LOCAL PARITY AND SYSTEMATIC PETERSON COUNTEREXAMPLES IN THE MOTIVIC HIT PROBLEM

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Abstract. The motivic hit problem asks for a minimal set of module generators of $H^{*,*}(BV_n;\mathbb{F}_2)$ over the mod 2 motivic Steenrod algebra. Kameko proved that the motivic Peterson-type analogue of Wood's theorem fails by constructing monomials z_k which are not hit even when the corresponding topological degree may satisfy $\beta(d) > n$. His proof passes to $N_n = M_n/(\tau)$ and analyzes, in degree $d = k + 2d_1$ with $d_1 = (n-1)(2^k - 1)$, a distinguished summand whose basis consists of the monotone translates of z_k. In this work, we isolate the local content of this summand before quotienting by hit elements. More precisely, we construct a linear projection

$$\vartheta : N_n^{d,*} \longrightarrow V,$$

where V is the M_1–summand spanned by the images of the monomials $\sigma(z_k)$, and define a parity functional $\varepsilon : V \to \mathbb{F}_2$ by summing the coefficients of these basis vectors. We prove that the local image of the hit subspace is exactly the parity-zero hyperplane:

$$\vartheta\big(A_+^{\sharp}(N_n) \cap N_n^{d,*}\big) = \ker(\varepsilon).$$

Consequently, every element whose local M_1–component has odd parity is non-hit, and every odd-parity linear combination of the monotone translates of z_k determines a nonzero class in the motivic hit quotient. We also obtain a systematic arithmetic family. For every integer $m \geq 3$, set $n = 2^r + 1$ and $k = n - m$. If

$$r \geq m + \alpha(m-3),$$

then the degree $d = (n-1)(2^{k+1} - 2) + k$ satisfies $\beta(d) > n$. Hence, for every fixed $m \geq 3$, these classes give infinitely many motivic Peterson-type counterexamples with $k = n - m$. The local parity theorem holds over every algebraically closed field of characteristic different from 2, and naturality under extension of the base field carries its non-hit consequences to every field of characteristic different from 2.

1. Introduction

Let $P_n = \mathbb{F}_2[x_1, \ldots, x_n]$ be the graded polynomial algebra with $|x_i| = 1$, regarded as a module over the mod 2 Steenrod algebra A. If $A^+ \subset A$ denotes the positive-degree part, the classical hit problem asks for a basis of

$$QP_n^d = P_n^d/(A^+(P_n) \cap P_n^d).$$

The fundamental vanishing theorem in this problem is Wood's proof [13] of the Peterson conjecture [7]. In the formulation used by Kameko, one defines

$$\beta(d) = \min\left\{s \in \mathbb{Z}_{\geq 0} \mid d = (2^{i_1} - 1) + \cdots + (2^{i_s} - 1),\ i_j \in \mathbb{Z}_{\geq 0}\right\}.$$

In [13], Wood proved that $\beta(d) > n$ implies $QP_n^d = 0$. This result is one of the basic inputs linking the hit problem with the cohomology of the Steenrod algebra and with Singer's algebraic transfer [8]. Later work has developed the hit problem in several directions; see, for example, Ault [1], Pengelley–Williams [6], and the two monographs of Walker–Wood [11, 12].

The motivic analogue begins with Voevodsky's construction [9] of Steenrod operations in motivic cohomology. Over the complex numbers, Kameko [5] considered

$$M_n(\mathbb{C}) = H^{*,*}(BV_n/\mathbb{C};\mathbb{F}_2)$$

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and its module structure over the mod 2 motivic Steenrod algebra. In this setting

$$M_n(\mathbb{C}) \cong \mathbb{F}_2[\tau, x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_1^2 + \tau y_1, \ldots, x_n^2 + \tau y_n),$$

where $|\tau| = (0,1)$, $|x_i| = (1,1)$, and $|y_i| = (2,1)$. The motivic Steenrod algebra is generated by the Bockstein Q_0 and by the reduced powers $\mathcal{P}^a$. Following Kameko, let $A_+^{*,*}$ denote its homogeneous positive-degree part, consisting of operations of bidegree (p,q) with $p+q>0$, and write $A_+^{*,*}(M_n)$ for the subspace generated by their values on M_n. The corresponding hit quotient is

$$QM_n^{d,*}(\mathbb{C}) = M_n(\mathbb{C})_{d,*}/\Big(A_+^{*,*}(M_n(\mathbb{C})) \cap M_n(\mathbb{C})_{d,*}\Big).$$

Kameko [5] formulated the motivic Peterson-type assertion

$$\beta(d) > n \quad \Longrightarrow \quad QM_n^{d,*}(\mathbb{C}) = 0$$

and proved that it is false.

