infty((t_0,T);\mathbb{R}^k)$.

Theorem 3.10 guarantees the existence and uniqueness of a fixed point of the synthesis map $\mathcal{S}_i$ on $\mathcal{F}(y_i)$. To connect this fixed point with L^2–norm minimality, we invoke a Lagrange multiplier. We will need preparatory results stated in Lemma 3.12 below. To this end, we introduce for any fixed $x^0 \in \mathbb{R}^d$ the endpoint map

$$\mathcal{E} := \mathcal{E}_{x^0,T} : u \in L^\infty((t_0,T);\mathbb{R}^k) \longmapsto \mathbb{R}^d, \quad \mathcal{E}(u) = x_u(T) \tag{3.24}$$

and for $i \in \{1,2\}$ and any fixed $y_i \in \mathbb{R}^d$, the feasible map

$$G_{\tau_i} : L^\infty((t_0,T);\mathbb{R}^k) \to \mathbb{R}^d, \quad G_{\tau_i}(u) = L_{u,\tau_i}u - y_i \tag{3.25}$$

where $x_u(\cdot)$ is the solution of (Σ) and L_{u,τ_i} is defined in (3.13). The proof of the following is given in Section A.4.

LEMMA 3.12. *One has $\mathcal{E}, G_{\tau_i} \in C^1(\mathcal{Y};\mathbb{R}^d)$. Furthermore, let $D\mathcal{E}(u)$, $DG_{\tau_i}(u) : L^2((t_0,T);\mathbb{R}^k) \to \mathbb{R}^d$ are, respectively the Fréchet derivative of $\mathcal{E}$ and G_{τ_i} at any fixed $u \in \mathcal{Y}$. Then, one has the following identity*

$$DG_{\tau_i}(u) = D\Phi_{T,\tau_i}\big(x_u(T)\big)\, D\mathcal{E}(u). \tag{3.26}$$

Moreover, let $\mathcal{N}_{\tau_i}(u)$ and $M(u) := D\mathcal{E}(u)\, D\mathcal{E}(u)^*$ are respectively defined in (3.10) and (3.12).

1. *If $\mathcal{N}_{\tau_i}(u)$ is invertible and $\mathrm{Id} + K_{u,\tau_i}L^*_{u,\tau_i}\mathcal{N}_{\tau_i}(u)^{-1} \in \mathbb{R}^{d\times d}$ is invertible, then $DG_{\tau_i}(u)L^*_{u,\tau_i} \in \mathbb{R}^{d\times d}$ is invertible and $DG_{\tau_i}(u)$ is right-invertible;*
2. *If $M(u)$ is invertible and $\mathrm{Id} - K_{u,\tau_i}DG_{\tau_i}(u)^*M(u)^{-1} \in \mathbb{R}^{d\times d}$ is invertible, then $L_{u,\tau_i}DG_{\tau_i}(u)^* \in \mathbb{R}^{d\times d}$ is invertible and L_{u,τ_i} is right-invertible.*

Here, we let $K_{u,\tau_i} := DG_{\tau_i}(u) - L_{u,\tau_i}$.

The following results connect the fixed-point of Theorem 3.10 with L^2-nom minimality in the feasible set; its proof is presented in Section A.5.

THEOREM 3.13. *Fix $i \in \{1,2\}$, assume $\mathcal{F}(C_i) \neq \emptyset$, and $\mathrm{Id} + K_{u,\tau_i}L^*_{u,\tau_i}\mathcal{N}_{\tau_i}(u)^{-1} \in \mathbb{R}^{d\times d}$ is invertible for all $u \in \mathcal{F}(C_i)$. Let $y_i \in \mathbb{R}^d$ and the feasible set*

$$\mathfrak{F}_i := \{u \in \mathcal{F}(y_i) : \; G_{\tau_i}(u) = 0\}.$$

Then, the following are equivalent for any local minimizer $\bar{u}$ of $\frac{1}{2}\|u\|^2_{L^2}$ over $\mathfrak{F}_i$:

1. *$\bar{u}$ is a fixed point of $\mathcal{S}_i$ on $\mathcal{F}(y_i)$, i.e., $\bar{u} = L^*_{\bar{u},\tau_i}\mathcal{N}_{\tau_i}(\bar{u})^{-1}y_i$;*
2. *The following orthogonality condition holds*

$$\langle [L_{\bar{u},\tau_i}DG_{\tau_i}(\bar{u})^*]^{-1}y_i, DG_{\tau_i}(\bar{u})\, h\rangle_{\mathbb{R}^d} = 0 \qquad \forall\, h \in \ker L_{\bar{u},\tau_i}. \tag{3.27}$$

The following shows the existence of a local minimizer of $\frac{1}{2}\|u\|^2_{L^2}$ over $\mathfrak{F}_i$. The proof is given in Section A.6.

PROPOSITION 3.14. *Fix $i \in \{1,2\}$ and assume $\mathcal{F}(C_i) \neq \emptyset$. Let $y_i \in \mathbb{R}^d$, then the feasible set*

$$\mathfrak{F}_i := \{u \in \mathcal{F}(y_i) : \; G_{\tau_i}(u) = 0\} \tag{3.28}$$

is weakly sequential closed in $L^2((t_0,T);\mathbb{R}^k)$. Consequently, $\frac{1}{2}\|u\|^2_{L^2}$ attains a minimum on $\mathfrak{F}_i$.

From Theorems 3.10, and 3.13, and Proposition 3.14, we deduce the synthesis and optimal control results:

THEOREM 3.15. *Let $(x^0,\, x^1)\in\mathbb{R}^d\times\mathbb{R}^d$, $i\in\{1,2\}$, and set $y_i := \Phi_{T,\tau_i}(x^1)-\Phi_{t_0,\tau_i}(x^0)$. Assume $\mathcal{F}(C_i)\neq\emptyset$, (3.21) and (3.23) are satisfied, and let $u_i\in\mathcal{F}(y_i)$ be the fixed point of $\mathcal{S}_i$. Then u_i steers (Σ) from x^0 to x^1 on $[t_0,T]$ and*

$$u_i(t) = L^*_{u_i,\tau_i}\mathcal{N}_{\tau_i}(u_i)^{-1}y_i = B(t,x_{u_i}(t))^\top D\Phi_{t,\tau_i}(x_{u_i}(t))^\top\mathcal{N}_{\tau_i}(u_i)^{-1}y_i,\quad t\in[t_0,T],\quad \tau_1=t_0,\, \tau_2=T, \tag{3.29}$$

with $\|u_i\|^2_{L^2} = y_i^\top\mathcal{N}_{\tau_i}(u_i)^{-1}y_i$.

If the hypotheses of Theorem 3.13 hold for every local minimizer over $\mathfrak{F}_i := \{u\in\mathcal{F}(y_i) : L_{u,\tau_i}u = y_i\}$, and if (3.27) holds whenever needed, then $\frac{1}{2}\|u\|^2_{L^2}$ attains its minimum over $\mathfrak{F}_i$, and the unique minimizer is u_i. Equivalently,

$$\|u_i\|^2_{L^2}\le\|w\|^2_{L^2}\quad\forall\, w\in\mathfrak{F}_i,\qquad \textit{with equality iff } w=u_i.$$

Proof. Since $u_i = \mathcal{S}_i(u_i) = L^*_{u_i,\tau_i}\mathcal{N}_{\tau_i}(u_i)^{-1}y_i$, we obtain (3.29), feasibility $L_{u_i,\tau_i}u_i = y_i$, and $\|u_i\|^2_{L^2} = y_i^\top\mathcal{N}_{\tau_i}(u_i)^{-1}y_i$. By Proposition 3.14, $\frac{1}{2}\|u\|^2_{L^2}$ attains a minimum on $\mathfrak{F}_i$. Let $\bar u$ be a minimizer. Under the stated hypotheses, Theorem 3.13 implies that $\bar u$ is a fixed point of $\mathcal{S}_i$. By uniqueness, $\bar u = u_i$. ∎

Remark 3.16 (A posteriori check of the minimizer). When $\mathcal{S}_i$ admits a unique fixed point $u_i\in\mathcal{F}(y_i)$, it suffices to verify whether the orthogonality condition (3.27) holds at u_i, viz.

$$\langle[L_{u_i,\tau_i}DG_{\tau_i}(u_i)^*]^{-1}y_i, DG_{\tau_i}(u_i)\,h\rangle_{\mathbb{R}^d} = 0\qquad\forall\, h\in\ker L_{u_i,\tau_i}. \tag{3.30}$$

In this case, Theorem 3.13 implies that u_i is a local minimizer of $\frac{1}{2}\|u\|^2_{L^2}$ over $\mathfrak{F}_i$, and that if the global minimizer $\bar u\in\mathfrak{F}_i$ satisfies (3.27), then it coincides with u_i; hence u_i is the global minimum–energy control on $\mathfrak{F}_i$.

Remark 3.17. If $B(t,x)\equiv B(t)$ and there exists $C_i>0$ such that $\mathcal{F}(C_i) = L^\infty((t_0,T);\mathbb{R}^k)$, then the conclusions of Theorem 3.15 hold on any bounded set $\mathcal{B}\subset L^\infty$ containing $\mathcal{F}(y_i)$. In particular, if (3.30) holds on $\mathcal{B}$, the fixed point u_i is the unique global minimizer of $\frac{1}{2}\|u\|^2_{L^2}$ among all $u\in\mathcal{B}$ satisfying $L_{u,\tau_i}u = y_i$.

Remark 3.18 (Relation to reachability for fixed $T>t_0$). Theorem 3.15 is fundamentally a *synthesis result*: given the fixed point $u_i\in\mathcal{F}(y_i)$ of $\mathcal{S}_i$, it provides an explicit representation of a control that steers x^0 to x^1. It does not assert that such a fixed point exists for every pair (x^0,x^1), nor that the reachable set coincides with $\mathbb{R}^d$. Reachability depends on the structure of the feasible coercivity class $\mathcal{F}(C_i)$ and on whether the self-mapping property (3.21) is satisfied. Three distinct situations may occur:

(a) Complete controllability. If there exists $C_i>0$, independent of the initial state $x^0\in\mathbb{R}^d$, such that $\mathcal{F}(C_i) = L^\infty((t_0,T);\mathbb{R}^k)$, then (3.21) holds automatically and $\mathcal{R}_\Sigma(T,x^0) = \mathbb{R}^d$ for all $x^0\in\mathbb{R}^d$. In particular, system (Σ) is completely controllable on $[t_0,T]$. The converse is not true: complete controllability does not imply the existence of $C_i>0$ such that $\mathcal{F}(C_i) = L^\infty((t_0,T);\mathbb{R}^k)$.

(b) Controllability from a fixed x^0. If for some $C_i = C_i(x^0)>0$ depending on the initial condition x^0, one has $\mathcal{F}(C_i) = L^\infty((t_0,T);\mathbb{R}^k)$, then (3.21) again holds, and $\mathcal{R}_\Sigma(T,x^0)=\mathbb{R}^d$. Thus, the system is controllable from that specific x^0, although the property may not extend to all other initial states.

(c) Restricted synthesis. If for some $C_i = C_i(x^0)>0$ one only has $\emptyset\neq\mathcal{F}(C_i)\subsetneq L^\infty((t_0,T);\mathbb{R}^k)$, then Theorem 3.15 ensures that any fixed point $u_i\in\mathcal{F}(y_i)$ steers x^0 to x^1, provided (3.21) is verified. Thus, the synthesis produces a family of feasibly controls, but the reachable set $\mathcal{R}_\Sigma(T,x^0)$ may be a proper subset of $\mathbb{R}^d$; its precise description depends on the estimates involving $|\Phi_{T,t_0}(x^1)-x^0|$ or $|x^1-\Phi_{t_0,T}(x^0)|$ that guarantee (3.21).

Directly checking invertibility of $\mathcal{N}_{\tau_i}(u)$ is generally a difficult task since it depends on the full trajectory $x_u(\cdot)$. This motivates the search for structural assumptions under which, for some $C_i>0$, $\mathcal{F}(C_i)\neq\emptyset$ or coincides with $L^\infty((t_0,T);\mathbb{R}^k)$. The first such condition is presented below. The proof is presented in Section A.7.

PROPOSITION 3.19. *Fix $i\in\{1,2\}$. Suppose $d=k$ and there exists a nonzero $b\in L^1((t_0,T);\mathbb{R}_+)$ such that*

$$|B(t,x)^\top y|\ \ge\ b(t)\,|y|\qquad \textit{for a.e. } t\in(t_0,T),\ \forall (x,y)\in\mathbb{R}^d\times\mathbb{R}^d. \tag{3.31}$$

Then $\mathcal{N}_{\tau_i}(u) := \mathcal{N}_{\tau_i}(u, x^0)$ is uniformly coercive, viz.

$$y^\top \mathcal{N}_{\tau_i}(u)\, y \;\geq\; \frac{e^{-2\Lambda_1 \Delta t_0}\, \|b\|_1^2}{\Delta t_0}\, |y|^2 \qquad \forall (y,\, x^0,\, u) \in \mathbb{R}^d \times \mathbb{R}^d \times L^\infty((t_0,T);\mathbb{R}^k). \tag{3.32}$$

Hence, one may take $C_i = \Delta t_0\, e^{2\Lambda_1 \Delta t_0}\, \|b\|_1^{-2}$, so that $\mathcal{F}(C_i) = L^\infty((t_0,T);\mathbb{R}^k)$. In particular, (3.21) is satisfied.

