Rank-Preserving Index-Dependent Matrix Transformations: Applications to Clockwork and Deconstruction Theory Space Models

Aadarsh Singh¹

Centre for High Energy Physics, Indian Institute of Science, C. V. Raman Avenue, Bengaluru 560 012, India

(*Electronic mail: aadarshsingh@iisc.ac.in)

(Dated: 25 June 2025)

We introduce a versatile framework of index-dependent element-wise matrix transformations, $b_{ij} = a_{ij}/g_f(i,j)$, with direct applications to hierarchy generating mass hierarchies in high-energy physics. This paper establishes the precise mathematical conditions on $g_f(i,j)$ that preserve the rank and nullity of the original matrix. Our study reveals that such transformations provide a powerful method for engineering specific properties of a matrix's null space; by appropriately selecting the function $g_f(i,j)$, one can generate null vectors (or eigenvectors) with diverse and controllable localization patterns. The broad applicability of this technique is discussed, with detailed examples drawn from high-energy physics. We demonstrate how our framework can be used to tailor 0-mode profiles and fermionic mass spectra in clockwork and dimensional deconstruction models —showing that the standard clockwork mechanism arises as a particular case $(g_f(i,j) = f^{(i-j)})$ — thereby offering new tools for particle physics BSM model building. This work illustrates the potential of these transformations in model building across various fields where localized modes or specific spectral properties are crucial.

I. INTRODUCTION

In linear algebra, the concepts of rank and nullity play fundamental roles in understanding the properties and behaviour of matrices and linear transformations. The rank of a matrix represents the dimension of its column space, while the nullity refers to the dimension of its null space or kernel space¹. The famous rank-nullity Theorem links these quantities with the number of columns for a matrix². These quantities also reveal information about the eigenvalues and eigenvectors of the matrix which are fundamental in understanding the matrix.

Matrices represent various objects across different domains, such as networks in graph theory via adjacency matrices, couplings among fields in high-energy physics via mass matrices, stiffness matrices in structural analysis, and inductance and capacitance matrices in electrical systems, etc. Therefore understanding a matrix's properties, including its range space and kernel space, is crucial for analyzing these objects.

This paper focuses on a specific type of matrix transformation which is similar to Hadamard or Schur product³ and its effects on rank and nullity. We consider a matrix B constructed by an indices-dependent element-wise transformation of another matrix A and study the specific properties of B from the properties of A. The 'indices-dependent element-wise transformation,' considered in this work is defined in the definition section. Our findings provide insights into how these transformations affect the fundamental structure of matrices. Since the application of Hadamard products is known in various fields such as in lossy compression, machine learning, image processing etc. ^{4,5,6}, the transformation considered in the paper can possibly contribute in those domains too. Beyond establishing the conditions for rank and nullity preservation, we will show that the specific functional form of $g_f(i,j)$ provides significant flexibility, allowing for the tailoring of resulting vector properties, such as localization patterns. We also try to implement this transformation in hierarchy-producing mechanisms known in high-energy physics, such as in clockwork or theory space models. In the SM of particle physics, the hierarchical masses of neutrinos are a mystery that people try to explain via various models such as the seesaw mechanism, radiative mechanism, the clockwork mechanism, extra dimension models (RS, UED), Dimension Deconstruction, etc.⁷⁻¹² Apart from mass hierarchy some of these mechanisms can be formalised to account for hierarchy observed in the strength of gravity and weak forces too.

The paper is organized as follows: In Section 2, we state and prove the main Theorems and Corollaries governing the rank, nullity, and eigenvector properties under the index-dependent element-wise transformation. In Section 3, we explicitly give an application of this transformation in high-energy physics to generate a variety of profiles for null modes in theory space models. Crucially, we demonstrate the framework's flexibility to generate independent profiles for the wavefunctions of left and right chiral fermions and explicitly compare the distinct localization profiles generated by different valid transformations. We demonstrate how the clockwork and its variants can be considered as a specific case of the general transformation $g_f(i,j)$. Finally, we implement the transformation in deconstruction models and show how it modifies the KK tower mass spectra. Section 4 concludes with a summary and outlook on future directions.

MAIN THEOREMS AND PROOFS

Definitions and Notations

Definition 1: Index-dependent Element-wise Transformation - Let $A = [a_{i,j}]$ be an $m \times n$ matrix. The index-dependent element-wise transformation of A, denoted T(A), is defined as a new matrix $B = [b_{i,j}]$ where:

$$b_{i,j} = h_f(a_{i,j}, i, j)$$

we are considering a specific case of this transformation in this work namely,

$$b_{i,j} = \frac{a_{i,j}}{g_f(i,j)}$$

for all i = 1, ..., m and j = 1, ..., n. The functions h_f and g_f are defined below.

This transformation can be seen as a type of Hadamard product between matrix A and matrix C to give matrix B with elements of matrix C being dependent on the elements of matrix A along with their position.

Notation: Let $f \in \mathbb{F}$ be an element of the field \mathbb{F} . We define $g_f : \mathbb{N} \times \mathbb{N} \to \mathbb{F}$ as a family of functions parameterized by f, where $g_f(i,j)$ takes as input the indices i and j corresponding to an element $a_{i,j}$ of the original matrix, and produces an output in the field \mathbb{F} . Here, \mathbb{F} denotes the field from which the elements of the original matrix are drawn. The subscript f in g_f indicates that the function's definition depends on the choice of f. For example, $g_2(i,j) = 2^{i+j}$ and $g_3(i,j) = 3^{i+j}$ when \mathbb{F} is the real number field.

Similarly, $h_f : \mathbb{F} \times \mathbb{N} \times \mathbb{N} \to \mathbb{F}$ is a function $h_f(a_{i,j},i,j)$ which takes as input an element $a_{i,j}$ of the original matrix and its corresponding indices i and j and produces an output in the field \mathbb{F} .