For $1 \le k < n$, Kameko [5] defines the monomial

$$z_k = x_1 \cdots x_k \left(\prod_{j=1}^{k} y_j^{2^k - 2^{k-j} - 1}\right)\left(\prod_{j=k+1}^{n} y_j^{2^k - 1}\right),$$

whose topological degree is

$$d = (n-1)(2^{k+1}-2) + k.$$

He proves that $z_k \notin A_+^{*,*}(M_n(\mathbb{C}))$ for every $1 \le k < n$. Moreover, if $k = n-3$ and $\alpha(n-2) \ge 3$, where $\alpha(t)$ denotes the number of 1's in the binary expansion of t, then $\beta(d) > n$; hence $QM_n^{d,*}(\mathbb{C}) \neq 0$ despite the Peterson-type inequality. Kameko's proof passes to

$$N_n(\mathbb{C}) = M_n(\mathbb{C})/(\tau) \cong \Lambda(x_1, \ldots, x_n) \otimes \mathbb{F}_2[y_1, \ldots, y_n]$$

and to $A^\sharp = A^{*,*}/(\tau)$. In degree $d = k + 2d_1$, where $d_1 = (n-1)(2^k - 1)$, he constructs a basis for a top part of

$$\Lambda_n^k \otimes (Y_n/G_n)^{2d_1}$$

which splits into two blocks M_0 and M_1. The block M_1 is precisely the set of monotone translates of z_k. The image of Q_0 in this block is controlled by pairwise sums of those translates. The present paper makes this local statement exact by defining a projection on the raw degree–d component of N_n, before dividing by the hit subspace.

The main construction is a linear map

$$\vartheta : N_n^{d,*} \longrightarrow V,$$

where V denotes the $\mathbb{F}_2$–span of the images of the M_1–basis elements, together with a parity functional $\varepsilon : V \to \mathbb{F}_2$. The central theorem is

$$\vartheta\Big(A_+^\sharp(N_n) \cap N_n^{d,*}\Big) = \ker(\varepsilon).$$

This gives the exact local quotient

$$V/\vartheta\Big(A_+^\sharp(N_n) \cap N_n^{d,*}\Big) \cong \mathbb{F}_2.$$

It is important that this is a statement about the local top-layer image of hit elements. It is not the assertion that the even-parity vectors in V are exactly those vectors which are themselves hit inside $N_n^{d,*}$. Rather, an even-parity vector is precisely a possible local M_1–component of a hit element, whereas an odd-parity vector cannot occur as such a component.

The obstruction is intrinsic to the reduction modulo τ. In N_n the classes x_i are exterior and Q_0 lowers exterior degree by one, so the top-layer projection separates the pairwise Q_0–relations from all reduced-power contributions. Under classical realization one instead sets $\tau = 1$ and $y_i = x_i^2$; the exterior grading and the local projection then disappear. Thus the parity obstruction does not descend to the classical hit quotient and does not conflict with Wood's theorem [13]. A precise comparison is given in Remark 3.10.

The arithmetic consequence is systematic. Fix $m \geq 3$ and put

$$n = 2^r + 1, \qquad k = n - m.$$

A direct binary calculation gives

$$\alpha(d+n) = r + n - m + 1 - \alpha(m-3).$$

It follows that $\beta(d) > n$ whenever

$$r \geq m + \alpha(m-3).$$

Combining this calculation with the local parity theorem gives, for every fixed $m \geq 3$, an infinite family of motivic Peterson-type counterexamples with $k = n - m$. The case $m = 3$ contains a subfamily of Kameko's original examples, while $m = 4$ recovers the family $n = 2^r + 1$, $k = n - 4$, $r \geq 5$.