Proposition 3.19 addresses the fully actuated case $k = d$ under a structural assumption on $B(t,x)$ ensuring uniform coercivity. If this assumption fails, or more generally in the underactuated regime $k < d$, such global bounds are no longer guaranteed. The following analysis provides sufficient conditions for reachability and synthesis in this configuration. The proof is provided in Section A.8.

THEOREM 3.20. *Let $x^0 \in \mathbb{R}^d$ and fix $i \in \{1,2\}$. Assume the existence of a reference control $u_i^\flat := u_i^\flat(x^0) \in L^\infty((t_0,T);\mathbb{R}^k)$ such that $\mathcal{N}_{\tau_i}(u_i^\flat)$ is invertible. Let $C_i := (1+\theta)/\lambda_{\min}\big(\mathcal{N}_{\tau_i}(u_i^\flat)\big)$ where $\theta \in (0,1]$. Furthermore, assume that the following estimate holds*

$$\|u_i^\flat\|_\infty < \frac{\theta\, \lambda_{\min}(\mathcal{N}_{\tau_i}(u_i^\flat))}{(1+\theta)\, L_{\mathcal{N}_{\tau_i}}}. \tag{3.33}$$

Then $\mathcal{F}(C_i) \neq \emptyset$, and (3.21) holds if $y_i \in \mathbb{R}^d$ satisfies

$$|y_i| \leq \vartheta(u_i^\flat) \;:=\; \frac{\lambda_{\min}(\mathcal{N}_{\tau_i}(u_i^\flat))}{(1+\theta)\, \|B\|_\infty\, e^{\Lambda_1 \Delta t_0}} \left(\frac{\theta\, \lambda_{\min}(\mathcal{N}_{\tau_i}(u_i^\flat))}{(1+\theta)\, L_{\mathcal{N}_{\tau_i}}} - \|u_i^\flat\|_\infty \right). \tag{3.34}$$

Here $L_{\mathcal{N}_{\tau_i}} > 0$ is the Lipschitz constant of the map $u \in \{u \in\, L^\infty((t_0,T);\mathbb{R}^k) : \|u\|_\infty \leq\, \zeta_i\} \mapsto \mathcal{N}_{\tau_i}(u)$.

In contrast to Proposition 3.19, Theorem 3.20 applies under Assumption 1.2 alone, requiring no additional structure on N_t or $B(t,x)$. Before discussing its implications, we first establish the Lipschitz constant $L_{\mathcal{N}_{\tau_i}} > 0$ for $\mathcal{N}_{\tau_i}(\cdot)$ under Assumptions 1.2 and 3.1. The proof of the following result is given in Section A.9.

LEMMA 3.21. *Fix $i \in \{1,2\}$. Then $\mathcal{N}_{\tau_i}$ defined by (3.10) is continuous from $L^\infty((t_0,T);\mathbb{R}^k)$ to $\mathcal{S}_d^+(\mathbb{R})$ and globally Lipschitz from $\mathcal{B}_i := \{u \in\, L^\infty((t_0,T);\mathbb{R}^k) : \|u\|_\infty \leq\, \zeta_i\}$ to $\mathcal{S}_d^+(\mathbb{R})$. For all $(u,\, v) \in \mathcal{B}_i^2$,*

$$\begin{aligned}\frac{\|\mathcal{N}_{\tau_1}(u) - \mathcal{N}_{\tau_1}(v)\|}{\|u-v\|_\infty} \leq L_{\mathcal{N}_{\tau_1}} &:= \frac{L_B \|B\|_\infty^2 e^{\Lambda_1 \Delta t_0}}{\Lambda_1} \left[\frac{e^{(2\Lambda_1 + L_B \zeta_1)\Delta t_0} - e^{\Lambda_1 \Delta t_0}}{\Lambda_1 + L_B \zeta_1} + \frac{e^{(2\Lambda_1 + L_B \zeta_1)\Delta t_0} - e^{-\Lambda_1 \Delta t_0}}{L_B \zeta_1 + 3\Lambda_1} \right] \\ &+ \frac{2\|B\|_\infty^3 \Lambda_2 e^{\Lambda_1 \Delta t_0}}{\Lambda_1(\Lambda_1 + L_B \zeta_1)} \left[\frac{e^{(3\Lambda_1 + L_B \zeta_1)\Delta t_0} - e^{-\Lambda_1 \Delta t_0}}{4\Lambda_1 + L_B \zeta_1} - \frac{e^{(2\Lambda_1 + L_B \zeta_1)\Delta t_0} - e^{-\Lambda_1 \Delta t_0}}{3\Lambda_1 + L_B \zeta_1} \right. \\ &\left. + \frac{e^{\Lambda_1 \Delta t_0} - e^{-\Lambda_1 \Delta t_0}}{2\Lambda_1} - \frac{e^{2\Lambda_1 \Delta t_0} - e^{-\Lambda_1 \Delta t_0}}{3\Lambda_1} \right].\end{aligned} \tag{3.35}$$

$$\begin{aligned}\frac{\|\mathcal{N}_{\tau_2}(u) - \mathcal{N}_{\tau_2}(v)\|}{\|u-v\|_\infty} \leq L_{\mathcal{N}_{\tau_2}} &:= \frac{L_B \|B\|_\infty^2 e^{\Lambda_1 \Delta t_0}}{\Lambda_1} \left[\frac{e^{\Lambda_1 \Delta t_0} - e^{L_B \zeta_2 \Delta t_0}}{\Lambda_1 - L_B \zeta_2} - \frac{e^{L_B \zeta_2 \Delta t_0} - e^{-\Lambda_1 \Delta t_0}}{L_B \zeta_2 + \Lambda_1} \right] \\ &+ \frac{2\|B\|_\infty^3 \Lambda_2 e^{\Lambda_1 \Delta t_0}}{\Lambda_1(\Lambda_1 + L_B \zeta_2)} \left[\frac{e^{2\Lambda_1 \Delta t_0} - e^{L_B \zeta_2 \Delta t_0}}{2\Lambda_1 - L_B \zeta_2} + \frac{e^{\Lambda_1 \Delta t_0} - e^{L_B \zeta_2 \Delta t_0}}{\Lambda_1 - L_B \zeta_2} \right. \\ &\left. + \frac{e^{\Lambda_1 \Delta t_0} - e^{-\Lambda_1 \Delta t_0}}{2\Lambda_1} - \frac{e^{2\Lambda_1 \Delta t_0} - e^{-\Lambda_1 \Delta t_0}}{3\Lambda_1} \right].\end{aligned} \tag{3.36}$$

Remark 3.22. When $B(t,x) \equiv B(t)$, (3.35) and (3.36) holds for $u,\, v \in L^\infty((t_0,T);\mathbb{R}^k)$, and

$$L_{\mathcal{N}_{\tau_1}} := \frac{\Lambda_2\,\|B\|_\infty^3}{6}\left(3e^{\Lambda_1\Delta t_0}+1\right)\left(\frac{e^{\Lambda_1\Delta t_0}-1}{\Lambda_1}\right)^3, \qquad L_{\mathcal{N}_{\tau_2}} := \frac{\Lambda_2\,\|B\|_\infty^3}{3}\left(\frac{e^{\Lambda_1\Delta t_0}-1}{\Lambda_1}\right)^3.$$

The following result is then an immediate consequence.

Corollary 3.23. *By Lemma 3.21 and spectral stability*

$$|\lambda_{\min}(C)-\lambda_{\min}(D)| \le \|C-D\| \qquad \forall (C,\, D)\in \mathcal{S}_d(\mathbb{R})^2,$$

the map $u \mapsto \lambda_{\min}\big(\mathcal{N}_{\tau_i}(u)\big)$ is continuous on $L^\infty((t_0,T);\mathbb{R}^k)$ and Lipschitz on $\{u\in L^\infty((t_0,T);\mathbb{R}^k) : \|u\|_\infty \le \zeta_i\}$. In particular, the feasible coercivity class $\mathcal{F}(C_i)$ defined in (3.17) is closed in $L^\infty((t_0,T);\mathbb{R}^k)$.

Remark 3.24. The bounds in (3.34), (3.35), and (3.36), while "sharp" in a general sense (under Assumption 1.2 alone), are conservative for specific vector fields N_t. In particular, the exponential factor $e^{\Lambda_1\Delta t_0}$ reflects the global Λ_1–Lipschitz bound on N_t (cf. Lemma 2.1), whereas for certain dynamics the differential $\|D\Phi_{t,s}(\cdot)\|$ can in fact decay exponentially. We provide illustrative examples later in Section 3.2.4.

3.2.2. A practical control reference: the zero control. In practice, to check the assumptions of Theorem 3.20, a natural first choice is $u_i^\flat \equiv 0 =: u_0$, for which the corresponding trajectory reads

$$x_{u_0}(t) = \Phi_{t_0,t}(x^0), \qquad t\in[t_0,T].$$

For $i\in\{1,2\}$ and $\tau_1 = t_0,\ \tau_2 = T$, this yields the symmetric positive semi-definite matrices

$$W_i(T) := \mathcal{N}_{\tau_i}(0,x^0) = \int_{t_0}^T D\Phi_{t,\tau_i}\big(\Phi_{t_0,t}(x^0)\big)\, B(t,\Phi_{t_0,t}(x^0))B(t,\Phi_{t_0,t}(x^0))^\top\, D\Phi_{t,\tau_i}\big(\Phi_{t_0,t}(x^0)\big)^\top\, dt. \tag{3.37}$$

It is immediate from (2.3) and (2.5) that $W_2(T)\in\mathbb{R}^{d\times d}$ solves the time-varying Lyapunov differential equation

$$\dot W(t) = B(t,\Phi_{t_0,t}(x^0))B(t,\Phi_{t_0,t}(x^0))^\top + DN_t\big(\Phi_{t_0,t}(x^0)\big)\,W(t) + W(t)\,DN_t\big(\Phi_{t_0,t}(x^0)\big)^\top, \quad W(t_0)=0. \tag{3.38}$$

Moreover, by Remark 3.5 one has,

$$W_2(t) = S_{t_0,x^0}(t,t_0)\,W_1(t)\,S_{t_0,x^0}(t,t_0)^\top, \qquad t\in[t_0,T] \tag{3.39}$$

so $W_1(t)$ is invertible iff $W_2(t)$ is invertible.

The following result is then an immediate consequence of Theorem 3.20.

Corollary 3.25. *Let $x^0\in\mathbb{R}^d$, $T>t_0$, and fix $i\in\{1,2\}$. Assume that $W_2(T)$ is invertible, and let $C_i := (1+\theta)/\lambda_{\min}(W_i(T))$ where $\theta\in(0,1]$. Then $\mathcal{F}(C_i)\neq\emptyset$, and (3.21) holds if $y_i\in\mathbb{R}^d$ satisfies*

$$|y_i| \le \frac{\theta\,\lambda_{\min}^2(W_i(T))}{(1+\theta)^2\,L_{\mathcal{N}_{\tau_i}}\,\|B\|_\infty\,e^{\Lambda_1\Delta t_0}}. \tag{3.40}$$

Here $L_{\mathcal{N}_{\tau_i}} > 0$ is the Lipschitz constant of the map $u\in\{u\in L^\infty((t_0,T);\mathbb{R}^k) : \|u\|_\infty\le\zeta_i\}\mapsto\mathcal{N}_{\tau_i}(u)$. Furthermore, for any $y_i\in\mathbb{R}^d$ satisfying (3.40), the following energy estimates hold

$$\int_{t_0}^T |u_i(t)|^2\,dt \le \frac{(1+\theta)}{\lambda_{\min}(W_i(T))}|y_i|^2 \tag{3.41}$$

where $u_i\in\mathcal{F}(y_i)$ is the fixed point of the synthesis map $\mathcal{S}_i : \mathcal{F}(y_i)\to\mathcal{F}(y_i)$ defined by (3.19).

Proof. Inequality (3.40) follows from (3.34). Next (3.29) and (3.10) yields

$$\int_0^T |u_i(t)|^2\,dt = \int_{t_0}^T u_i(t)^\top u_i(t)\,dt = y_i^\top \mathcal{N}_{\tau_i}(u_i)^{-1} y_i \le \frac{|y_i|^2}{\lambda_{\min}(\mathcal{N}_{\tau_i}(u_i))} \le \frac{(1+\theta)}{\lambda_{\min}(W_i(T))}|y_i|^2, \tag{3.42}$$

since $\lambda_{\min}(\mathcal{N}_{\tau_i}(u_i)) \ge \lambda_{\min}(W_i(T))/(1+\theta)$. ∎

Remark 3.26. It follows from Remark (3.5) that $W_2(T) = \mathcal{N}_{\tau_2}(0, x^0)$ is invertible iff the LTV system

$$\dot y(t) = DN_t\big(\Phi_{t_0,t}(x^0)\big)\, y(t) + B(t, \Phi_{t_0,t}(x^0))\, u(t), \qquad y(t_0) = y^0 \in \mathbb{R}^d, \quad t \in [t_0, T], \tag{3.43}$$

is controllable with inputs $u \in L^1((t_0,T);\mathbb{R}^k)$. This is the Kalman criterion for LTV systems via the controllability Gramian [10, Theorem. 1.16]. Under additional regularity, invertibility of $W_2(T)$ can also be characterized by algebraic rank conditions; see, e.g., [7, Theorem 1], [10, Theorem 1.18] and [25, Proposition 3.5.15].

3.2.3. Optimizing the admissible radius: balancing coercivity and control size. The trivial choice $u_i^\flat = 0$ might not be the optimal one in practice to apply Theorem 3.20 if $W_2(T)$ is invertible. In fact, it follows from the admissible target displacement estimate (3.34) that a good reference $u_i^\flat \in L^\infty((t_0,T);\mathbb{R}^k)$ is one that maximizes $\vartheta(u)$, which jointly increases the positive coercivity contribution and controls the penalty induced by $\|u\|_\infty$. The aim of this section is to illustrate how we can choose a reference control that maximizes $\vartheta(u)$.