Theorems and Proofs

Theorem 1 - For any matrix A of size $N \times M$ with the following element-wise transformation,

$$b_{i,j} = \frac{a_{i,j}}{g_f(i,j)}$$

the nullity and rank of newly formed matrix B will be the same as of A if $g_f(i,j)$ satisfies the following equality

$$\frac{g_f(i,j)}{g_f(k_0,j)} = G_f(i)$$
 or $\frac{g_f(i,j)}{g_f(i,k_0)} = G'_f(j)$

i.e., when the ratio of the function evaluated at a common column or a common row is independent of the column index or row index respectively, with $G_f(i)$ denoting some function parameterized by f that varies with row index and $G'_f(j)$ some other function varying with column index, to make sure the new elements $b_{i,j}$ don't blow up, the following constraint is applied.

$$g_f(i,j) \neq 0, \infty, \quad \forall i \in \{1,2,...,N\} \& j \in \{1,2,...,M\}$$

Proof - To prove this, we need to show that a nullity in A will lead to a nullity in B and vice versa under this transformation. Since the dimensions of the matrix are preserved under this transformation, the necessary nullities in rectangular matrices with N < M will always be present in both matrices (underdetermined system). So, we will focus on the additional nullities. Let's take the row linear dependence of matrix A. Consider v_{i_0} , denoting the i_0^{th} row of matrix A of dimensions $N \times M$, to be linearly dependent on other rows i.e.,

$$v_{i_0} = \sum_{j \neq i_0}^{N} \alpha_j v_j \tag{1}$$

Now, consider v_i to be the row of B corresponding to the v_i row of A. Then showing the emergence of the following equality from the above equality

$$v'_{i_0} = \sum_{j \neq i_0}^{N} \alpha'_j v'_j \tag{2}$$

for $\alpha_i' \in \mathbb{F}$ will prove corresponding row linear dependence in B. Take the k^{th} element of i_0

$$v_{i_0,k} = \sum_{j \neq i_0}^{N} \alpha_j v_{j,k} \tag{3}$$

 α_i is the same for a given row i.e. α_i must not vary with the column elements k for a fixed row, and the same goes for α_i as we are checking for linear dependence of rows. Then from the definition of the elements of matrix B,

$$v'_{i_0,k} = \frac{v_{i_0,k}}{g_f(i_0,k)}$$

$$v_{i_0,k} = v'_{i_0,k} g_f(i_0,k) \tag{4}$$

Using this in eq. 1 for matrix A,

$$v'_{i_0,k}g_f(i_0,k) = \sum_{j \neq i_0}^N \alpha_j v'_{j,k}g_f(j,k), \qquad \forall \ k \in \{1,2,\ldots,N\}$$
 (5)

$$v'_{i_0,k} = \frac{1}{g_f(i_0,k)} \sum_{j \neq i_0}^{N} \alpha_j v'_{j,k} g_f(j,k) \qquad \forall \ k \in \{1,2,\dots,N\}$$
 (6)

$$v'_{i_0,k} = \sum_{j \neq i_0}^{N} \alpha_j v'_{j,k} G_f(j) \qquad \forall k \in \{1, 2, \dots, N\}$$
 (7)

hence.

$$v'_{i_0} = \sum_{j \neq i_0}^{N} \alpha'_j v'_j \tag{8}$$

with $\alpha'_i = \alpha_i G_f(j)$, it is not dependent on column indices k. Hence, a linearly dependent row in the A matrix leads to a linearly dependent row in the B matrix. Similarly, repeating the proof starting from the linearly dependent row in the B matrix will lead to the linearly dependent row in the A matrix. So we can conclude that the number of linearly dependent rows in the A and B matrices will be the same under this transformation. As the matrix dimensions are preserved in this transformation, the number of linearly independent rows in the A and B matrices will be N - r, r is assumed to be the number of linearly dependent rows in the A matrix and hence in the B matrix. Then, using the fundamental row-column rank Theorem¹³, the row rank for any matrix is always equal to its column rank i.e.

number of linearly independent column = number of linearly independent rows

we get the number of linearly independent columns in the A and B matrix = N - r. So, the number of linearly dependent columns in matrix A and B = M - (N - r) = nullity of the matrix A and B. Hence

$$Nullity \ of \ A = Nullity \ of \ B$$

Finally, from the Rank-Nullity Theorem, the Rank of matrix A = M - nullity of A = M - nullity of B = rank of matrix B

$$Rank \ of A = Rank \ of B$$

Hence proved. \square

Theorem 2 - Any function $g_f(x,y)$ which is separable, satisfies the condition of Theorem 1 and vice versa. **Proof** - From Theorem 6 in 14 , we know that a function $g_f(x, y)$ is separable iff

$$g_f(i,j)g_f(x,y) = g_f(x,j)g_f(i,y)$$

this leads to

$$\frac{g_f(x,y)}{g_f(i,y)} = \frac{g_f(x,j)}{g_f(i,j)} \qquad \text{or} \qquad \frac{g_f(x,y)}{g_f(x,j)} = \frac{g_f(i,y)}{g_f(i,j)}$$

hence satisfies the desired condition on $g_f(x, y)$

$$\frac{g_f(x,y)}{g_f(i,y)} = G_f(x)$$
 or $\frac{g_f(x,y)}{g_f(x,j)} = G_f'(y)$

where i and j represent some value of x and y in domain of $g_f(x, y)$. Now take

$$\frac{g_f(x,y)}{g_f(i,y)} = G_f(x)$$

Here $G_f(x)$ has to satisfy the condition that for x = i, $G_f(i) = 1 \forall y$, and also it is independent of any value of y, hence WLOG

$$G_f(x) = \frac{g_f(x,j)}{g_f(i,j)}$$

similarly,

$$G'_f(y) = \frac{g_f(i, y)}{g_f(i, j)}$$

which leads to

$$g_f(i,j)g_f(x,y) = g_f(x,j)g_f(i,y)$$

and hence separability. \square

C. Corollaries

Corollary 1 - For any matrix A with $\{v^1, v^2, \dots, v^n\}$ as eigenvectors of its nullspace, the corresponding eigenvectors for the nullspace of matrix B, constructed by above transformation, are given by $\{v'^1, v'^2, \dots, v'^n\}$ with

$$v_j^{\prime i} = v_j^i g_f^{\prime \prime}(j)$$

where v_j^i represents the j^{th} component of i^{th} null basis vector and $g_f(x,y) = g_f'(x)g_f''(y)$ from the above Theorem. $g_f'(x), g_f''(y)$ denotes any two functions parameterized by f and depends on x and y respectively.