The classical hit problem is also an input to Singer's algebraic transfer. Kameko explicitly identifies the construction of motivic counterparts of the transfer as a natural problem [5]. Gregersen and Rognes have since constructed motivic Singer constructions and proved that their evaluation maps are Ext–equivalences [3]. These results do not by themselves furnish an algebraic transfer whose source is the motivic hit quotient. Moreover, the individual odd-parity classes constructed here are not asserted to be $GL_n(\mathbb{F}_2)$–invariant. Section 3 therefore formulates a transfer question only after taking the relevant invariant subquotient. The motivic Adams-theoretic target is understood in the sense developed by Dugger and Isaksen [2].

The paper is organized as follows. Section 2 establishes the algebraic framework, reviewing Kameko's weight filtration and the top-layer monomial basis of the quotient N_n. Section 3 is devoted to the core structural results: we construct explicit Bockstein preimages for the Johnson graph, prove the exact parity theorem, and discuss its implications for both classical realization and the algebraic transfer. Finally, Section 4 extends the parity obstruction to general base fields via base-change naturality, and carries out the binary arithmetic to establish the infinite families of Peterson-type counterexamples for $k = n - m$.

2. Preliminaries

This section fixes notation and recalls the precise ingredients from Kameko's work which enter the local argument. The emphasis is on the quotient $N_n = M_n/(\tau)$, the subspace $G_n \subset Y_n$, and the top-layer basis $\pi(M_0 \cup M_1)$ in degree $d = k + 2d_1$.

2.1. The functions α and β.

For a non-negative integer t, let $\alpha(t)$ be the number of 1's in the binary expansion of t. Kameko proves the following equivalence, which is the form of Wood's numerical criterion used below.

Proposition 2.1 (Kameko [5])**.** *For positive integers d and n, one has*

$$\beta(d) > n \iff \alpha(d+n) > n.$$

2.2. The quotient N_n and the reduced power subalgebra.

Over $\mathbb{C}$, Kameko's presentation gives

$$M_n = M_n(\mathbb{C}) \cong \mathbb{F}_2[\tau, x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_1^2 + \tau y_1, \ldots, x_n^2 + \tau y_n).$$

We write

$$N_n = M_n/(\tau) = \Lambda_n \otimes Y_n, \qquad \Lambda_n = \Lambda(x_1, \ldots, x_n), \qquad Y_n = \mathbb{F}_2[y_1, \ldots, y_n].$$

The quotient motivic Steenrod algebra is denoted

$$A^\sharp = A^{*,*}/(\tau),$$

and its homogeneous positive-degree part is denoted by $A^\sharp_+$. It is generated by Q_0 and by the reduced powers $\mathcal{P}^a$, $a \geq 1$. In N_n one has

$$Q_0(x_i) = y_i, \qquad Q_0(y_i) = 0,$$

and

$$\mathcal{P}^a(x_i) = 0 \quad (a \geq 1), \qquad \mathcal{P}^1(y_i) = y_i^2, \qquad \mathcal{P}^a(y_i) = 0 \quad (a \geq 2).$$

After quotienting by (τ) the Cartan formula for the reduced powers has no additional τ–term. Consequently, for $\lambda \in \Lambda_n$ and $y \in Y_n$,

$$\mathcal{P}^a(\lambda y) = \lambda \, \mathcal{P}^a(y) \qquad (a \geq 1). \tag{2.1}$$

Let $P \subset A^\sharp$ be the subalgebra generated by the reduced powers, and let P^+ be its positive-degree part.