Let $m \in \mathbb{N}$, $m \ge 1$, and $\{\varphi_j(t) : t \in [t_0,T]\}_{1\le j\le m}$ be a chosen (e.g., with respect to the properties of the vector field N_t and the input matrix B) m-linearly independent familly of functions in $L^\infty((t_0,T);\mathbb{R}^k)$. Fix $i \in \{1,2\}$, $\theta \in (0,1]$ and introduce the set

$$\mathcal{U}_{\rm ad} := \left\{ u \in \operatorname{Span}\{\varphi_j\} : g(u) := \frac{\theta\,\lambda_{\min}(\mathcal{N}_{\tau_i}(u))}{(1+\theta)\,L_{\mathcal{N}_{\tau_i}}} - \|u\|_\infty \ge 0 \right\}. \tag{3.44}$$

PROPOSITION 3.27. *Let $x^0 \in \mathbb{R}^d$, $T > t_0$, and fix $i \in \{1,2\}$. Assume that $W_2(T)$ is invertible, and let*

$$\vartheta(u) \;:=\; \frac{\lambda_{\min}(\mathcal{N}_{\tau_i}(u))}{(1+\theta)\,\|B\|_\infty\, e^{\Lambda_1 \Delta t_0}} \left(\frac{\theta\,\lambda_{\min}(\mathcal{N}_{\tau_i}(u))}{(1+\theta)\,L_{\mathcal{N}_{\tau_i}}} - \|u\|_\infty \right), \qquad u \in L^\infty((t_0,T);\mathbb{R}^k).$$

Then, the following program

$$\max_{u\in\mathcal{U}_{\rm ad}} \;\vartheta(u) \tag{3.45}$$

admits at least one solution $u_i^\flat \in \mathcal{U}_{\rm ad}$. Furthermore, if $\vartheta(u_i^\flat) > 0$ then $\lambda_{\min}(\mathcal{N}_{\tau_i}(u_i^\flat)) > 0$, $\mathcal{F}(C_i) \ne \emptyset$, and (3.21) holds if $y_i \in \mathbb{R}^d$ satisfies $|y_i| \le \vartheta(u_i^\flat)$ where $C_i := (1+\theta)/\lambda_{\min}\big(\mathcal{N}_{\tau_i}(u_i^\flat)\big)$.

Proof. Since $W_2(T)$ is invertible, $0 \in \mathcal{U}_{\rm ad}$ so that $\mathcal{U}_{\rm ad} \ne \emptyset$, and $\mathcal{U}_{\rm ad}$ is closed since $\mathcal{U}_{\rm ad} = g^{-1}([0,\infty))$ and $u \mapsto g(u)$ is continuous. Furthermore, since $\lambda_{\min}(\mathcal{N}_{\tau_i}(u)) \le \|\mathcal{N}_{\tau_i}(u)\| \le \|B\|_\infty^2 (e^{2\Lambda_1\Delta t_0} - 1)/(2\Lambda_1)$, one deduce that any $u \in \mathcal{U}_{\rm ad}$ satisfies

$$\|u\|_\infty \le \frac{\theta\,\|B\|_\infty^2}{(1+\theta)\,L_{\mathcal{N}_{\tau_i}}} \frac{e^{2\Lambda_1\Delta t_0} - 1}{2\Lambda_1},$$

so that $\mathcal{U}_{\rm ad}$ is bounded, and therefore compact. Since $u \mapsto \vartheta(u)$ is continuous, it follows that (3.45) has at least one solution $u_i^\flat \in \mathcal{U}_{\rm ad}$ by the Weierstrass extreme value theorem. The last part of the proposition follows from Theorem 3.20, and the following equivalence

$$\vartheta(u_i^\flat) > 0 \quad\Longleftrightarrow\quad \left(\|u_i^\flat\|_\infty < \frac{\theta\,\lambda_{\min}(\mathcal{N}_{\tau_i}(u_i^\flat))}{(1+\theta)\,L_{\mathcal{N}_{\tau_i}}} \quad \text{and} \quad \lambda_{\min}(\mathcal{N}_{\tau_i}(u_i^\flat)) > 0 \right).$$

∎

Remark 3.28. In Proposition 3.27, the assumption that $W_2(T)$ is invertible provides the simplest guarantee that $\mathcal{U}_{\rm ad} \ne \emptyset$. More generally, one may only require the existence of $u_i^* \in L^\infty((t_0,T);\mathbb{R}^k)$ such that $\mathcal{N}_{\tau_i}(u_i^*)$ is invertible and $g(u_i^*) \ge 0$. Verifying the latter condition, however, may be nontrivial in concrete models.

3.2.4. Refined estimates for Hopfield-type dynamics. We now return to a key point: the estimates for $\|D\Phi_{t,s}(\cdot)\|$ (Lemma 2.1) and for $L_{\mathcal{N}_{\tau_i}}$ (Lemma 3.21) are somehow "sharp" under Assumption 1.2, but may be overly conservative for concrete models since they use the global estimates (1.2) and (1.3). We illustrate this fact in this section. Consider the vector field from Hopfield-type recurrent neural networks [15], widely used in theoretical neuroscience and large-scale brain modeling, viz.

$$N(x) = -D\,x + W\sigma(x), \qquad x \in \mathbb{R}^d, \tag{3.46}$$

where $D \in \mathbb{R}^{d\times d}$ is diagonal and positive, $W \in \mathbb{R}^{d\times d}$ encodes connectivity, and σ is the neural activation,

$$\sigma(x) := (\sigma_1(x_1), \dots, \sigma_d(x_d))^\top, \qquad x := (x_1, \cdots, x_d)^\top \in \mathbb{R}^d,$$

with $\sigma_i(\cdot)$ is of sigmoid-type, say, $\sigma_i(s) = \tanh(s)$. Although $W = W(t)$ may vary analytically in time, e.g. to capture neuron–astrocyte interactions [13], it is commonly assumed to be constant, a simplification that we adopt. As a sigmoid function, σ_i is infinitely differentiable and globally bounded on $\mathbb{R}$ with all its successive derivatives, often $\sigma_i'(\cdot) \geq 0$. It follows that N satisfies all the hypotheses of Assumption 1.2. Since N is autonomous, one assumes that $t_0 = 0$. In particular, its generated flow $\phi_t := \Phi_{0,t}$, $t \in \mathbb{R}$, is a one parametric subgroup of $\mathrm{Diff}(\mathbb{R}^d)$, the group of all diffeomorphisms on $\mathbb{R}^d$. In this framework, for $i \in \{1, 2\}$, one has

$$\mathcal{N}_{\tau_i}(u) := \mathcal{N}_{\tau_i}(u, x^0) = \int_0^T D\phi_{\tau_i - t}(x_u(t))\, B(t)\, B(t)^\top\, D\phi_{\tau_i - t}(x_u(t))^\top\, dt, \qquad \tau_1 := 0,\ \tau_2 := T, \tag{3.47}$$

where the input matrix $B(t, x) \equiv B(t) \in L^\infty((t_0, T); \mathbb{R}^{d\times k})$, and x_u is the corresponding solution to (Σ) with the vector field in (3.46). For ease in notation, we introduce

$$\|\sigma'\|_\infty := \sup_{x\in\mathbb{R}^d} \|W\, D\sigma(x)\|, \quad \Gamma := -\lambda_{\min}(D) + \|\sigma'\|_\infty, \quad \Gamma_1 := \lambda_{\max}(D) + \|\sigma'\|_\infty, \quad \Gamma_2 := \sup_{x\in\mathbb{R}^d} \|W\, D^2\sigma(x)\|.$$

Then, one has the following result, the proof of which is given in Section A.10.

Proposition 3.29. *Let $x^0 \in \mathbb{R}^d$ and fix $i \in \{1, 2\}$. Then $\mathcal{N}_{\tau_i} : L^\infty((0, T); \mathbb{R}^k) \to \mathcal{S}_d^+(\mathbb{R})$ is globally Lipschitz,*

$$\|\mathcal{N}_{\tau_1}(u) - \mathcal{N}_{\tau_1}(v)\| \leq \frac{\Gamma_2\, \|B\|_\infty^3}{6} \left(3e^{\Gamma_1 T} + 1\right) \left(\frac{e^{\Gamma_1 T} - 1}{\Gamma_1}\right)^3 \|u - v\|_\infty, \qquad u, v \in L^\infty((0, T); \mathbb{R}^k). \tag{3.48}$$

$$\|\mathcal{N}_{\tau_2}(u) - \mathcal{N}_{\tau_2}(v)\| \leq \begin{cases} \dfrac{\Gamma_2\, \|B\|_\infty^3}{3} \left(\dfrac{e^{\Gamma T} - 1}{\Gamma}\right)^3 \|u - v\|_\infty & \text{if } \Gamma \neq 0, \\[2ex] \dfrac{\Gamma_2\, \|B\|_\infty^3\, T^3}{3}\, \|u - v\|_\infty & \text{if } \Gamma = 0, \end{cases} \qquad u, v \in L^\infty((0, T); \mathbb{R}^k). \tag{3.49}$$

Remark 3.30. The bounds related to $\mathcal{N}_{\tau_2}$ may involve a rate $\Gamma \leq 0$, and are therefore sharper than the general estimates of Remark 3.22, where a direct application necessarily involves $\Gamma_1 > 0$ as the exponential rate.

Example 3.31. Consider the 2D Hopfield-type recurrent neural network

$$\dot{x} = -D\,x + W\,\sigma(x) + Bu, \qquad x(0) = x^0 \in \mathbb{R}^2 \tag{3.50}$$

where $x = (x_1, x_2)^\top \in \mathbb{R}^2$, $u \in L^\infty((0, T); \mathbb{R})$, $B = (1, 0)^\top$, $D = \mathrm{diag}(d_1, d_2)$ with $d_i > 0$, $W = (w_{ij})_{1\leq i,j\leq 2}$, and $\sigma_i(s) = \tanh(s)$. By [26, Theorem 2.1], the necessary and sufficient condition for the complete controllability of (3.50) is that the criterion function

$$C(\gamma(t)) = -d_2\gamma_2(t) + w_{21}\tanh(\gamma_1(t)) + w_{22}\tanh(\gamma_2(t))$$

changes its sign over any control curve $\gamma(t) := (\gamma_1(t), \gamma_2(t))$ solution of

$$\dot{\gamma} = 1, \quad \dot{\gamma}_2 = 0 \quad \Longleftrightarrow \quad \gamma_1(t) = t + \gamma_1^0, \quad \gamma_2(t) = \gamma_2^0, \quad t \in \mathbb{R}, \quad (\gamma_1^0, \gamma_2^0) \in \mathbb{R}^2.$$

Since $d_2 > 0$, and tanh is bounded, $C(\gamma(t))$ (as a function of $t \in \mathbb{R}$) does not change its sign for large γ_2^0. Therefore, (3.50) is not completely controllable. It follows that, neither the HCM [8, 22, 29, 16] nor [26, Theorem 2.1] can be directly applied for the motion planning of (3.50). However, one proves using the Kalman-rank condition that $\mathcal{N}_{\tau_i}(u)$ is invertible for all $(x^0, u) \in \mathbb{R}^2 \times L^\infty((0,T); \mathbb{R})$ whenever $w_{21} \neq 0$. In particular, Corollary 3.25 applies to (3.50) for all $(x^0, x^1) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that $y_2 := x^1 - \phi_T(x^0)$ satisfies (3.40).

4. Drift–modulated control-affine dynamics with time-varying and state–dependent input matrix. In this section, we consider the more general control-affine system

$$\dot{x}(t) = A(t, x(t)) N_t(x(t)) + B(t, x(t)) u(t), \quad x(t_0) = x^0, \quad t \in [t_0, T]. \tag{4.1}$$

Throughout the following, $(t, x) \mapsto N_t(x)$ satisfies Assumption 1.2. Whereas, the matrices A and B satisfy the following relaxed regularity assumptions:

Assumption 4.1. $A : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is an element of $L^\infty((t_0, T) \times \mathbb{R}^d; \mathbb{R}^{d\times d})$, the input matrix $B : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ is an element of $L^\infty((t_0, T) \times \mathbb{R}^d; \mathbb{R}^{d\times k})$. Additionally, we assume that $x \mapsto A(\cdot, x)$, $B(\cdot, x)$ are continuous and locally Lipschitz.

Note that under Assumptions 1.2 and 4.1, the Cauchy-Lipschitz theory guarantees the existence of a unique absolutely continuous solution $x \in C^0([t_0, T]; \mathbb{R}^d)$ to (4.1) for any control $u \in L^\infty((t_0, T); \mathbb{R}^k)$; see, for instance, [4, Chapter 2]. Observe also that in contrast to Section 3, here we have only mild smoothness assumptions on A and B. Still, under the trajectory freezing and compactness argument, we illustrate how the control analysis and synthesis of the baseline system (Σ) developed in Section 3.2 can be leveraged to provide some insight into the controllability properties of the general control-affine system (4.1).