Proof - Consider the $v^i th$ null basis vector of matrix A, $Av^i = \vec{0}$. This implies

$$\sum_{j=1}^{M} a_{l,j} v_j^i = 0 \qquad \forall l \in \{1, 2, \dots, N\}$$
(9)

now using the element-wise transformation of matrix A by the function in the above corollary,

$$a_{l,j} = b_{l,j} \times g_f'(l)g_f''(j)$$
 (10)

$$\sum_{j=1}^{M} b_{l,j} \times g_f'(l)g_f''(j)v_j^i = 0 \qquad \forall l \in \{1, 2, \dots, N\}$$
(11)

without loss of generality, the factor of $g'_f(l)$ can be absorbed to 0 in the R.H.S.

$$\sum_{j=1}^{M} b_{l,j} \times g_f''(j) v_j^i = 0 \qquad \forall l \in \{1, 2, \dots, N\}$$
(12)

$$\sum_{j=1}^{M} b_{l,j} v_j^{l} = 0 \qquad \forall l \in \{1, 2, \dots, N\}$$
(13)

with $v_i^{li} = v_i^{li} g_f''(j)$. Hence all of the null basis vectors of A with their elements scaled by $g_f''(j)$, will behave as null basis vectors

Corollary 2 - For any diagonalizable square matrix A with $\{\mu_1, \mu_2, \dots, \mu_n\}$ eigenvalues and the corresponding eigenvectors $\{v^1, v^2, v^3, \dots, v^N\}$, the matrix B, constructed by above transformation, will also be diagonalizable with same eigenvalues as the eigenvalues of matrix A and with eigenvectors $\{v'^1, v'^2, \dots, v'^N\}$ given by

$$v_j^{i} = v_j^i \times g_f''(j) \tag{14}$$

iff the function $g_f(i, j)$ satisfies

$$g'_f(k)g''_f(k) = 1 \qquad \forall k \in \{1, 2, \dots, N\}$$
 (15)

Proof - Consider the $v^i th$ eigenvector of matrix A, $Av^i = \mu_i v^i$. This implies

$$\sum_{j=1}^{N} a_{l,j} v_j^i = \mu_i v_l^i \qquad \forall \ l \in \{1, 2, \dots, N\}$$
 (16)

$$\sum_{i=1}^{N} (a_{l,j} - \mu_i \delta_l^j) v_j^i = 0 \qquad \forall l \in \{1, 2, \dots, N\}$$
(17)

now using the element-wise transformation of matrix A by the operator in the above corollary,

$$a_{l,j} = b_{l,j} \times g_f(l,j) \tag{18}$$

$$\sum_{i=1}^{N} (b_{l,j} \times g_f'(l)g_f''(j) - \mu_i \delta_l^j) v_j^i = 0 \qquad \forall l \in \{1, 2, \dots, N\}$$
(19)

$$\sum_{i \neq l}^{N} b_{l,j} \times g_f'(l) g_f''(j) v_{i,j} + (b_{l,l} g_f'(l) g_f''(l) - \mu_i) v_l^i = 0 \qquad \forall l \in \{1, 2, \dots, N\}$$
(20)

using the property $g'_f(l)g''_f(l) = 1$,

$$\sum_{j\neq l}^{N} b_{l,j} \times g_f'(l)g_f''(j)v_{i,j} + (b_{l,l} - \mu_i)v_l^i = 0 \qquad \forall l \in \{1, 2, \dots, N\}$$
(21)

$$\sum_{j\neq l}^{N} b_{l,j} \times g_f''(j) v_{i,j} + (b_{l,l} - \mu_l) \frac{v_l^i}{g_f'(l)} = 0 \qquad \forall l \in \{1, 2, \dots, N\}$$
 (22)

$$\sum_{i \neq l}^{N} b_{l,j} \times g_f''(j) v_{i,j} + (b_{l,l} - \mu_i) v_l^i \times g_f''(l) = 0 \qquad \forall l \in \{1, 2, \dots, N\}$$
(23)

$$\sum_{j=1}^{N} (b_{l,j} - \mu_l \delta_l^j) v_j^i \times g_f''(j) = 0 \qquad \forall l \in \{1, 2, \dots, N\}$$
 (24)

with $v_i^{\prime i} = v_i^i g_f^{\prime \prime}(j)$. Hence, the eigenvalues of matrix B are the same as the eigenvalues of matrix A. The converse of this can also be proved easily, starting from eq. 17 and using the transformation eq. 18 gives eq. 20 that needs to be equal to eq. 24 as per assumption which would demand the function to satisfy $g'_f(l)g''_f(l) = 1$. \square

Corollary 3 - Any matrix B produced from matrix A by the index-dependent element-wise transformation function of corollary 2 will be similar to each other.

Proof - From the Theorem¹⁵, we know any two diagonalizable matrices with the same eigenvalues are similar i.e.,

$$B = P^{-1}AP \qquad \Box \tag{25}$$

APPLICATIONS TO RELEVANT FIELDS

In high-energy physics, the hierarchical small values of neutrino mass among the standard model fields are a challenge. There are several mechanisms already proposed, such as seesaw, clockwork, Randomness etc., to account for the natural emergence of such small scales. The seesaw mechanism still demands a field at a very high energy scale (GUT scale) to work, but clockwork and Randomness models can achieve the small masses naturally with all O(1) parameters in the model, but they require a mass matrix of a certain type to work. The Clockwork model can be analysed to have its successful functioning relying on two important facts, 1) - the presence of 0-mode or Nullity in the mass matrix, and 2) - the localisation of these 0-modes on some particular sites. The localised 0-modes or eigenmodes can be used to produce highly suppressed coupling between left and right chiral neutrinos, which produces the observed hierarchical small scale.

The element-wise transformations developed in this work offer a flexible framework capable of accommodating a variety of theoretical models. In particular, they enable the construction of scenarios that can outperform the traditional clockwork mechanism in generating naturally small scales, as demonstrated in 16. In this context, the clockwork model matrix appears as just one specific case within a broader space of possible transformed matrices. This connection opens the door to new possibilities in model building, particularly in the context of neutrino mass generation and related particle physics phenomena.

The full Lagrangian for the class of models considered is given by:

$$\mathcal{L} = \mathcal{L}_{SM} + \mathcal{L}_{NP} + \mathcal{L}_{int}. \tag{26}$$

with \mathcal{L}_{SM} representing the lagrangian for the standard model given by

$$\mathcal{L}_{SM} = \mathcal{L}_{Gauge} + \mathcal{L}_{Fermion} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa}. \tag{27}$$

 \mathcal{L}_{NP} represents the physics of new fields and \mathcal{L}_{int} is the interaction lagrangian between SM fields and new fields. For the new physics Lagrangian, we will consider chiral fermionic fields with only Dirac couplings.

$$\mathcal{L}_{NP} = \mathcal{L}_{kin} - \sum_{i,j=1}^{n} \overline{L_i} \mathcal{H}_{i,j} R_j + h.c.$$
(28)

with $\mathcal{H}_{i,j}$ being the Hamiltonian representing the connection between different chiral fields. L_i and R_i represent the left and right-handed clockwork gear fields, respectively.