For a monomial

$$z = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} y_1^{e_1} \cdots y_n^{e_n} \in N_n,$$

where $\varepsilon_i \in \{0, 1\}$, write

$$\varepsilon_i + 2e_i = \sum_{j \geq 0} \alpha_{ij}(z) 2^j, \qquad \alpha_{ij}(z) \in \{0, 1\},$$

and set

$$\alpha_i(z) = \sum_{j \geq 0} \alpha_{ij}(z), \qquad \omega_j(z) = \sum_{i=1}^n \alpha_{ij}(z), \qquad u_j(z) = (\alpha_{1j}(z), \ldots, \alpha_{nj}(z)).$$

For fixed k, Kameko defines $G_n \subset Y_n$ to be the subspace spanned by $P^+(Y_n)$ and by those monomials $y \in Y_n$ satisfying

$$(\omega_1(y), \ldots, \omega_k(y)) < (n-1, \ldots, n-1)$$

in lexicographic order. Thus

$$P^+(Y_n) \subset G_n. \tag{2.2}$$

The map $\psi : Y_n \to P_n$, $y_i \mapsto x_i$, identifies the quotient $(Y_n/G_n)^{2b}$ with Kameko's corresponding classical quotient in degree b.

2.3. **Kameko's top-layer basis.** Fix integers $n \geq 2$ and $1 \leq k < n$. Put

$$d_1 = (n-1)(2^k - 1), \qquad d = k + 2d_1 = (n-1)(2^{k+1} - 2) + k.$$

Let

$$\pi : \Lambda_n^k \otimes Y_n^{2d_1} \longrightarrow \Lambda_n^k \otimes (Y_n/G_n)^{2d_1}$$

be the quotient map. Kameko constructs two sets of monomials $M_0, M_1 \subset \Lambda_n^k \otimes Y_n^{2d_1}$ such that

$$\pi(M_0 \cup M_1) \tag{2.3}$$

is a basis of $\Lambda_n^k \otimes (Y_n/G_n)^{2d_1}$. Moreover,

$$M_1 = \{\sigma(z_k) \mid \sigma \in \mathrm{Mono}(k)\},$$

where $\mathrm{Mono}(k)$ is the set of increasing maps $\{1, \ldots, k\} \to \{1, \ldots, n\}$. Explicitly, if $\sigma(1) < \cdots < \sigma(k)$, then

$$\sigma(z_k) = x_{\sigma(1)} \cdots x_{\sigma(k)} \left(\prod_{j=1}^k y_{\sigma(j)}^{2^k - 2^{k-j} - 1} \right) \left(\prod_{i \notin \mathrm{im}(\sigma)} y_i^{2^k - 1} \right).$$

We write

$$U_0 = \langle \pi(m) \mid m \in M_0 \rangle, \qquad V = \langle \pi(m) \mid m \in M_1 \rangle.$$

Then (2.3) gives the direct sum

$$\Lambda_n^k \otimes (Y_n/G_n)^{2d_1} = U_0 \oplus V. \tag{2.4}$$

We shall use the following form of Kameko's Bockstein calculation.

Proposition 2.2 (Kameko [5], Proposition 5.1). *Let $z \in N_n^{d-1,*}$ be a monomial with $\omega_0(z) = k+1$. Then*

$$\pi(Q_0 z) \in \Lambda_n^k \otimes (Y_n/G_n)^{2d_1}$$

is a linear combination of elements $\pi(m)$ with $m \in M_0$ and elements of the form

$$\pi(\sigma_1(z_k) + \sigma_2(z_k)), \qquad \sigma_1, \sigma_2 \in \mathrm{Mono}(k).$$

The reverse inclusion in the parity theorem requires explicit control of which pairwise sums occur. This is supplied by a direct construction in Lemma 3.4.

3. A parity functional on Kameko's local top layer

This section proves the structural result. The point is to keep the degree–d component of N_n before quotienting by hits, project only to Kameko's local M_1–summand, and then compute exactly which vectors in that summand can arise from hit elements.

3.1. The local projection and the parity functional.

Since

$$N_n^{d,*} = \bigoplus_{a+2b=d} \Lambda_n^a \otimes Y_n^{2b},$$

there is a canonical projection

$$\mathrm{pr}_k : N_n^{d,*} \longrightarrow \Lambda_n^k \otimes Y_n^{2d_1}.$$

Let

$$p_{M_1} : \Lambda_n^k \otimes (Y_n/G_n)^{2d_1} \longrightarrow V$$

be the projection with kernel U_0 under the direct sum decomposition (2.4).