Fix $z \in C^0([t_0, T]; \mathbb{R}^d)$ and define N_t^z and B_z as

$$N_t^z(x) := A(t, z(t)) N_t(x), \quad B_z(t) := B(t, z(t)) \qquad \forall (t, x) \in [t_0, T] \times \mathbb{R}^d. \tag{4.2}$$

Then, $(t, x) \mapsto N_t^z(x)$ satisfies the same regularity assumptions as $(t, x) \mapsto N_t(x)$, as stated in Assumption 1.2 while B_z satisfies Assumption 3.1. Additionally, the following estimates hold

$$\|DN_t^z(w)\| \leq \Lambda_3 := \Lambda_1 \|A\|_\infty, \quad \|D^2 N_t^z(w)\| \leq \Lambda_4 := \Lambda_2 \|A\|_\infty \quad \forall (t, w, z) \in [t_0, T] \times \mathbb{R}^d \times C^0([t_0, T]; \mathbb{R}^d) \tag{4.3}$$

showing that $w \mapsto N_t^z(w)$ is globally Λ_3-Lipschitz continuous on $\mathbb{R}^d$, uniformly with respect to (t, z).

If $t \in [t_0, T] \to \Phi_{t_0,t}^z$ denotes the nonautonomous flow associated with N_t^z, then $\{\Phi_{s,t}^z \mid (s,t) \in [t_0, T]\}$ is a two-parameter family of diffeomorphisms which satisfies the same properties as the flow $\Phi_{t_0,t}$ of N_t recalled in Section 2. In particular, we have a similar lemma related to $\Phi_{s,t}^z$ as that given in Lemma 2.1.

Consider the following control-affine system

$$\dot{x}(t) = N_t^z(x(t)) + B_z(t) u(t), \quad x(t_0) = x^0, \quad t \in [t_0, T]. \tag{Σ_z}$$

As for (Σ), system (Σ_z) has unique absolutely continuous solution $x_u^z \in C^0([t_0, T]; \mathbb{R}^d)$ that can be represented as in Theorem 3.2 with $b(t, x) = B_z(t) u(t)$. We use the backward representation and get

$$x_u^z(t) = \Phi_{T,t}^z \left(\Phi_{t_0,T}^z(x^0) + \int_{t_0}^t D\Phi_{s,T}^z\big(x_u^z(s)\big)\, B_z(s)\, u(s)\, ds \right), \qquad t \in [t_0, T]. \tag{4.4}$$

Fix $x^0 \in \mathbb{R}^d$, $u \in L^\infty((t_0, T); \mathbb{R}^k)$ and let $x_u^z := x_{u,x^0}^z \in C^0([t_0, T]; \mathbb{R}^d)$ be the corresponding solution of (Σ_z). We define $\mathcal{N}_{\tau_2}^z(u) := \mathcal{N}_{\tau_2}^z(u, x^0)$ similarly as was defined $\mathcal{N}_{\tau_2}(u)$ in Section 3.2, viz.

$$\mathcal{N}_{\tau_2}^z(u) = \int_{t_0}^T D\Phi_{t,T}^z(x_u^z(t)) B_z(t) B_z(t)^\top D\Phi_{t,T}^z(x_u^z(t))^\top\, dt, \tag{4.5}$$

which is a symmetric positive-semidefinite matrix. In particular, Remark 3.22 yields

$$\|\mathcal{N}^z_{\tau_2}(u) - \mathcal{N}^z_{\tau_2}(v)\| \leq \frac{\Lambda_4\,\|B\|^3_\infty}{3}\left(\frac{e^{\Lambda_3\Delta t_0}-1}{\Lambda_3}\right)^3 \|u-v\|_\infty \qquad \forall\, u,\, v \in\, L^\infty((t_0,T);\mathbb{R}^k), \tag{4.6}$$

showing that $\mathcal{N}^z_{\tau_2} : L^\infty((t_0,T);\mathbb{R}^k) \to \mathcal{S}^+_d(\mathbb{R})$ is uniformly globally Lipschitz w.r.t. $z \in C^0([t_0,T];\mathbb{R}^d)$.

Similarly, as for $W_2(T)$ defined in (3.37), one introduces the matrix

$$W^z_2(T) := \mathcal{N}^z_{\tau_2}(0) = \int_{t_0}^T D\Phi^z_{t,T}(\Phi^z_{t_0,t}(x^0))B_z(t)B_z(t)^\top D\Phi^z_{t,T}(\Phi^z_{t_0,t}(x^0))^\top\,dt, \tag{4.7}$$

which is symmetric positive semi-definite.

The main theorem of this section is then the following.

THEOREM 4.2. *Let $x^0 \in \mathbb{R}^d$. Assume that there exists $C>0$ such that*

$$\lambda_{\min}(W^z_2(T)) \geq C^{-1} \qquad \forall z \in C^0([t_0,T];\mathbb{R}^d), \tag{4.8}$$

and that, for each frozen system, the corresponding synthesis map satisfies the contractivity condition of Theorem 3.10. Then, there exist $z_* \in C^0([t_0,T];\mathbb{R}^d)$ such that (4.1) is controllable on $[t_0,T]$ from x^0 to any target state $x^1 \in \mathbb{R}^d$ satisfying for some $\theta \in (0,1)$ the following estimate

$$|x^1 - \Phi^{z_*}_{t_0,T}(x^0)| \leq \frac{\theta\,\lambda_{\min}\big(W^{z_*}_2(T)\big)}{(1+\theta)\,L_*\,\|B\|_\infty\,e^{\Lambda_3\Delta t_0}}. \tag{4.9}$$

Here $L_*>0$ is the Lipschitz constant in (4.6).

Remark 4.3. Following Theorem 3.20, the assumption in Theorem 4.2 can be relaxed as follows: assume the existence of a reference control $u^\flat_2 \in L^\infty((t_0,T);\mathbb{R}^k)$ and a constant $C>0$ such that

$$\lambda_{\min}\big(\mathcal{N}^z_{\tau_2}(u^\flat_2)\big) \;\geq\; C^{-1}, \qquad \forall\, z \in C^0([t_0,T];\mathbb{R}^d).$$

Moreover, as illustrated in Section 3.2.4, the constant $L_*>0$ is often conservative, and can be sharpened in concrete models by exploiting structural properties of the vector field N^z_t.

Proof of Theorem 4.2. Let $z \in C^0([t_0,T];\mathbb{R}^d)$. By (4.8), $W^z_2(T)$ is invertible. Then, letting $C_2 := (1+\theta)/C$ with $C>0$ as in (4.8), and for some $\theta \in (0,1]$, the feasible coercivity class

$$\mathcal{F}(C_2) = \{u \in L^\infty((t_0,T);\mathbb{R}^k) : \lambda_{\min}(\mathcal{N}^z_{\tau_2}(u)) \geq C_2^{-1}\} \tag{4.10}$$

is nonempty since it contains $u=0$. For $x^1 \in \mathbb{R}^d$, if the following estimate holds

$$|x^1 - \Phi^z_{t_0,T}(x^0)| \leq \frac{\theta\,\lambda_{\min}\big(W^z_2(T)\big)}{(1+\theta)\,L_*\,\|B\|_\infty\,e^{\Lambda_3\Delta t_0}}, \tag{4.11}$$

then by Remark 3.22, and Theorems 3.10 and 3.15, the fixed point $u_z \in \mathcal{F}(C_2)$ of

$$\mathcal{S}^z_2(u)(t) = B_z(t)^\top D\Phi^z_{t,T}(x_u(t))^\top(\mathcal{N}^z_{\tau_2}(u))^{-1}\left(x^1 - \Phi^z_{t_0,T}(x^0)\right)$$

exists, is unique, and the corresponding solution x_{u_z} to system (Σ_z) satisfies $x_{u_z}(T) = x^1$.

To complete the proof of the theorem, let us define the map

$$\mathcal{Z} : z \in C^0([t_0,T];\mathbb{R}^d) \mapsto \mathcal{Z}(z) = x_{u_z} \in C^0([t_0,T];\mathbb{R}^d) \tag{4.12}$$

where $x_{u_z} \in C^0([t_0,T];\mathbb{R}^d)$ is the solution of (Σ_z) corresponding to the control $u_z = \mathcal{S}^z_2(u_z)$.

If $\mathcal{Z}$ has a fixed point $z_* \in C^0([t_0,T];\mathbb{R}^d)$, then $z_* = x_{u_{z_*}}$, and by construction, u_{z_*} steers (4.1) from x^0 to any target state $x^1 \in \mathbb{R}^d$ satisfying (4.9), i.e., $x_{u_{z_*}}$ is the solution to system (4.1) and satisfies $x_{u_{z_*}}(T) = x^1$.

First, $\mathcal{Z}$ is clearly well-defined and continuous. Moreover, by Lemma 2.1 and (4.8), it holds that

$$\|u_z\|_\infty \le C\|B\|_\infty e^{\Lambda_3 \Delta t_0}(e^{\Lambda_3 \Delta t_0}|x^1| + |x^0|) \quad \forall z \in C^0([t_0,T];\mathbb{R}^d) \tag{4.13}$$

so that for some $M > 0$ independent of z, we find

$$\|\mathcal{Z}(z)\|_\infty = \|x_{u_z}\|_\infty \le e^{\Lambda_1 \Delta t_0}(|x^0| + \Delta t_0(\|B\|_\infty \|u_z\|_\infty)) \le M \quad \forall z \in C^0([t_0,T];\mathbb{R}^d). \tag{4.14}$$

Next, $\mathcal{Z}$ is compact. In fact, let $\mathcal{B} \subset C^0([t_0,T];\mathbb{R}^d)$ be a bounded set, let us show that $\overline{\mathcal{Z}(\mathcal{B})}$ is compact. First, $\mathcal{Z}(\mathcal{B}) \subset C^0([t_0,T];\mathbb{R}^d)$ is bounded by (4.14). It remains to prove that for all $t_* \in [t_0,T]$, the following holds

$$\sup_{z \in \mathcal{B}} |\mathcal{Z}(z)(t) - \mathcal{Z}(z)(t_*)| \xrightarrow[t \to t_*]{} 0. \tag{4.15}$$

One has immediately from (4.13) and (4.14) that for some $M_1 > 0$, it holds that

$$|\mathcal{Z}(z)(t) - \mathcal{Z}(z)(t_*)| = |x_{u_z}(t) - x_{u_z}(t_*)| \le (\Lambda_3 \|x_{u_z}\|_\infty + \|B\|_\infty \|u_z\|_\infty)|t - t_*| \le M_1|t - t_*| \tag{4.16}$$

for every $t \in [t_0,T]$. This proves (4.15) and, therefore, that $\mathcal{Z}$ is compact by the Ascoli-Arzelà theorem. Finally, the set $\{z \in C^0([t_0,T];\mathbb{R}^d) \mid z = \theta\mathcal{Z}(z) \text{ for some } \theta \in [0,1]\}$ is bounded by (4.13) and (4.14). According to Schaefer's fixed point theorem, $\mathcal{Z}$ admits at least one fixed point $z_* \in C^0([t_0,T];\mathbb{R}^d)$, i.e., $z_* = \mathcal{F}(z_*) = x_{u_{z_*}}$. ☐

Remark 4.4. Theorem 4.2 generalizes [10, Theorem 3.40] beyond the trivial case $N_t(x) = x$. In this setting, $N_t^z(x) = A(t,z(t))x$, so $\mathcal{N}_{\tau_2}^z(u)$ is independent of u and the admissible class in (4.10) reduces to $\mathcal{F}(C_2) = L^\infty((t_0,T);\mathbb{R}^k)$ for all initial states and trajectories by letting $C_2 := C$ with $C > 0$ as in (4.8). Hence, by arguments analogous to Proposition 3.19, the baseline system (Σ_z) is globally controllable for all z, and global controllability of (4.1) follows via Schaefer's fixed point theorem.

5. Conclusion. We developed a constructive framework of control synthesis for control–affine system $\dot{x} = N_t(x) + B(t,x)u$ based on trajectory–dependent Gramians and fixed–point synthesis maps. On a ball $\mathcal{F}(y_i)$ of the feasible coercivity class $\mathcal{F}(C_i)$, sufficient conditions ensure that the synthesis map is a self–map and admits a unique fixed point with the associated control satisfying an explicit energy certificate. Beyond synthesis, we established existence of minimizers: the feasible set $\mathfrak{F}_i = \{u \in \mathcal{F}(y_i) : L_{u,\tau_i}u = y_i\}$ is weakly sequentially closed in $L^2((t_0,T);\mathbb{R}^k)$, hence $\frac{1}{2}\|u\|_{L^2}^2$ attains a minimum on $\mathfrak{F}_i$ (Proposition 3.14). We then linked optimality to fixed points: under the orthogonality condition at the fixed point (Theorem 3.13) and a finite-dimensional invertibility assumption, every local minimizer over $\mathfrak{F}_i$ equals the fixed point u_i; therefore, the minimizer on $\mathfrak{F}_i$ is unique and global. If, in addition, $\mathcal{F}(C_i) = L^\infty((t_0,T);\mathbb{R}^k)$, the same conclusion holds among all bounded controls satisfying the endpoint constraint.

The framework extends the classical Gramian theory for linear systems and, for (1.1), recovers [10, Theorem. 3.40] in the case of $N_t(x) = x$. Moreover, our results apply to Hopfield–type networks [15] and Lur'e systems [11] in general. It is constructive and amenable to numerical implementation. The analysis is pointwise in the initial state; natural next steps include deriving uniform state–space conditions paralleling the linear equivalence between controllability and Gramian invertibility, investigating robustness, and obtaining a sharper quantitative characterization of the reachable set.