Clockwork Model Generalisations - Mass spectrum, 0-mode Profile

The Hamiltonian and new physics Lagrangian for the generalized CW are given below ¹⁷:-

$$\mathcal{H}_{i,j} = m_i \delta_{i,j} + m_i q_i \delta_{i+1,j}. \tag{29}$$

$$\mathcal{L}_{NP} = \mathcal{L}_{kin} - \sum_{i}^{n} m_{i} \overline{L_{i}} R_{i} - \sum_{i}^{n} m_{i} q_{i} \overline{L_{i}} R_{i+1} + h.c.$$
(30)

with m_i as the fundamental mass parameters and q_i as the coupling strength parameters of the model. For uniform clockwork case, $m_i = m$ with unity coupling strength parameters $q_i = q = 1$, the Dirac mass matrix for the fermions in the basis $\{\overline{L}_1,\overline{L}_2,\ldots,\overline{L}_n\}$ and $\{R_1,R_2,\ldots,R_{n+1}\}$ is given by

$$M_{CW} = m \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}_{n \times n+1}$$
(31)

The original matrix possesses a null mode due to its rectangular structure. However, the zero mode corresponding to the null eigenvalue lacks localization.

$$\Lambda_0 = \{(-1)^{n+1}, (-1)^n, ..., (-1)^2, (-1)^1\}$$

As a result, once the Higgs field acquires a vacuum expectation value v, the mass modes generated are of the same order as the underlying fundamental scale, O(m), owing to the absence of localization. Upon applying an element-wise transformation of the form $b_{i,j} = \frac{a_{i,j}}{g_f(i,j)}$, the components of the new null vector are determined by Corollary 1. The newly transformed Hamiltonian is

$$\mathcal{H}'_{i,j} = \mathcal{H}_{i,j} g_f(i,j) \tag{32}$$

Hence for $g_f(i, j) = q^{i-j}$, the null mode is given by

$$\Lambda_0 = \{(-q)^{n+1}, (-q)^n, ..., (-q)^2, (-q)^1\}$$

which corresponds to the well-known clockwork profile—a widely studied structure in particle physics that naturally generates hierarchies for $|q|>1^{12}$. One can consider an alternative function $\bar{g}_f(i,j)$ for the transformation. For example, we can define $\bar{g}_f(i,j) = q^{(-1)^j j}$ in the matrix eq.(31). Since this function is independent of the *i*-indices, it trivially preserves the Nullity. We observe that the localized 0-mode produced by this transformation exhibits a qualitatively different profile compared to the clockwork profile, as illustrated in Fig.2. The profile for 0-modes for the scenario of even n

$$\vec{0}_n^k = \frac{1}{\mathcal{N}_0} (-1)^k q^{n+1-(-1)^k k}$$

$$\vec{0}_{n}^{k} = \{0_{n}^{1}, 0_{n}^{2}, ..., 0_{n}^{n+1}\}$$

for $k \in \{1, 2, 3, ..., n, n+1\}$, with $n \in \mathbb{N}$ with \mathcal{N}_0 being the normalization factor. This transformation can be further modified to generate 0-modes with a variety of interesting localization profiles.

An extension from the local to non-local clockwork models can also be achieved by allowing non-zero couplings between left- and right-chiral fields that are not immediate neighbors, thereby modifying the interaction structure and enabling new localization patterns. This additional structure in the mass matrix alters the localization pattern of the mass modes. However, the element-wise transformation can still be applied to recover localized zero modes.

Hamiltonian for this extension can be written as

$$\mathcal{H}_{i,j} = \sum_{k=1}^{n+1} a_{i,k} \delta_{i,j-k+1}$$
(33)

with $i \in \{1, 2, ...n\}$ and $j \in \{1, 2, ...n + 1\}$. Using the CW notation to write the new physics Lagrangian¹⁶, one gets

$$\mathcal{L}_{NP} = \mathcal{L}_{kin} - \sum_{i=1}^{n} m_{i} \overline{L_{i}} R_{i} - \sum_{i=1}^{n} m_{i} q_{i}^{(1)} \overline{L_{i}} R_{i+1} - \sum_{i=1}^{n-1} m_{i} q_{i}^{(2)} \overline{L_{i}} R_{i+2} - \sum_{i=1}^{n-2} m_{i} q_{i}^{(3)} \overline{L_{i}} R_{i+3} - \sum_{i=1}^{n-3} m_{i} q_{i}^{(4)} \overline{L_{i}} R_{i+4} + \dots - \sum_{i=1}^{n-(n-1)} m_{i} q_{i}^{(n)} \overline{L_{i}} R_{i+n} + h.c.$$

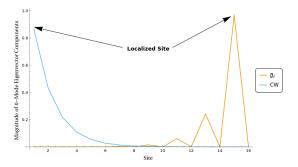
$$= \mathcal{L}_{kin} - \sum_{i=1}^{n} m_{i} \overline{L_{i}} R_{i} - \sum_{k=1}^{n} \sum_{i=1}^{n-k+1} m_{i} q_{i}^{(k)} \overline{L_{i}} R_{i+k} + h.c.$$
(34)

In CW notations, $a_{i,i} = m_i$, $a_{i,i+1} = m_i q_i^{(1)}$, $a_{i,i+2} = m_i q_i^{(2)}$, ..., $a_{i,i+n} = m_i q_i^{(n)}$. The K^{th} component for null space basis of CN-CW (completely non-local clockwork) in the limiting case, $a_{i,i+k} = a_k \ \forall \ i$, is given by

$$\Lambda_0^K = \sum_{\{k_1, \dots, k_n\}} \frac{(k_1 + k_2 + \dots + k_n)!}{k_1! \dots k_n!} \frac{(-a_n)^{k_n} \dots (-a_1)^{k_1}}{a_0^{k_1 + k_2 + \dots + k_n}}$$
with
$$nk_n \dots + 2k_2 + k_1 = n + 1 - K$$
(36)

with
$$nk_n ... + 2k_2 + k_1 = n + 1 - K$$
 (36)
 $\Lambda_0 = \{\Lambda_0^1, \Lambda_0^2, ..., \Lambda_0^{n+1}\}$

and $k_1, k_2, ..., k_n \in \mathbb{N}_0 = \{0, 1, 2, 3, 4, ...\}$. The normalized 0-mode $\tilde{\Lambda}_0$ is given by $\mathcal{N}_0\Lambda_0$, with \mathcal{N}_0 representing the normalizing factor. In this case, the zero-mode components exhibit a localized profile even when the coupling strength is unity, i.e., $|q_i^k|$ $|q^k|=1$ for all k. This localization can be significantly enhanced by applying the element-wise transformation $g_f(i,j)=q^{i-j}$, enabling the generation of mass scales that are suppressed by several orders of magnitude. Under a uniform scenario where $m_i = m$, the coupling terms $a_{i,i+k}$ in the transformed matrix take the form $a_{i,i+k} = mq^k$.



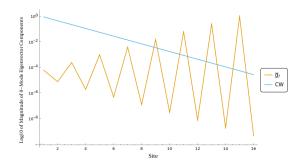


FIG. 1. The left plot displays the magnitude of the components of 0-modes generated under two different transformation scenarios: 1) $g_f(i,j) = q^{i-j}$, i.e., the clockwork(CW) scenario, and 2) $\bar{g}_f(i,j) = q^{(-1)^j j}$, in matrix eq. (31), for n = 15 with q = 2.0. The qualitative difference in the localization of the 0-mode between these two scenarios is clearly evident. The right plot shows the same data, but with the Y-axis presented on a log₁₀ scale.

Deconstruction Models: Zero mode localisation and mass spectrum

Beyond the clockwork mechanism and its variants, the proposed transformation can also be applied within the framework of deconstructed extra dimensions to localize the wavefunction profile for the Kaluza–Klein (KK) modes of bulk fields on branes.

An interesting feature of localizing the 0-modes in the model by these transformations is that the chiral left and right massless modes can have independent localization profiles. This can be easily seen from the fact that for a Dirac mass matrix M_{Dirac} in \bar{L} and R chiral basis, the right 0-modes are null space of M_{Dirac} whereas left modes are obtained as eigenbasis of matrix $M_{Dirac}M_{Dirac}^{\dagger}$. Hence, the right massless chiral modes are null vectors of M_{Dirac} , whereas the left chiral mass modes are of M_{Dirac}^{\dagger} . Now, from Corollary 1, for a matrix A with null vector v^i , the null vector components of transformed matrix B are given by $\vec{0}_{j,R} = v_i^i g_f''(j)$ with $g_f(i,j) = g_f'(i)g_f''(j)$ i.e, they entirely depend on the part of the separable function that varies with column. So for the left massless chiral mode, the null space will be formed by null vectors of B^{\dagger} since

$$B^{\dagger}w_{i} = \vec{0} \implies BB^{\dagger}w_{i} = \vec{0}$$

now the matrix B^{\dagger} is created from transformation of A^{\dagger} by the element-wise transformation $g_f(i,j)$ but the i^{th} row and j^{th} column element of matrix A^{\dagger} are complex conjugate of j^{th} row and i^{th} column element of A. As we have seen in above corollary only the column indices dependent part $g''_f(j)$ of the total separable function $g_f(i,j)$ determines the profile of the null vectors, the profile for 0-mode of M_{Dirac} will be determined by the row indices dependent part $g'_f(i)$ of the function $g_f(i,j)$ as $\vec{0}_{j,L} = v^i_{ij}g'_f(j)$. We can use this feature to come up with matrices whose left and right chiral modes behave completely non-trivially and independently from each other. Consider the following scenario of local theory space

$$\mathcal{H}_{i,j} = m\delta_{i,j} + m\delta_{i+1,j} + m\delta_{i-1,j}. \tag{37}$$

This will form a tridiagonal matrix, which will have a 0-mode when the number of chiral fields is

$$n \equiv 2 \pmod{3} \Rightarrow 0$$
-mode appears for $n = 2, 5, 8, 11, \dots$

The null vector of this matrix will be the same for left and right chiral modes since it is symmetric. The components of the null vector are given by

$$\vec{0}_n^k = \begin{cases} -1 & \text{if } k \text{ mod } 3 = 1, \\ 1 & \text{if } k \text{ mod } 3 = 2, \\ 0 & \text{if } k \text{ mod } 3 = 0. \end{cases}$$

$$\vec{0}_n^k = \frac{1}{N_0} \{0_n^1, 0_n^2, ..., 0_n^n\}$$

for $k \in \{1, 2, 3, ..., n\}$, with $n \in \mathbb{N}$ with \mathcal{N}_0 . As is evident, the profile of this null mode is not localized. We can again use the separable theorem to create the Dirac mass matrix M_D from the transformation of M_{Dirac} with $g_f(i,j) = g_f'(i)g_f''(j)$. The plot shows the profile of left and right chiral null modes for arbitrary $g'_f(i)$ and $g''_f(j) = a^{(-1)^j j}$. The transformed Hamiltonian is given by

$$\mathscr{H}'_{i,j} = \mathscr{H}_{i,j} g_f(i) a^{(-1)^j j} \tag{38}$$

The profile of right chiral 0-modes for the case where n = 2 + 3(2h), with $h \in \mathbb{N}$, follows a repeating pattern with elements

$$\vec{0}_n^k = \begin{cases} -a^{n-(-1)^{n-k}(k)} & \text{if } k \text{ mod } 3 = 2, \\ a^{n-(-1)^{n-k}(k)} & \text{if } k \text{ mod } 3 = 1, \\ 0 & \text{if } k \text{ mod } 3 = 0. \end{cases}$$

and for the scenario of n = 2 + 3(2h - 1) with $h \in \mathbb{N}$, the k^{th} component is given by

$$\vec{0}_n^k = \begin{cases} \frac{-1}{a^{n-(-1)^{n-k}k}} & \text{if } k \text{ mod } 3 = 2, \\ \frac{1}{a^{n-(-1)^{n-k}k}} & \text{if } k \text{ mod } 3 = 1, \\ 0 & \text{if } k \text{ mod } 3 = 0. \end{cases}$$

The plot shows the profile of left and right chiral null modes for $g_f'(i) = Sin(2ai)$ and $g_f''(j) = a^{(-1)^j j}$. The transformed Hamiltonian is given by

$$\mathscr{H}_{i,j}'' = \mathscr{H}_{i,j} Sin(2ai)a^{(-1)^{j}j}$$
(39)

The profile of left chiral 0-modes for the case where n = 2 + 3(h), with $h \in \mathbb{N}$, follows a repeating pattern with elements

$$\vec{0}_{n}^{k} = \begin{cases} \frac{-Sin(2na)}{Sin(2ka)} & \text{if } k \text{ mod } 3 = 2, \\ \frac{Sin(2na)}{Sin(2ka)} & \text{if } k \text{ mod } 3 = 1, \\ 0 & \text{if } k \text{ mod } 3 = 0. \end{cases}$$

for $k \in \{1, 2, 3, ..., n\}$, with $n \in \mathbb{N}$. As can be seen, the components of right chiral massless modes are independent of the row transformation function. Hence, starting from a local deconstruction model with symmetric left-right chiral field profiles, we can choose the row transformation function $g'_f(i)$ independently from the column transformation function $g''_f(j)$ according to the desired profile of the chiral modes. In the above example, consider a delocalized row transformation function $g'_f(i) = Sin(2ai)$, then the left chiral modes will be delocalized in contrast to right modes, as is shown in Fig. 2 for left and right chiral 0-mode.