When complete controllability on $[t_0,T]$ is known *a priori* for system $\dot{x} = N_t(x) + B(t,x)u$, our framework suggests a systematic procedure: certify on short windows that (i) $\mathcal{F}(C_i)$ is nonempty (via an invertible reference Gramian), and (ii) the synthesis map is a self–map and satisfies an iterate-contractivity condition; then concatenate the windowed fixed points along a partition of $[t_0,T]$ to steer arbitrary targets, with additive energy bounds. This "windowed Gramian synthesis" provides an implementable alternative to homotopy continuation, equipped with an energy certificate and an *a posteriori* optimality upgrade at the concatenated fixed point via the orthogonality condition test. Establishing uniform window conditions from structural data and extending the patching argument to underactuated regimes are promising directions for future work.

Appendix A. Proofs of results stated in the main text. In this section, we present proofs of some of the results from the main text. We start with the following.

A.1. Proof of Theorem 3.2. We present in this section the proof of the solution representation (3.2).

Proof. Under the assumptions of Theorem 3.2, system

$$\dot{x}(t) = N_t(x(t)) + b(t, x(t)), \qquad x(t_0) = x^0 \tag{A.1}$$

has (see, for instance, [4, Chapter 2]) a unique absolutely continuous solution $x \in C^0([t_0, T]; \mathbb{R}^d)$. Let us prove that this solution can be represented by (3.2). Let $t \in [t_0, T] \mapsto y(t) \in \mathbb{R}^d$ be such that $t \mapsto x(t) = \Phi_{\tau_i,t}(y(t))$ is the solution of (A.1). Then y is derivable almost everywhere w.r.t. t and it holds that

$$N_t(x(t)) + b(t, x(t)) = \dot{x}(t) = N_t(x(t)) + D\Phi_{\tau_i,t}(y(t))\, \dot{y}(t) \tag{A.2}$$

so that using (2.6), we find that y solves the following.

$$\dot{y}(t) = D\Phi_{t,\tau_i}(x(t))\, b(t, x(t)), \qquad y(t_0) = \Phi_{t_0,\tau_i}(x^0). \tag{A.3}$$

Integrating (A.3) yields (3.2). Conversely, if (3.2) holds, then $x(t_0) = \Phi_{\tau_i,t_0}(\Phi_{t_0,\tau_i}(x^0)) = x^0$ and $x \in C^0([t_0, T]; \mathbb{R}^d)$ by composition. Otherwise, there exists $(t_n) \subset [t_0, T]$, $t_* \in [t_0, T]$ with $t_n \to t_*$ and $\varepsilon > 0$ such that $|\Phi_{t,\tau_i}(x(t_n)) - \Phi_{t_*,\tau_i}(x(t_*))| \geq \varepsilon$. In fact, $x \in C^0([t_0, T]; \mathbb{R}^d)$ if and only if $t \mapsto y(t) := \Phi_{t,\tau_i}(x(t))$ belongs to $C^0([t_0, T]; \mathbb{R}^d)$ since Φ_{t,τ_i} is invertible and C^1 w.r.t. $t \in \mathbb{R}$. However, we have from (3.2) that

$$\Phi_{t_n,\tau_i}(x(t_n)) - \Phi_{t_*,\tau_i}(x(t_*)) = \int_{t_*}^{t_n} D\Phi_{s,\tau_i}(x(s))\, b(s, x(s))\, ds$$

which implies $|\Phi_{t_n,\tau_i}(x(t_n)) - \Phi_{t_*,\tau_i}(x(t_*))| \leq \|b\|_\infty e^{\Lambda_1|\tau_i - t_*|}|t_n - t_*|$ by using (2.7) with $\beta(s) = x(s)$. It follows that $|\Phi_{t_n,\tau_i}(x_u(t_n)) - \Phi_{t_*,\tau_i}(x_u(t_*))| \to 0$ as $n \to \infty$, which is inconsistent. Letting now

$$z(t) := \Phi_{t_0,\tau_i}(x^0) + \int_{t_0}^{t} D\Phi_{s,\tau_i}(x(s))\, b(s, x(s))\, ds \tag{A.4}$$

and deriving (3.2) almost everywhere w.r.t. t yields

$$\dot{x}(t) = \partial_t \Phi_{\tau_i,t}(z(t)) + D\Phi_{\tau_i,t}(z(t))\, D\Phi_{t,\tau_i}(x(t))\, b(t, x(t)) = N_t(\Phi_{\tau_i,t}(z(t))) + b(t, x(t)) = N_t(x(t)) + b(t, x(t)) \tag{A.5}$$

by $x(t) = \Phi_{\tau_i,t}(z(t))$ and $D\Phi_{\tau_i,t}(z(t))\, D\Phi_{t,\tau_i}(x(t)) = \mathrm{Id}$. It follows that (3.2) solves (A.1). ∎

A.2. Proof of Lemma 3.4.

Proof. Recall from [4, Theorem 3.2.6] that the Fréchet derivative $D_u x_u(t)h$ of $x_u(t)$ w.r.t. u in the direction of $h \in L^\infty((t_0, T); \mathbb{R}^k)$ is given by

$$D_u x_u(t)h = \int_{t_0}^{t} R_u(t, s)B(s, x_u(s))h(s)\, ds. \tag{A.6}$$

Using the solution representation (3.2) with $b(t, x(t)) = B(t, x(t))u(t)$ and letting $y_u(t) := \Phi_{t,\tau_i}(x_u(t))$, one finds $D_u x_u(t)h = \left[D\Phi_{t,\tau_i}(x_u(t))\right]^{-1} D_u y_u(t)h$ and

$$\frac{d}{dt} D_u y_u(t)h = \mathcal{A}_{\tau_i,u}(t) D_u y_u(t)h + D\Phi_{t,\tau_i}(x_u(t))B(t, x_u(t))h(t), \qquad D_u y_u(t)h|_{t=t_0} = 0$$

where $\mathcal{A}_{\tau_i,u}(t)$ is defined as in the statement of Lemma 3.4. It follows that

$$D_u y_u(t)h = \int_{t_0}^{t} M_{\tau_i,u}(t, s) D\Phi_{s,\tau_i}(x_u(s))B(s, x_u(s))h(s)\, ds \tag{A.7}$$

where $M_{\tau_i,u}(t,s)$ is defined as in the statement of Lemma 3.4. One deduces that

$$D_u x_u(t)h = \big[D\Phi_{t,\tau_i}(x_u(t))\big]^{-1}\int_{t_0}^{t} M_{\tau_i,u}(t,s)D\Phi_{s,\tau_i}(x_u(s))B(s,x_u(s))h(s)\,ds. \tag{A.8}$$

Next, (A.6) and (A.8) suggest the factorization $R_u(t,s) = \big[D\Phi_{t,\tau_i}(x_u(t))\big]^{-1}M_{\tau_i,u}(t,s)D\Phi_{s,\tau_i}(x_u(s))$. Using this factorization, one checks immediately that

$$R_u(t_1,t_1) = \mathrm{Id}, \quad R_u(t_1,t_2)R_u(t_2,t_3) = R_u(t_1,t_3), \quad R_u(t_1,t_2)R_u(t_2,t_1) = \mathrm{Id}, \qquad \forall (t_1,t_2,t_3) \in [t_0,T]^3$$

since $M_{\tau_i,u}(t,s)$ is a state-transition matrix. Finally, using the following identities

$$\begin{aligned}\frac{d}{dt}\big[D\Phi_{t,\tau_i}(x_u(t))\big]^{-1} &= -\big[D\Phi_{t,\tau_i}(x_u(t))\big]^{-1}\left(\frac{d}{dt}D\Phi_{t,\tau_i}(x_u(t))\right)\big[D\Phi_{t,\tau_i}(x_u(t))\big]^{-1}\\ &= DN_t(x_u(t))\big[D\Phi_{t,\tau_i}(x_u(t))\big]^{-1} + D^2\Phi_{\tau_i,t}\big(\Phi_{t,\tau_i}(x_u(t))\big)D\Phi_{t,\tau_i}(x_u(t))B(t,x_u(t))u(t),\end{aligned} \tag{A.9}$$

$$D^2\Phi_{\tau_i,t}\big(\Phi_{t,\tau_i}(x_u(t))\big)D\Phi_{t,\tau_i}(x_u(t))D\Phi_{t,\tau_i}(x_u(t)) + D\Phi_{\tau_i,t}\big(\Phi_{t,\tau_i}(x_u(t))\big)D^2\Phi_{t,\tau_i}(x_u(t)) = 0, \tag{A.10}$$

and the fact that $M_{\tau_i,u}(t,s)$ is a state-transition matrix of (3.7), one finds that

$$\frac{\partial R_u}{\partial t}(t,s) = \big[DN_t(x_u(t)) + D_xB(t,x_u(t))u(t)\big]R_u(t,s), \qquad (t,s) \in [t_0,T]^2. \tag{A.11}$$

This completes the proof of the Lemma. ∎

A.3. Proof of Theorem 3.10. In this section, we present the proof of Theorem 3.10 that we split into several steps for the reader's convenience. Since the maps $\mathcal{S}_1$ and $\mathcal{S}_2$ play similar roles, we only focus on $\mathcal{S}_2$. Throughout this section, we set $\mathcal{Y} := L^\infty((t_0,T);\mathbb{R}^k)$. Under the assumptions of Theorem 3.10,

$$\mathcal{S}_2(u) = B_u(t)^\top Q_u(t)^\top \mathcal{N}_{\tau_2}(u)^{-1} y_2 \qquad \text{and} \qquad \mathcal{S}_2(u) \in \mathcal{F}(y_2) \qquad \forall u \in \mathcal{F}(y_2),$$

where $y_2 := x^1 - \Phi_{t_0,T}(x^0)$, $B_u(t) := B(t,x_u(t))$ and $Q_u(t) := D\Phi_{t,T}(x_u(t))$. For ease in notation, for $s \in [0,\Delta t_0]$, where $\Delta t_0 := T - t_0$, we let

$$E_0 = e^{L_B\,\zeta_2\,\Delta t_0},\ \ E_1(s) = e^{\Lambda_1 s},\ \ E_2(s) = e^{2\Lambda_1 s} - e^{\Lambda_1 s},\ \ E_{1,\infty} = \max_{s\in[0,\Delta t_0]} E_1(s),\ \ E_{2,\infty} = \max_{s\in[0,\Delta t_0]} E_2(s). \tag{A.12}$$

We introduce the following Volterra and Fredholm-type bounded linear operators $V,\ F : \mathcal{Y} \to \mathcal{Y}$ defined by

$$(Vf)(t) = \int_{t_0}^{t} f(\tau)\,d\tau, \qquad Ff = \int_{t_0}^{T} f(\tau)\,d\tau \qquad \forall (t,\ f) \in [t_0,T]\times\mathcal{Y}. \tag{A.13}$$

Lemma A.1. *For every $(u,\ v) \in \mathcal{F}(y_2)^2$, the following estimates hold*

$$|x_u(t) - x_v(t)| \le \|B\|_\infty\, E_0\, E_{1,\infty}\,(V|u-v|)(t) \le \|B\|_\infty\, E_0\, E_{1,\infty}\,\Delta t_0\|u-v\|_\infty \qquad \forall t \in [t_0,T], \tag{A.14}$$

$$\|\mathcal{N}_{\tau_2}(u) - \mathcal{N}_{\tau_2}(v)\| \le 2\,E_0\,E_{1,\infty}\,\|B\|_\infty^2\left(\frac{\|B\|_\infty\Lambda_2\,E_{2,\infty}}{\Lambda_1} + L_B\,E_{1,\infty}\right)F\circ V|u-v|. \tag{A.15}$$

Proof. Recall that $\dot{x}_u(t) = N_t(x_u(t)) + B(t, x_u(t))u(t)$, $x_u(t_0) = x^0$ and $\dot{x}_v(t) = N_t(x_v(t)) + B(t, x_u(t))v(t)$, $x_v(t_0) = x^0$. Then, by the Cauchy-Schwarz inequality, one finds that

$$\frac{d}{dt}|x_u(t) - x_v(t)| = \frac{\langle \dot{x}_u(t) - \dot{x}_v(t), x_u(t) - x_v(t)\rangle}{|x_u(t) - x_v(t)|} \leq (\Lambda_1 + L_B\,\zeta_2)\,|x_u(t) - x_v(t)| + \|B\|_\infty\,|u(t) - v(t)|$$

which by Gronwall's lemma yields (A.14). Now using Lemma 2.1 respectively with $\beta \in \{x_u, x_v\}$, one finds

$$\|\mathcal{N}_{\tau_2}(u) - \mathcal{N}_{\tau_2}(v)\| \leq 2\bigg(\frac{\|B\|_\infty^2\Lambda_2\,E_{2,\infty}}{\Lambda_1} + L_B\|B\|_\infty\,E_{1,\infty}\bigg)\int_{t_0}^T |x_u(s) - x_v(s)|\,ds \tag{A.16}$$

which by the first inequality in (A.14) yields (A.15). ∎

We introduce the positive constants

$$\alpha_1 = 2E_0C_2^2\|B\|_\infty^3E_{1,\infty}^2\bigg(\frac{\Lambda_2\|B\|_\infty E_{2,\infty}}{\Lambda_1} + L_BE_{1,\infty}\bigg)|y_2|,\;\; \alpha_2 = E_0C_2\|B\|_\infty E_{1,\infty}\bigg(\frac{\Lambda_2\|B\|_\infty E_{2,\infty}}{\Lambda_1} + L_BE_{1,\infty}\bigg)|y_2|, \tag{A.17}$$

and the bounded and linear operator $\mathcal{L} : \mathcal{Y} \to \mathcal{Y}$, defined by

$$(\mathcal{L}f)(t) = \alpha_1\,F \circ Vf + \alpha_2(Vf)(t) \qquad \forall (t,\,f) \in [t_0, T] \times \mathcal{Y}. \tag{A.18}$$

Lemma A.2. *Set*

$$K_2 := \Delta t_0(\alpha_1\Delta t_0 + \alpha_2), \qquad p_2 := \frac{\alpha_1\Delta t_0}{\alpha_1\Delta t_0 + \alpha_2} \in (0,1). \tag{A.19}$$

Let $\gamma_m(p_2) := z_m(1)$, where $z_0 \equiv 1$ on $[0,1]$ and

$$z_{m+1}(y) = (1 - p_2)\int_0^y z_m(r)\,dr + p_2\int_0^1 (1 - r)z_m(r)\,dr. \tag{A.20}$$

Then, for every $m \geq 1$,

$$\|\mathcal{L}^m\| \leq K_2^m\gamma_m(p_2). \tag{A.21}$$

Proof. Introduce the isometric rescaling $\mathcal{R}f(y) := f(t_0 + \Delta t_0 y)$ on $L^\infty(0,1)$, and let $(Wg)(y) = \int_0^y g(r)dr$ and $Jg = \int_0^1 g(r)dr$. Then $\mathcal{R}V\mathcal{R}^{-1} = \Delta t_0 W$ and $\mathcal{R}F\mathcal{R}^{-1} = \Delta t_0 J$. Hence $\widetilde{\mathcal{L}} := \mathcal{R}\mathcal{L}\mathcal{R}^{-1} = K_2T_{p_2}$, with $T_{p_2} := (1 - p_2)W + p_2JW$.