Now note that from Corollary 2 we know that for any separable transformation satisfying $g_f(i,i) = 1 \ \forall i \in \{1,2,...,N\}$, the eigenvalues of the new matrix will be same as the older one though the singular value can change. Hence, using these different transformation effects on the fermions' mass spectrum. The Fig. 3 shows the mass spectrum for some chosen transformations as a demonstration case. For certain scenarios, these transformations $g_f(i,j)$ can be associated with a higher-dimensional metric $g_{\mu\nu}$ in the continuum limit. Such as for the extra fifth dimension with orbifold topology S^1/\mathbb{Z}_2 or circular topology S^1 , their dimensional deconstruction and the mass spectrum of fermions and gauge fields are known in the literature. 11

The Lagrangian for a fermion in the five-dimensional bulk is given by: 11,18

$$\mathcal{L}_{5}(x^{\mu}, x^{5}) = \bar{\Psi}\left(i\gamma^{\mu}D_{\mu} - \gamma^{5}D_{5}\right)\Psi - \frac{1}{4}\operatorname{Tr}\left(F_{MN}F^{MN}\right) \tag{40}$$

with $\mu = 0,1,2,3$ and M,N = 0,1,2,3,5. Imposing Neumann boundary conditions on Ψ_L results in a massless left-handed fermionic zero mode localized on the brane, whereas Dirichlet conditions on Ψ_R lift the masses of all right-handed modes. After compactification, Ψ_L expands naturally in a cosine basis, while Ψ_R admits a sine mode decomposition in the compact dimension. The masses of the fermion Kaluza-Klein (KK) modes are given by: 11

$$M_{L/R,k} = \frac{k\pi}{R} \tag{41}$$

where k is a positive integer and R is the compactification radius of the extra dimension. This extra dimension can be deconstructed as a theory space model, giving the same physics at low energy with different UV completions. In the theory space model of N + 1 groups G(k), there are N+1 fermions Ψ_k ($k=0,\ldots,N$) corresponding to each group; these fermions are charged under their respective symmetries. The model also has various link fields $\Phi_{n,m}$, which transform as $(\mathbf{g}, \bar{\mathbf{g}})$ under two arbitrary G(n) and G(m) symmetries. The various possible non-zero vevs of link fields $\Phi_{n,m}$ lead to different scenarios of theory space,

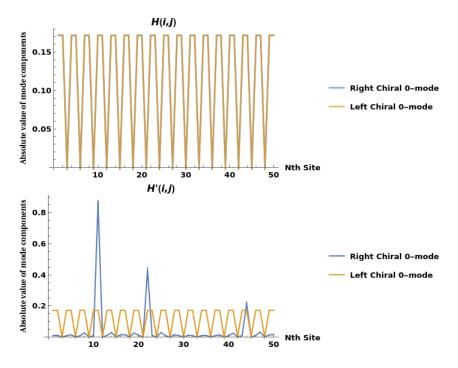


FIG. 2. The upper plot displays the magnitude of the profile of left and right chiral 0-modes for the deconstruction Hamiltonian $H_{i,j}$ and lower plot shows this for the transformed Hamiltonian $H''_{i,j}$ that is generated by the transformation $g_f(i,j) = Sin(2ai)a^{(-1)^{j}j}$, for n = 50 with a = 1.0. For $a \neq 1$, the left modes also start localising at a particular site. The profiles of chiral 0-modes can be controlled independently of each other under this transformation.

such as Aliphatic theory space, Disk theory space, etc. For the scenario of non-zero vevs for only consecutive symmetries, it provides the nearest neighbour couplings between the fermionic fields. The effective Lagrangian takes the form¹¹

$$\mathscr{L}_{fermion}(x^{\mu}) = \sum_{n=0}^{N} \left[\bar{\Psi}_{n,L/R} \not\!\!{D} \Psi_{n,L/R} + M_f \bar{\Psi}_{n,L} \left(\Phi_{n,n+1}^{\dagger} \Psi_{n+1,R} - \Psi_{n,R} \right) - \bar{\Psi}_{n,R} \left(\Psi_{n,L} - \Phi_{n,n-1} \Psi_{n-1,L} \right) \right]$$
(42)

where the explicit terms depend on the physics details of the model. The mass eigenvalues for this scenario are well known as

$$m_k = 2M_f \sin\left(\frac{k\pi}{2N}\right), \quad k = 1, 2, \dots, N-1$$

with M_f setting the mass scale of physics. Now, the transformation of this mass matrix with $g_f(i,j)$ will correspond to the link fields $\Phi_{i,j}$ taking different vevs as per the desired transformation. This also changes the mass spectrum of fermions. The analytical form for any arbitrary transformation is not known, but for some specific cases can be derived. For the transformation $g_f(i,j)$ such that $g_f(i,i) = g_f$ and $g_f(i,i+1) = g_f'$, the mass matrix will transform to:

$$M = M_f \begin{pmatrix} g_f & 0 & 0 & \cdots & 0 \\ -g'_f & g_f & 0 & \cdots & 0 \\ 0 & -g'_f & g_f & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & g_f \\ 0 & 0 & 0 & \cdots & -g'_f \end{pmatrix}$$

$$(43)$$

Diagonalisation of the matrix $M^{\dagger}M$ yields the mass-squared values of the fermions.

$$M^{\dagger}M = |M_{f}|^{2} \begin{pmatrix} g_{f}^{2} + g_{f}^{\prime 2} & -g_{f}g_{f}^{\prime} & 0 & \cdots & 0 \\ -g_{f}g_{f}^{\prime} & g_{f}^{2} + g_{f}^{\prime 2} & -g_{f}g_{f}^{\prime} & \cdots & 0 \\ 0 & -g_{f}g_{f}^{\prime} & g_{f}^{2} + g_{f}^{\prime 2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & -g_{f}g_{f}^{\prime} \\ 0 & 0 & 0 & -g_{f}g_{f}^{\prime} & g_{f}^{2} + g_{f}^{\prime 2} \end{pmatrix}$$

$$(44)$$

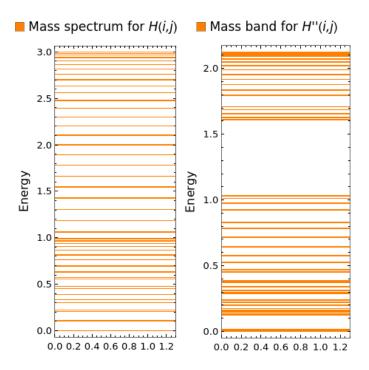


FIG. 3. The left plot displays the mass spectrum of the KK tower obtained by considering the deconstruction Hamiltonian as $H_{i,j}$, and the right plot shows the KK tower for the transformed Hamiltonian $H''_{i,j}$ as in eq. 39 with n=50 and a =1. This transformation significantly influences the KK tower spectrum and can be utilised as a tool to achieve the desired mass spectrum in the system. For a > 1, the states become dense for low mass values and get diluted for high mass eigenvalues.

The eigenvalues for this tridiagonal matrix can be found out as

$$m_k'^2 = M_f^2 \left(g_f^2 + g_f'^2 + 2g_f g_f' \cos\left(\frac{k\pi}{N}\right) \right)^2, \quad k = 1, 2, \dots, N-1$$
 (45)

In the large N limit, k << N, the original theory space gives linear mass gaps Δm_k for the consecutive KK towers similar to what was obtained in the extra dimension scenarios, but the transformed matrix scenario has increasing mass gaps $\Delta m'_k$ proportional to the mode number k, as is demonstrated in equations below:

$$m_k = 2M_f \sin\left(\frac{k\pi}{2N}\right) \approx M_f \frac{k\pi}{N}$$
 (46)

$$\Delta m_k = m_{k+1} - m_k \approx M_f \frac{(k+1)\pi}{N} - M_f \frac{k\pi}{N} = M_f \frac{\pi}{N}$$
(47)

$$m'_{k} = M_{f} \sqrt{g_{f}^{2} + g_{f}^{2} + 2g_{f}g_{f}^{\prime} \left(1 - \frac{k^{2}\pi^{2}}{2N^{2}}\right)} \approx M_{f}(g_{f} + g_{f}^{\prime}) - M_{f} \frac{g_{f}g_{f}^{\prime}}{g_{f} + g_{f}^{\prime}} \frac{k^{2}\pi^{2}}{2N^{2}}$$

$$(48)$$

$$\Delta m'_{k} = m'_{k-1} - m'_{k} \approx -M_{f} \frac{g_{f}g'_{f}}{g_{f} + g'_{f}} \left(\frac{(k-1)^{2}\pi^{2}}{2N^{2}} - \frac{k^{2}\pi^{2}}{2N^{2}} \right) = M_{f} \frac{g_{f}g'_{f}}{g_{f} + g'_{f}} \left(\frac{(2k-1)\pi^{2}}{2N^{2}} \right)$$
(49)

Now, since the right and left massless modes have independent profiles, we can easily modify only one part of the function to keep the profile of one chirality massless field fixed while modifying the Dirac mass spectrum of the fields. Note that while chiral massless modes have independent profiles, the profile for massive fields can depend on both row and column transformation parts of the function.

Certain configurations of these matrices, featuring localised zero modes, can serve as effective frameworks for tackling key hierarchy problems in physics, including the relative weakness of gravitational interactions and the Higgs naturalness problem.

Apart from high-energy physics (HEP), the matrices under consideration have versatile applications in graph theory and network analysis. In graph theory, a graph with nodes/vertices V and Edges E can be alternatively represented as a matrix. Hence, the transformed matrix will give a different graph but can also preserve some properties of the graph, depending on the transformation. Since the above-mentioned element-wise transformation does not convert any non-zero element to zero or vice-versa, the structure of the underlying graph is also preserved. This transformation only changes the weights assigned to the edges in such a way that it can produce localized 0-mode if the initial graph had a 0-mode or the other way around. Similarly, by reversing the process, one can produce a delocalized 0-mode too. As mentioned in 19, in a quantum system, a localized mode represents a bounded state. This bounded state is not due to the presence of a potential well but because of the underlying geometry. Hence, these localizing transformations can be used to create bound states in the system. The exact properties of the wave function will depend on whether the transformed matrix has exact duplication or partial duplication. These null-eigenvectors are also useful in continuous-time quantum walk (CTQW) models, which describe coherent transport on complex networks. Apart from these domains, null vectors also play various important roles in condensed matter physics, such as in Haldane's null vector criterion 20.

IV. CONCLUSION AND OUTLOOK

In this work, we have introduced and rigorously analysed a class of rank-preserving, index-dependent element-wise matrix transformations. Our central finding establishes that separability of the transformation function, $g_f(i,j)$, is a necessary and sufficient condition to maintain the rank-nullity structure of the original matrix. Furthermore, we identified specific constraints on $g_f(i,j)$ that ensure the invariance of the matrix eigenvalues, providing a powerful toolkit for manipulating matrix structure while preserving key spectral properties.

The significance of this framework is particularly evident in its application to high-energy physics. We have demonstrated that established mechanisms for generating mass hierarchies, such as the clockwork model, emerge as specific limiting cases within our more general transformation paradigm. This not only offers a new perspective on these models but also opens avenues for constructing a broader class of theories with controlled hierarchical structures. A crucial outcome of our analysis is the analytical demonstration of how left and right chiral null modes can be independently shaped by these transformations, offering unprecedented flexibility in model building, for instance, in scenarios involving deconstructed extra dimensions.

Moreover, by deriving explicit expressions for fermionic Kaluza-Klein mass towers under specific transformations, we have shown that this methodology can be directly employed to engineer desired mass spectra. This capability could be instrumental in addressing phenomenological challenges or exploring new theoretical possibilities in areas like neutrino physics or beyond the Standard Model scenarios.

Looking ahead, this work paves the way for several exciting research directions. Within HEP, investigating the continuum limit of these discrete transformations and their potential connection to metric engineering in higher-dimensional theories warrants further exploration. Beyond particle physics, the ability of these transformations to modify edge weights while preserving graph structure and systematically localize or delocalize zero modes suggests intriguing applications in graph theory, network analysis, and the study of continuous-time quantum walks on complex networks. The potential to engineer localized states, analogous to geometrically induced bound states in quantum systems¹⁹, or to explore connections with criteria like Haldane's null vector criterion²⁰ in condensed matter, also merits further investigation. Finally, a systematic classification of transformation functions $g_f(i,j)$ and their corresponding physical interpretations remains an open and promising avenue for future study, potentially uncovering new mechanisms for generating scale separation or specific phenomenological signatures across various disciplines.

V. ACKNOWLEDGEMENTS

AS thanks CSIR, Govt. of India for SRF fellowship No. 09/0079(15487)/2022-EMR-I. The author also acknowledges the open-source software tools and community resources, such as Python, and the various scientific computing libraries. All code used to perform the analysis and generate the results presented in this manuscript is publicly available at: https://github.com/AadarshSinghO/Nullity-Preserving-Hadmard-Product.

DATA AVAILABILITY AND CONFLICT OF INTEREST STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study. The author has no conflicts to disclose.