Since T_{p_2} is positive, for $\|g\|_\infty \leq 1$ one has $|T_{p_2}^m g| \leq T_{p_2}^m\mathbf{1}$ a.e. Let $z_m := T_{p_2}^m\mathbf{1}$. Then $z_0 \equiv 1$ and $z_{m+1} = T_{p_2}z_m$, which gives exactly (A.20). The functions z_m are nonnegative and nondecreasing by induction; hence $\|z_m\|_\infty = z_m(1) = \gamma_m(p_2)$. Thus $\|T_{p_2}^m\| \leq \gamma_m(p_2)$, and since $\mathcal{R}$ is an isometry, $\|\mathcal{L}^m\| = \|\widetilde{\mathcal{L}}^m\| \leq K_2^m\gamma_m(p_2)$. ∎

Lemma A.3. *Let $p \in (0,1)$, and let $\lambda(p) > 0$ be the unique positive solution of*

$$p(e^\lambda - 1) - \lambda = 0. \tag{A.22}$$

Set $\Lambda(p) := \lambda(p)/(1-p)$. Then $\Lambda(p) > 2$. Moreover, if $0 < K < \Lambda(p)$, then $K^m\gamma_m(p) \to 0$ as $m \to \infty$.

Proof. For $T_p = (1-p)W + pJW$, the same argument as in the proof of Lemma A.2 gives $\|T_p^m\| \leq \gamma_m(p)$, while testing on $\mathbf{1}$ gives the reverse inequality. Hence $\gamma_m(p) = \|T_p^m\|$. The compact operator T_p is strongly positive on the cone of nonnegative functions in $L^\infty(0,1)$, so the Krein–Rutman theorem [5, Theorem 6.13 & Problem 41] gives an eigenpair $\mu = r(T_p) > 0$ and $f \in \operatorname{Int} P$ such that $T_pf = \mu f$. It follows that f has an absolutely continuous representation on $[0,1]$ that we still denote by f. Differentiating $T_pf = \mu f$ gives $\mu f' = (1-p)f$, hence $f(y) = f(0)e^{\lambda y}$ with $\lambda = (1-p)/\mu$. Evaluating at $y = 0$ gives (A.22); hence $r(T_p) = 1/\Lambda(p)$. The inequality $\Lambda(p) > 2$ follows by setting $q = 1 - p$ and observing that the defining function is negative at $\lambda = 2q$, equivalently $2q < \log((1+q)/(1-q))$. The spectral-radius formula gives $\lim_{m\to\infty}\gamma_m(p)^{1/m} = 1/\Lambda(p)$, and therefore $K^m\gamma_m(p) \to 0$ whenever $K < \Lambda(p)$. ∎

Proof of Theorem 3.10. Let $(u,\, v) \in \mathcal{F}(y_2)^2$. Then, for every $t \in [t_0, T]$, it hols that

$$\begin{aligned}|\mathcal{S}_2(u)(t) - \mathcal{S}_2(v)(t)| &\le |y_2| C_2 \big(\|B\|_\infty \, E_{1,\infty}\, C_2 \|\mathcal{N}_{\tau_2}(u) - \mathcal{N}_{\tau_2}(v)\| + \|B\|_\infty \|Q_u(t) - Q_v(t)\| + E_{1,\infty} \|B_u(t) - B_v(t)\| \big) \\ &\le \alpha_1\, F \circ V|u - v| + \alpha_2 (V|u - v|)(t) = (\mathcal{L}|u - v|)(t) \qquad \text{(A.23)}\end{aligned}$$

by Lemma 2.1, (A.14), (A.15), (A.17) and (A.18). It follows that for every $m \in \mathbb{N}$, with $m \ge 1$, it holds that

$$|\mathcal{S}_2^m(u)(t) - \mathcal{S}_2^m(v)(t)| \le (\mathcal{L}^m |u - v|)(t) \le \|(\mathcal{L}^m |u - v|)\|_\infty \le \|\mathcal{L}^m\| \|u - v\|_\infty \le K_2^m \gamma_m(p_2) \|u - v\|_\infty, \qquad \text{(A.24)}$$

for all $t \in [t_0, T]$, by Lemma A.2. This proves (3.22) with $\varrho_m := K_2^m \gamma_m(p_2)$. If there exists $m_i \ge 1$ such that $\varrho_{m_i} < 1$, one completes the proof of the theorem by the Banach fixed-point theorem [3, Theorem 2.4]. ∎

A.4. Proof of Lemma 3.12. For clarity, we split the proof into several steps. Throughout this section, we set $\mathcal{X} := L^2((t_0, T); \mathbb{R}^k)$, $\mathcal{Y} := L^\infty((t_0, T); \mathbb{R}^k)$, and let $\mathcal{F}(C_i)$ be defined by (3.17). Fix $u \in \mathcal{Y}$, and consider the following linear and bounded operators H_{u,τ_i}, W_{u,τ_i}, and J_{u,τ_i} defined by

$$(H_{u,\tau_i} h)(t) = R_u(\tau_i, t) B(t, x_u(t)) h(t), \qquad (W_{u,\tau_i} h)(t) = R_u(t, \tau_i) \int_{t_0}^t h(s)\, ds,$$

and

$$J_{u,\tau_i} h = \int_{t_0}^T D^2 \Phi_{t,\tau_i}(x_u(t)) B(t, x_u(t)) u(t) h(t)\, dt + \int_{t_0}^T D\Phi_{t,\tau_i}(x_u(t)) D_x B(t, x_u(t)) u(t) h(t)\, dt.$$

Here $R_u(t,s)$ is defined in Lemma 3.4. Since R_u is a state transition matrix, one has

$$(W_{u,\tau_i} H_{u,\tau_i} h)(t) = R_u(t, \tau_i) \int_{t_0}^t (H_{u,\tau_i} h)(s)\, ds = \int_{t_0}^t R_u(t,s) B(s, x_u(s)) h(s)\, ds.$$

We set $K_{u,\tau_i} := J_{u,\tau_i} W_{u,\tau_i} H_{u,\tau_i}$. It is clear by Assumptions 1.2 and 3.1 that $\mathcal{E} \in C^1(\mathcal{Y}; \mathbb{R}^d)$; see, for instance, [4, Theorem 3.2.6]. Hence $G_{\tau_i} \in C^1(\mathcal{Y}; \mathbb{R}^d)$ by composition. Moreover, for any $h \in \mathcal{X}$,

$$D\mathcal{E}(u) h = \int_{t_0}^T R_u(T, t) B(t, x_u(t)) h(t)\, dt, \qquad DG_{\tau_i}(u) h = L_{u,\tau_i} h + K_{u,\tau_i} h. \qquad \text{(A.25)}$$

Using the forward–backward representation (3.2) with $i = 2$, one obtains

$$\mathcal{E}(u) = x_u(T) = \Phi_{t_0,T}(x^0) + \int_{t_0}^T D\Phi_{t,T}(x_u(t)) B(t, x_u(t)) u(t)\, dt = G_2(u) + \Phi_{t_0,T}(x^0) + y_2,$$

and therefore $D\mathcal{E}(u) = DG_2(u)$. The case $i = 1$ follows similarly from the same representation, which gives $DG_{\tau_i}(u) = D\Phi_{T,\tau_i}(x_u(T))\, D\mathcal{E}(u)$. This proves (3.26).

Next, suppose that $\mathcal{N}_{\tau_i}(u)$ is invertible. Then, the canonical right inverse of L_{u,τ_i} satisfies

$$R_{u,\tau_i} := L_{u,\tau_i}^* \mathcal{N}_{\tau_i}(u)^{-1} \in \mathscr{L}(\mathbb{R}^d, \mathcal{X}), \qquad L_{u,\tau_i} R_{u,\tau_i} = \mathrm{Id}.$$

Hence, if $\mathrm{Id} + A_{u,\tau_i}$ is invertible with $A_{u,\tau_i} := K_{u,\tau_i} L_{u,\tau_i}^* \mathcal{N}_{\tau_i}(u)^{-1}$, one uses $DG_{\tau_i}(u) = L_{u,\tau_i} + K_{u,\tau_i}$ to get

$$(L_{u,\tau_i} + K_{u,\tau_i})\, R_{u,\tau_i}\, (\mathrm{Id} + A_{u,\tau_i})^{-1} = \mathrm{Id} \quad \text{which is equivalent to} \quad DG_{\tau_i}(u)\, L_{u,\tau_i}^* \left[DG_{\tau_i}(u)\, L_{u,\tau_i}^* \right]^{-1} = \mathrm{Id} \qquad \text{(A.26)}$$

showing that $DG_{\tau_i}(u)\, L_{u,\tau_i}^* \in \mathbb{R}^{d \times d}$ is invertible and $DG_{\tau_i}(u)$ is right-invertible. So $DG_{\tau_i}(u)$ is onto, which implies that $M(u) := DG_{\tau_i}(u) DG_{\tau_i}(u)^*$ is invertible.

Suppose now that $M(u) := DG_{\tau_i}(u)DG_{\tau_i}(u)^*$ is invertible. The canonical right-inverse of $DG_{\tau_i}(u)$ satisfies

$$T_{u,\tau_i} := DG_{\tau_i}(u)^* M(u)^{-1} \in \mathscr{L}(\mathbb{R}^d, \mathcal{X}), \qquad DG_{\tau_i}(u)T_{u,\tau_i} = \mathrm{Id}.$$

If $\mathrm{Id} - A_{u,\tau_i}$ is invertible with $A_{u,\tau_i} := K_{u,\tau_i} DG_{\tau_i}(u)^* M(u)^{-1}$, one uses $L_{u,\tau_i} = DG_{\tau_i}(u) - K_{u,\tau_i}$ to obtain

$$(DG_{\tau_i}(u) - K_{u,\tau_i})\, T_{u,\tau_i}\, (\mathrm{Id} - A_{u,\tau_i})^{-1} = \mathrm{Id} \quad \Longleftrightarrow \quad L_{u,\tau_i}\, DG_{\tau_i}(u)^* \left[L_{u,\tau_i}\, DG_{\tau_i}(u)^*\right]^{-1} = \mathrm{Id}. \tag{A.27}$$

This shows that $L_{u,\tau_i}\, DG_{\tau_i}(u)^* \in \mathbb{R}^{d\times d}$ is invertible and L_{u,τ_i} is right-invertible. So L_{u,τ_i} is onto, which implies that $\mathcal{N}_{\tau_i}(u) := L_{u,\tau_i}\, L^*_{u,\tau_i}$ is invertible. This completes the proof of the lemma.

A.5. Proof of Theorem 3.13. In this section, we use the same notations introduced in Section A.4.

Proof of Theorem 3.13. Since $\mathrm{Id} + K_{u,\tau_i} L^*_{u,\tau_i} \mathcal{N}_{\tau_i}(u)^{-1}$ and $\mathcal{N}_{\tau_i}(u)$ are invertible for $u \in \mathcal{F}(y_i) \subset \mathcal{F}(C_i) \subset L^\infty \subset L^2$, one deduces that $DG_{\tau_i}(u)$ is onto by Lemma 3.12. Then, for any local minimizer $\bar{u}$ on $\mathfrak{F}_i := \{u \in \mathcal{F}(y_i) : \ G_{\tau_i}(u) = 0\}$, the Lagrange multiplier theorem in Hilbert spaces [32, Theorem 43.D, p. 290] (applied with $F(u) = \frac{1}{2}\|u\|_2^2$ and the submersion $G_{\tau_i} : \mathcal{F}(y_i) \to \mathbb{R}^d$, $G_{\tau_i}(u) = L_{u,\tau_i} u - y_i$) yields $\lambda \in \mathbb{R}^d$ such that

$$\bar{u} = DG_{\tau_i}(\bar{u})^* \lambda, \qquad L_{\bar{u},\tau_i} \bar{u} = y_i. \tag{A.28}$$

Since $DG_{\tau_i}(\bar{u})$ is onto, one has that $M(\bar{u}) = DG_{\tau_i}(\bar{u})DG_{\tau_i}(\bar{u})^* \in \mathbb{R}^{d\times d}$ is invertible, and Lemma 3.12 ensures that $L_{\bar{u},\tau_i}\, DG_{\tau_i}(\bar{u})^* \in \mathbb{R}^{d\times d}$ is invertible. Next, left-multiplying the first identity in (A.28) by $L_{\bar{u},\tau_i}$ and using the second one yields

$$\lambda = \left[L_{\bar{u},\tau_i}\, DG_{\tau_i}(\bar{u})^*\right]^{-1} y_i \tag{A.29}$$

Substituting back to (A.28) leads to

$$\bar{u} = P_{\bar{u},\tau_i} y_i \qquad \text{where} \quad P_{\bar{u},\tau_i} := DG_{\tau_i}(\bar{u})^* \left[L_{\bar{u},\tau_i}\, DG_{\tau_i}(\bar{u})^*\right]^{-1}. \tag{A.30}$$

Recall from (A.27) that

$$L_{\bar{u},\tau_i}\, P_{\bar{u},\tau_i} = \mathrm{Id}$$

showing that $P_{\bar{u},\tau_i}$ is a right-inverse of $L_{\bar{u},\tau_i}$. Next, since the canonical right inverse $R_{\bar{u},\tau_i} = L^*_{\bar{u},\tau_i}\mathcal{N}_{\tau_i}(\bar{u})^{-1}$ of $L_{\bar{u},\tau_i}$ satisfies $L_{\bar{u},\tau_i} R_{\bar{u},\tau_i} = \mathrm{Id}$, one deduces that $L_{\bar{u},\tau_i}\,(P_{\bar{u},\tau_i} - R_{\bar{u},\tau_i}) = 0$, which implies

$$(P_{\bar{u},\tau_i} - R_{\bar{u},\tau_i})y \in \ker L_{\bar{u},\tau_i} \qquad \forall y \in \mathbb{R}^d.$$

Therefore, (A.30) recast as

$$\bar{u} = R_{\bar{u},\tau_i} y_i + z_i, \qquad z_i := (P_{\bar{u},\tau_i} - R_{\bar{u},\tau_i}) y_i \in \ker L_{\bar{u},\tau_i}. \tag{A.31}$$

Now, let us prove the equivalence in the theorem. Assume that

$$\langle [L_{\bar{u},\tau_i} DG_{\tau_i}(\bar{u})^*]^{-1} y_i, DG_{\tau_i}(\bar{u})\, h\rangle_{\mathbb{R}^d} = 0 \qquad \forall\, h \in \ker L_{\bar{u},\tau_i}$$

On one hand, for $h \in \ker\, L_{\bar{u},\tau_i}$, one get from (A.30) that $\langle \bar{u}, h\rangle_{L^2} = \langle [L_{\bar{u},\tau_i} DG_{\tau_i}(\bar{u})^*]^{-1} y_i, DG_{\tau_i}(\bar{u})\, h\rangle_{\mathbb{R}^d} = 0$, which implies that

$$\bar{u} \in (\ker\, L_{\bar{u},\tau_i})^\perp. \tag{A.32}$$

On the other hand, by (A.31), (A.32) and $R_{\bar{u},\tau_i} y_i = L^*_{\bar{u},\tau_i}\mathcal{N}_{\tau_i}(\bar{u})^{-1} y_i \in \mathrm{Ran}(L^*_{\bar{u},\tau_i}) = (\ker L_{\bar{u},\tau_i})^\perp$, one finds that $z_i = \bar{u} - R_{\bar{u},\tau_i} y_i \in (\ker L_{\bar{u},\tau_i})^\perp$. Therefore, $z_i \in \ker(L_{\bar{u},\tau_i}) \cap (\ker L_{\bar{u},\tau_i})^\perp = \{0\}$ so $z_i = 0$, and

$$\bar{u} = R_{\bar{u},\tau_i} y_i = L^*_{\bar{u},\tau_i}\mathcal{N}_{\tau_i}(\bar{u})^{-1} y_i = \mathcal{S}_i(\bar{u}).$$

Conversely, if $\bar{u} = L^*_{\bar{u},\tau_i}\mathcal{N}_{\tau_i}(\bar{u})^{-1} y_i$, then (A.31) yields $P_{\bar{u},\tau_i} y_i = R_{\bar{u},\tau_i} y_i \in \mathrm{Ran}(L^*_{\bar{u},\tau_i}) = (\ker L_{\bar{u},\tau_i})^\perp$. Let $h \in \ker\, L_{\bar{u},\tau_i}$, then

$$\langle [L_{\bar{u},\tau_i} DG_{\tau_i}(\bar{u})^*]^{-1} y_i, DG_{\tau_i}(\bar{u})\, h\rangle_{\mathbb{R}^d} = \langle [DG_{\tau_i}(\bar{u})^* L_{\bar{u},\tau_i} DG_{\tau_i}(\bar{u})^*]^{-1} y_i\, h\rangle_{L^2} = \langle P_{\bar{u},\tau_i} y_i, h\rangle_{L^2} = 0.$$

This completes the proof of the theorem. ∎

A.6. Proof of Proposition 3.14.

Proof. For ease in notation, set $L^p := L^p((t_0,T);\mathbb{R}^k)$ for $p \in \{1,2,\infty\}$. By definition, $\mathfrak{F}_i \subset L^\infty \subset L^2$. Let $u \in L^2$ and $(u_n) \subset \mathfrak{F}_i$ are such that $u_n \rightharpoonup u$ in L^2. Let us show that $u \in \mathfrak{F}_i$.

Step 1. One has $u \in L^\infty$ and $\|u\|_\infty \le \zeta_i$. In fact, since $(u_n) \subset L^\infty$, by Banach–Alaoglu, there exists a subsequence (not relabeled) and $\tilde{u} \in L^\infty$ with $\|\tilde{u}\|_{L^\infty} \le \zeta_i$ such that $u_n \overset{*}{\rightharpoonup} \tilde{u}$ in L^∞, i.e. $\int_{t_0}^T \varphi(t)^\top u_n(t)\,dt \to \int_{t_0}^T \varphi(t)^\top \tilde{u}(t)\,dt$ for all $\varphi \in L^1$. On the other hand, $u_n \rightharpoonup u$ in L^2, so $\int_{t_0}^T \psi(t)^\top u_n(t)\,dt \to \int_{t_0}^T \psi(t)^\top u(t)\,dt$ for all $\psi \in L^2$. It holds that

$$\int_{t_0}^T \phi(t)^\top u(t)\,dt = \lim_{n\to\infty} \int_{t_0}^T \phi(t)^\top u_n(t)\,dt = \int_{t_0}^T \phi(t)^\top \tilde{u}(t)\,dt \qquad \forall \phi \in L^2 \cap L^1 = L^2.$$

In particular, for $\phi := u - \tilde{u} \in L^2$, one finds $\int_{t_0}^T |u(t)-\tilde{u}(t)|^2\,dt = 0$, which implies $u = \tilde{u}$, a.e., and thus $u \in L^\infty$ and $\|u\|_{L^\infty} = \|\tilde{u}\|_{L^\infty} \le \zeta_i$.

Step 2. One has $G_{\tau_i}(u) := L_{u,\tau_i} u - y_i = 0$. We split the proof of this fact into three steps.

Step 2.1. For $n \in \mathbb{N}$, and $t \in [t_0,T]$, set $\delta_n(t) = \int_{t_0}^t B(s,x_u(s))\,(u_n(s)-u(s))\,ds$. It is clear that $(\delta_n) \subset C^0([t_0,T];\mathbb{R}^d)$. Consider the bounded and linear operator $T_t : L^2 \to \mathbb{R}^d$, $T_t h = \int_{t_0}^T \mathbf{1}_{[t_0,t]}(s)\,B(s,x_u(s))\,h(s)\,ds$. Its adjoint $T_t^* : \mathbb{R}^d \to L^2$ is $T_t^* z = \mathbf{1}_{[t_0,t]}(\cdot)\,B(\cdot,x_u(\cdot))^\top z \in L^2$ for all $z \in \mathbb{R}^d$. Then $\delta_n(t) = T_t(u_n - u)$ and the weak convergence in L^2 gives $\langle \delta_n(t), z\rangle_{\mathbb{R}^d} = \langle u_n - u, T_t^* z\rangle_{L^2} \to 0$ for all $z \in \mathbb{R}^d$, hence $\delta_n(t) \to 0$ in $\mathbb{R}^d$.

For all $n \in \mathbb{N}$, one has by Cauchy-Schwarz inequality,

$$|\delta_n(t)| \le \|B\|_\infty\,(\zeta_i + \|u\|_2)\,\sqrt{\Delta t_0} \qquad \forall\, t \in [t_0,T]$$

showing that $(\delta_n)_n$ is uniformly bounded. For $(t,t') \in [t_0,T]^2$ with $t \ge t'$ (the case of $t < t'$ is identical), one has by Cauchy-Schwarz inequality,

$$|\delta_n(t) - \delta_n(t')| \le \|B\|_\infty\,(\zeta_i + \|u\|_2)\,\sqrt{t-t'} \qquad \forall\, n \in \mathbb{N}$$

showing that $(\delta_n)_n$ is equicontinuous. Consequently, $(\delta_n) \subset C^0([t_0,T];\mathbb{R}^d)$ is relatively compact, and by Ascoli-Arzèla, it admits a uniformly convergent subsubsequence; its limit must be the pointwise limit, viz.

$$\|\delta_n\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2.2. Let x_{u_n} and x_u be the solution of $\dot{x} = N_t(x) + B(t,x)\,u_n$ and $\dot{x} = N_t(x) + B(t,x)\,u$, respectively with the same intitial state $x^0 \in \mathbb{R}^d$. Gronwall's lemma yields

$$\|x_{u_n} - x_u\|_\infty \le \Delta t_0\, e^{(\Lambda_1 + L_B \zeta_i)\Delta t_0} \|\delta_n\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2.3. Write $K_i(u)(t) := D\Phi_{t,\tau_i}(x_u(t))\,B(t,x_u(t)) \in \mathbb{R}^{d\times k}$. Lemma 2.1 and (A.12) yield

$$\|K_i(u_n) - K_i(u)\|_\infty \le \left(\frac{\Lambda_2}{\Lambda_1}\,E_{2,\infty}\|B\|_\infty + L_B\,E_{1,\infty}\right)\|x_{u_n} - x_u\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Overall,

$$\begin{aligned} G_{\tau_i}(u_n) - G_{\tau_i}(u) &= \int_{t_0}^T \big(K_i(u_n) - K_i(u)\big)\,u_n\,dt \;+\; \int_{t_0}^T K_i(u)\,(u_n - u)\,dt \\ &\xrightarrow[n\to\infty]{} 0 + 0, \end{aligned}$$

since the first term vanishes by $\|K_i(u_n) - K_i(u)\|_{L^\infty} \to 0$ and L^2-boundedness of (u_n), and the second by weak convergence against the fixed $K_i(u) \in L^\infty \subset L^2$. As $G_{\tau_i}(u_n) = 0$, we obtain $G_{\tau_i}(u) = 0$.

Step 3. One has $\lambda_{\min}\big(\mathcal{N}_{\tau_i}(u)\big) \geq C_i^{-1}$. In fact, $\mathcal{N}_{\tau_i}(u) = \int_{t_0}^{T} K_i(u)(t)\, K_i(u)(t)^\top\, dt$, and spectral stability,

$$\big|\lambda_{\min}(\mathcal{N}_{\tau_i}(u_n)) - \lambda_{\min}(\mathcal{N}_{\tau_i}(u))\big| \;\leq \|\mathcal{N}_{\tau_i}(u_n) - \mathcal{N}_{\tau_i}(u)\| \leq 2\Delta t_0\, \|B\|_\infty\, e^{\Lambda_1 \Delta t_0}\, \|K_i(u_n) - K_i(u)\|_\infty \xrightarrow[n\to\infty]{} 0.$$

Since $\lambda_{\min}(\mathcal{N}_{\tau_i}(u_n)) \geq C_i^{-1}$ for all n, we obtain $\lambda_{\min}(\mathcal{N}_{\tau_i}(u)) \geq C_i^{-1}$.

If $(u_n) \subset \mathfrak{F}_i$ is a minimizing sequence, then $(\frac{1}{2}\|u_n\|_{L^2}^2)_n$ is a bounded sequence. After extraction, $u_{n_j} \rightharpoonup \bar{u}$ in L^2. By weak sequential closedness, $\bar{u} \in \mathfrak{F}_i$. Then $\frac{1}{2}\|\bar{u}\|_{L^2}^2 \leq \liminf_j \frac{1}{2}\|u_{n_j}\|_{L^2}^2$, so $\bar{u}$ attains the infimum. □

A.7. Proof of Proposition 3.19.

Proof. Let $(x^0, u) \in \mathbb{R}^d \times L^\infty((t_0,T);\mathbb{R}^k)$ and x_u be the solution to (Σ). Then, applying (2.7) with $\beta(t) = \Phi_{t,t_0}(x_u(t))$, one immediately finds that

$$|D\Phi_{t,\tau_i}(x_u(t))z| \geq e^{-\Lambda_1|\tau_i - t|}|z| \qquad \forall (t,\, z) \in [t_0,T]\times\mathbb{R}^d, \quad \tau_1 = t_0,\, \tau_2 = T.$$

It follows that

$$y^\top \mathcal{N}_{\tau_i}(u)\, y = \int_{t_0}^{T} |D\Phi_{t,\tau_i}(x_u(t))^\top\, B(t,x_u(t))^\top y|^2\, dt \geq e^{-2\Lambda_1\Delta t_0}\, |y|^2 \int_{t_0}^{T} b(t)^2\, dt \geq \frac{e^{-2\Lambda_1 \Delta t_0}\, \|b\|_1^2}{\Delta t_0}\, |y|^2,$$

by the Cauchy-Schwarz inequality. This proves (3.32). So, letting $C_i > 0$ as in the proposition, one gets

$$\mathcal{F}(C_i) = \big\{u \in L^\infty((t_0,T);\mathbb{R}^k) : \lambda_{\min}\big(\mathcal{N}_{\tau_i}(u)\big) \geq e^{-2\Lambda_1\Delta t_0}\|b\|_1^2/\Delta t_0\big\} = L^\infty((t_0,T);\mathbb{R}^k),$$

and (3.21) is automatically satisfied. □

A.8. Proof of Theorem 3.20.