Appendix A: Detailed Examples

1. Example: Illustration of Theorem

In the following cases, we are considering a few scenarios to check Theorem 1.

Case 1 - $g_f(i, j)$ = constant.

In this scenario, both the conditions of $\frac{g_f(i,k)}{g_f(j,k)}$ and $\frac{g_f(i,k)}{g_f(i,j)}$ being independent of k^{th} column and i^{th} row is satisfied. Hence, we expect the nullity to be preserved. The matrix B obtained in this scenario will be a constant times the matrix A. It is trivial to show

$$Null(A) = Null(cA)$$
 $c \neq 0$

e.g., For

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \qquad B = c \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

Null(A) = 2 = Null(B).

Case 2 - $g_f(i, j) = g_f(i)$ i.e., the function depends only on the row indices.

In this scenario, the condition $\frac{g_f(i,k)}{g_f(j,k)}$ being independent of k^{th} column is always satisfied for any general function $g_f(i)$. Hence again we expect the nullity to be preserved. The matrix B obtained in this scenario will have its rows as rows of matrix A multiplied by $g_f(i)$ for i^{th} row.

e.g., For
$$g_f(i) = \frac{1}{f+i^2}$$
,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{f+1} & \frac{2}{f+1} & \frac{3}{f+1} \\ \frac{2}{f+4} & \frac{4}{f+4} & \frac{6}{f+4} \\ \frac{3}{f+9} & \frac{6}{f+9} & \frac{9}{f+9} \end{pmatrix}$$

Null(A) = 2 = Null(B).

Case 3 - $g_f(i, j) = g_f(j)$ i.e., the function depends only on the column indices.

In this scenario, the condition $\frac{g_f(i,k)}{g_f(i,j)}$ being independent of i^{th} row is always satisfied for any general function $g_f(j)$. Hence again we expect the nullity to be preserved. The matrix B obtained in this scenario will have its columns as columns of matrix A multiplied by $g_f(j)$ for j^{th} column.

e.g., For
$$g_f(j) = \frac{1}{\sqrt{f+j^2}}$$
,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{\sqrt{f+1}} & \frac{2}{\sqrt{f+4}} & \frac{3}{\sqrt{f+9}} \\ \frac{2}{\sqrt{f+1}} & \frac{4}{\sqrt{f+4}} & \frac{6}{\sqrt{f+9}} \\ \frac{3}{\sqrt{f+1}} & \frac{6}{\sqrt{f+4}} & \frac{9}{\sqrt{f+9}} \end{pmatrix}$$

Null(A) = 2 = Null(B).

Case 4 - $g_f(i,j) = g_f(i-j)$ i.e., the function depends on the difference between row and column indices.

In this scenario, the condition $\frac{g_f(i,k)}{g_f(j,k)}$ or $\frac{g_f(i,k)}{g_f(i,j)}$ being independent of k^{th} column and i^{th} row is not satisfied for any general function $g_f(i-j)$ such as for

$$g_f(i-j) = f+i-j$$

$$\frac{g_f(i-j)}{g_f(k-j)} = \frac{f+i-j}{f+k-j} \quad or \quad \frac{g_f(i-j)}{g_f(i-k)} = \frac{f+i-j}{f+i-k}$$

being independent of j^{th} column or i^{th} row respectively is not true. e.g., For $g_f(i-j) = f + i - j$,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} f & 2(f-1) & 3(f-2) \\ 2(f+1) & 4f & 6(f-1) \\ 3(f+2) & 6(f+1) & 9f \end{pmatrix}$$

 $Null(A) = 2 \neq Null(B) = 1$. Nullity is not preserved. But for special function $g_f(i-j)$ such as

$$g_f(i-j) = f^{i-j}$$

$$\frac{g_f(i-j)}{g_f(k-j)} = f^{i-k}$$
 or $\frac{g_f(i-j)}{g_f(i-k)} = f^{k-j}$

being independent of j^{th} column or i^{th} row respectively is true. e.g., For $g_f(i-j) = f^{i-j}$,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & \frac{2}{f} & \frac{3}{f^2} \\ 2f & 4 & \frac{6}{f} \\ 3f^2 & 6f & 9 \end{pmatrix}$$

Null(A) = 2 = Null(B). Nullity is preserved.

2. Example: Illustration of Corollaries

The following examples are considered to check the corollaries. The matrix for these cases is explicitly written in the above example section.

Case 1 - $g_f(i, j)$ = constant.

Null vectors for matrix A and matrix B are

$$\Lambda_A = \begin{pmatrix} -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \qquad \Lambda_B = \begin{pmatrix} -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$
 (A1)

Case 2 - $g_f(i,j) = g_f(i) = \frac{1}{f+i^2}$

Null vectors for matrix A and matrix B are

$$\Lambda_A = \begin{pmatrix} -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \qquad \Lambda_B = \begin{pmatrix} -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$
 (A2)

Case 3 - $g_f(i, j) = g_f(j) = \frac{1}{\sqrt{f + j^2}}$

Null vectors for matrix A and matrix B are

$$\Lambda_A = \begin{pmatrix} -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \qquad \Lambda_B = \begin{pmatrix} -\frac{3\sqrt{f+1}}{\sqrt{f+9}} & 0 & 1 \\ -\frac{2\sqrt{f+1}}{\sqrt{f+4}} & 1 & 0 \end{pmatrix}$$
 (A3)

Case 4 - $g_f(i, j) = g_f(i - j) = f^{i-j}$

Null vectors for matrix A and matrix B are

$$\Lambda_A = \begin{pmatrix} -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \qquad \Lambda_B = \begin{pmatrix} -\frac{3}{f^2} & 0 & 1 \\ -\frac{2}{f} & 1 & 0 \end{pmatrix}$$
 (A4)

All of these examples are in agreement with the null eigenvector corollary 1.

Case 5 - Consider $g_f(i,j) = f^{(i-j)}$, then clearly it satisfies condition of corollary 2 i.e.,

$$g_f(k,k) = f^{k-k} = 1 \quad \forall k$$

For matrix A, matrix B from the element-wise transformation is given by

$$A = \begin{pmatrix} a & b & c \\ d & e & h \\ k & l & m \end{pmatrix} \qquad B = \begin{pmatrix} a & bf & cf^2 \\ \frac{d}{f} & e & fh \\ \frac{k}{f^2} & \frac{l}{f} & m \end{pmatrix}$$

Alternatively, B can be obtained from the similarity condition matrix P given by

$$P = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f^2 \end{array}\right)$$

P matrix for general case is given by $P_{i,j} = \delta_i^j \times g_f(i,1)$.

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