Proof. Let $C_i > 0$ as in the statement of the proposition, and $y_i \in \mathbb{R}^d$. Then, (3.21) is satisfied if and only if $v \in \mathcal{F}(C_i)$ for every $v \in \mathcal{S}_i(\mathcal{F}(y_i))$. Observe that

$$\mathcal{F}(C_i) = \Big\{u \in L^\infty((t_0,T);\mathbb{R}^k) : \lambda_{\min}\big(\mathcal{N}_{\tau_i}(u)\big) \geq \lambda_{\min}\big(\mathcal{N}_{\tau_i}(u_i^\flat)\big)/(1+\theta)\Big\}.$$

Now, $v \in \mathcal{S}_i(\mathcal{F}(y_i))$ if and only if $v = \mathcal{S}_i(u) = B(t,x_u(t))^\top D\Phi_{t,\tau_i}(x_u(t))^\top\, \mathcal{N}_{\tau_i}(u)^{-1}\, y_i$ for some $u \in \mathcal{F}(C_i)$, where $\tau_1 := t_0$ and $\tau_2 := T$. Therefore, $\|v\|_\infty \leq \|B\|_\infty\, e^{\Lambda_1\Delta t_0}\lambda_{\min}^{-1}(\mathcal{N}_{\tau_i}(u))\,|y_i| \leq (1+\theta)\, e^{\Lambda_1\Delta t_0}\|B\|_\infty|y_i|/\lambda_{\min}\big(\mathcal{N}_{\tau_i}(u_i^\flat)\big)$. It remains to prove that

$$\lambda_{\min}(\mathcal{N}_{\tau_i}(v)) = \lambda_{\min}\big(\mathcal{N}_{\tau_i}(\mathcal{S}_i(u))\big) \geq \lambda_{\min}\big(\mathcal{N}_{\tau_i}(u_i^\flat)\big)/(1+\theta). \tag{A.33}$$

By the spectral stability and $L_{\mathcal{N}_{\tau_i}}$-Lipschitz of $u \in \{u \in L^\infty((t_0,T);\mathbb{R}^k) : \|u\| \leq \zeta_i\} \mapsto \mathcal{N}_{\tau_i}(u)$, one finds that

$$\Big|\lambda_{\min}\big(\mathcal{N}_{\tau_i}(\mathcal{S}_i(u))\big) - \lambda_{\min}\big(\mathcal{N}_{\tau_i}(u_i^\flat)\big)\Big| \leq L_{\mathcal{N}_{\tau_i}}\left(\frac{(1+\theta)\, e^{\Lambda_1\Delta t_0}\|B\|_\infty|y_i|}{\lambda_{\min}\big(\mathcal{N}_{\tau_i}(u_i^\flat)\big)} + \|u_i^\flat\|_\infty\right).$$

Since

$$\Big|\lambda_{\min}\big(\mathcal{N}_{\tau_i}(\mathcal{S}_i(u))\big) - \lambda_{\min}\big(\mathcal{N}_{\tau_i}(u_i^\flat)\big)\Big| \geq \frac{\lambda_{\min}\big(\mathcal{N}_{\tau_i}(u_i^\flat)\big)}{1+\theta} + \frac{\theta\lambda_{\min}\big(\mathcal{N}_{\tau_i}(u_i^\flat)\big)}{1+\theta} - \lambda_{\min}\big(\mathcal{N}_{\tau_i}(\mathcal{S}_i(u))\big),$$

one deduces that if

$$\frac{\theta\lambda_{\min}\big(\mathcal{N}_{\tau_i}(u_i^\flat)\big)}{1+\theta} \geq L_{\mathcal{N}_{\tau_i}}\left(\frac{(1+\theta)\, e^{\Lambda_1\Delta t_0}\|B\|_\infty|y_i|}{\lambda_{\min}\big(\mathcal{N}_{\tau_i}(u_i^\flat)\big)} + \|u_i^\flat\|_\infty\right) \tag{A.34}$$

then (A.33) is satisfied. Since (A.34) is equivalent to (3.34), this completes the proof of the proposition. □

A.9. Proof of Lemma 3.21.

Proof. Write $K_i(u)(t) := D\Phi_{t,\tau_i}(x_u(t))\,B(t,x_u(t)) \in \mathbb{R}^{d\times k}$ so that $\mathcal{N}_{\tau_i}(u) = \int_{t_0}^T K_i(u)(t)\,K_i(u)(t)^\top\,dt$. Clearly, $\mathcal{N}_{\tau_i}$ is continuous on L^∞ by Lebesgue dominated convergence theorem. Let us prove the second part of the lemma. Let $u,\, v \in \{u \in\, L^\infty((t_0,T);\mathbb{R}^k) : \|u\|_\infty \leq\, \zeta_i\}$. By the Cauchy-Schwarz inequality, one finds that

$$\frac{d}{dt}|x_u(t)-x_v(t)| = |\dot{x}_u(t)-\dot{x}_v(t)| \leq (\Lambda_1 + L_B\,\zeta_i)|x_u(t)-x_v(t)| + \|B\|_\infty\,|u(t)-v(t)|$$

which, by Gronwall's lemma, yields

$$|x_u(t)-x_v(t)| \leq \|B\|_\infty \frac{e^{(\Lambda_1+L_B\,\zeta_i)(t-t_0)}-1}{(\Lambda_1+L_B\,\zeta_i)}\|u-v\|_\infty \qquad \forall \in [t_0,T]. \tag{A.35}$$

Using Lemma 2.1, one finds (we let $\Delta t_0 := T-t_0$)

$$\begin{aligned}
\frac{\|\mathcal{N}_{\tau_2}(u)-\mathcal{N}_{\tau_2}(v)\|}{\|u-v\|_\infty} &\leq \frac{2\|B\|_\infty^3\Lambda_2}{\Lambda_1(\Lambda_1+L_B\,\zeta_2)}\int_{t_0}^T e^{\Lambda_1(T-t)}\,\left(e^{2\Lambda_1(T-t)}-e^{\Lambda_1(T-t)}\right)\,(e^{(\Lambda_1+L_B\,\zeta_2)(t-t_0)}-1)\,dt \\
&\quad+\frac{2L_B\|B\|_\infty^2}{(\Lambda_1+L_B\,\zeta_2)}\int_{t_0}^T e^{2\Lambda_1(T-t)}(e^{(\Lambda_1+L_B\,\zeta_2)(t-t_0)}-1)\,dt \\
&= \frac{L_B\|B\|_\infty^2 e^{\Lambda_1\Delta t_0}}{\Lambda_1}\left[\frac{e^{\Lambda_1\Delta t_0}-e^{L_B\zeta_2\Delta t_0}}{\Lambda_1-L_B\zeta_i}-\frac{e^{L_B\zeta_2\Delta t_0}-e^{-\Lambda_1\Delta t_0}}{L_B\zeta_2+\Lambda_1}\right]+ \\
\frac{2\|B\|_\infty^3\Lambda_2 e^{\Lambda_1\Delta t_0}}{\Lambda_1(\Lambda_1+L_B\,\zeta_2)}&\left[\frac{e^{2\Lambda_1\Delta t_0}-e^{L_B\zeta_2\Delta t_0}}{2\Lambda_1-L_B\zeta_2}+\frac{e^{\Lambda_1\Delta t_0}-e^{L_B\zeta_2\Delta t_0}}{\Lambda_1-L_B\zeta_2}+\frac{e^{\Lambda_1\Delta t_0}-e^{-\Lambda_1\Delta t_0}}{2\Lambda_1}-\frac{e^{2\Lambda_1\Delta t_0}-e^{-\Lambda_1\Delta t_0}}{3\Lambda_1}\right],
\end{aligned}$$

by a direct integration. The same proof applies to $\mathcal{N}_{\tau_1}$. ◻

A.10. Proof of Proposition 3.29.

Proof. Recall that $DN(x) = -D + W\,D\sigma(x)$ and $D^2N(x) = W\,D^2\sigma(x)$. Then, from

$$\frac{d}{dt}\phi_t(x) = N(\phi_t(x)), \quad \phi_0(x)=x; \qquad \frac{d}{dt}D\phi_t(x) = DN(\phi_t(x))\,D\phi_t(x), \quad D\phi_0(x)=\mathrm{Id} \quad \forall x\in\mathbb{R}^d,$$

Cauchy-Schwarz inequality yields

$$\frac{d}{dt}|D\phi_t(x)\,y| = \frac{\langle -D\,D\phi_t(x)\,y, D\phi_t(x)\,y\rangle + \langle W\,D\sigma(\phi_t(x))\,D\phi_t(x)\,y, D\phi_t(x)\,y\rangle}{|D\phi_t(x)\,y|} \leq \Gamma\,|D\phi_t(x)\,y| \quad \forall y \in \mathbb{R}^d\backslash\{0\},$$

which, by Gronwall's lemma, leads to

$$\|D\phi_t(x)\| \leq e^{\Gamma t} \qquad \forall\, x\in\mathbb{R}^d,\, \forall t\geq 0. \tag{A.36}$$

Similarly, one has

$$\frac{d}{dt}D^2\phi_t(x) = D^2N(\phi_t(x))\,\left[D\phi_t(x), D\phi_t(x)\right] + DN(\phi_t(x))\,D^2\phi_t(x), \quad D^2\phi_0(x)=0, \quad x\in\mathbb{R}^d,$$

so that letting $g(t) := D^2\phi_t(x)[y,z]$ for $y,z\in\mathbb{R}^d$, Cauchy-Schwarz inequality and (A.36) lead to

$$\begin{aligned}
\frac{d}{dt}|g(t)| &= \frac{\langle -D\,g(t), g(t)\rangle + \langle W\,D\sigma(\phi_t(x))\,g(t), g(t)\rangle + \langle W\,D^2\sigma(\phi_t(x))\,[D\phi_t(x)y, D\phi_t(x)z]\,, g(t)\rangle}{|g(t)|} \\
&\leq \Gamma\,|g(t)| + \Gamma_2\,e^{2\Gamma\,t}\,|y|\,|z|,
\end{aligned}$$

which, by Gronwall's lemma and immediate integration, yields

$$\|D^2\phi_t(x)\| \le \begin{cases} \Gamma_2\,\frac{e^{2\Gamma\,t}-e^{\Gamma\,t}}{\Gamma} & \text{if} \quad \Gamma \neq 0, \\ \Gamma_2\,t & \text{if} \quad \Gamma = 0, \end{cases} \qquad \forall\, x \in \mathbb{R}^d,\ \forall t \ge 0. \tag{A.37}$$

In particular as in Lemma 2.1, one can replace $x \in \mathbb{R}^d$ in (A.36) and (A.37) with $\beta(\cdot)$ for any $\beta \in C^0([0,T];\mathbb{R}^d)$. Finally, for $u, v \in L^\infty((0,T);\mathbb{R}^k)$, one uses the same technique to prove (A.36) and (A.37), and get

$$|x_u(t) - x_v(t)| \le \begin{cases} \|B\|_\infty\,\frac{e^{\Gamma\,t}-1}{\Gamma}\|u-v\|_\infty & \text{if} \quad \Gamma \neq 0, \\ \|B\|_\infty\,t\,\|u-v\|_\infty & \text{if} \quad \Gamma = 0, \end{cases} \qquad \forall t \ge 0. \tag{A.38}$$

Now, recall that $\mathcal{N}_{\tau_2}(u) = \int_0^T D\phi_t(x_u(T-t))\,B(T-t)\,B(T-t)^\top\,D\phi_t(x_u(T-t))^\top\,dt$, so that letting $Q_u(t) := D\phi_t(x_u(T-t))$, if $\Gamma = 0$, the estimates (A.36), (A.37) and (A.38) yield

$$\|\mathcal{N}_{\tau_2}(u) - \mathcal{N}_{\tau_2}(v)\| \le\ 2\,\|B\|_\infty^3\,\Gamma_2\|u-v\|_\infty \int_0^T t\,(T-t)\,dt = \frac{\Gamma_2\,\|B\|_\infty^3\,T^3}{3}\,\|u-v\|_\infty. \tag{A.39}$$

When $\Gamma \neq 0$, one finds that

$$\|\mathcal{N}_{\tau_2}(u) - \mathcal{N}_{\tau_2}(v)\| \le\ 2\,\|B\|_\infty^3\,\Gamma_2\,\|u-v\|_\infty \int_0^T \frac{(e^{3\Gamma\,t} - e^{2\Gamma\,t})(e^{\Gamma(T-t)}-1)}{\Gamma^2}\,dt = \frac{\Gamma_2\,\|B\|_\infty^3}{3}\left(\frac{e^{\Gamma\,T}-1}{\Gamma}\right)^3\|u-v\|_\infty,$$

which completes the proof of the inequality involving $\mathcal{N}_{\tau_2}$. The same arguments are used to prove those of $\mathcal{N}_{\tau_1}$. □

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