Definition 3.1. The local M_1–projection is the linear map

$$\vartheta = p_{M_1} \circ \pi \circ \mathrm{pr}_k : N_n^{d,*} \longrightarrow V.$$

Choose an enumeration

$$\mathrm{Mono}(k) = \{\sigma_1, \dots, \sigma_N\}, \qquad N = \binom{n}{k},$$

and put $m_i = \sigma_i(z_k)$. Since $\{\pi(m_1), \dots, \pi(m_N)\}$ is a basis of V, every $v \in V$ is written uniquely in the form

$$v = \sum_{i=1}^{N} c_i \pi(m_i), \qquad c_i \in \mathbb{F}_2.$$

Definition 3.2. The parity functional is

$$\varepsilon : V \longrightarrow \mathbb{F}_2, \qquad \varepsilon\left(\sum_{i=1}^{N} c_i \pi(m_i)\right) = \sum_{i=1}^{N} c_i.$$

Thus ε records the parity of the number of M_1–basis vectors which occur with nonzero coefficient.

3.2. Even parity and pairwise sums.

Lemma 3.3. *Let W be an N–dimensional vector space over $\mathbb{F}_2$ with basis $e_1, \dots, e_N$, where $N \geq 2$. Let $E \subset W$ be the subspace spanned by all vectors $e_i + e_j$ with $i \neq j$. Then*

$$E = \left\{ \sum_{i=1}^{N} a_i e_i \mid \sum_{i=1}^{N} a_i = 0 \right\}.$$

Proof. Each generator $e_i + e_j$ has coordinate sum 0, so E is contained in the right-hand side. Conversely, suppose $w = \sum_{i=1}^{N} a_i e_i$ and $\sum_i a_i = 0$. Then $a_1 = \sum_{i=2}^{N} a_i$, and hence

$$w = a_1 e_1 + \sum_{i=2}^{N} a_i e_i = \sum_{i=2}^{N} a_i (e_1 + e_i),$$

because the ground field is $\mathbb{F}_2$. Therefore $w \in E$. □

3.3. **The local Q_0–image.** We identify the basis vector $\pi(\sigma_i(z_k)) \in V$ with $e_i \in W$. Equivalently, we may index the basis by k–subsets of $\{1, \dots, n\}$. For a k–subset $I = \{i_1 < \cdots < i_k\}$, let $\sigma_I \in \mathrm{Mono}(k)$ be given by $\sigma_I(j) = i_j$, and write $z_I = \sigma_I(z_k)$.

Lemma 3.4. *Let I and J be adjacent vertices of the Johnson graph $J(n,k)$, that is, $|I| = |J| = k$ and $|I \cap J| = k-1$. Then there exists a monomial $w_{I,J} \in N_n^{d-1,*}$ such that*

$$\vartheta(Q_0 w_{I,J}) = \pi(z_I) + \pi(z_J).$$

Proof. Put $C = I \cap J$, and write

$$I = C \cup \{a\}, \qquad J = C \cup \{b\}, \qquad L = I \cup J = C \cup \{a,b\}.$$

Choose a bijection

$$\phi : C \longrightarrow \{2, \dots, k\};$$

when $k = 1$, both sets are empty. Define binary digits $\alpha_{ij} = \alpha_{ij}(w_{I,J})$ by

$$\alpha_{i0} = \begin{cases} 1, & i \in L, \\ 0, & i \notin L, \end{cases} \qquad \alpha_{i1} = \begin{cases} 0, & i \in \{a,b\}, \\ 1, & i \notin \{a,b\}, \end{cases}$$

and, for $2 \le j \le k$, by

$$\alpha_{ij} = \begin{cases} 0, & i = \phi^{-1}(j), \\ 1, & i \ne \phi^{-1}(j). \end{cases}$$

Set $\alpha_{ij} = 0$ for $j > k$. These digits determine a unique monomial $w_{I,J}$. Indeed, if

$$h_i = \sum_{j=0}^{k} \alpha_{ij} 2^j,$$

then the exponent of x_i is α_{i0} and the exponent of y_i is $(h_i - \alpha_{i0})/2$, which is a non-negative integer.

Every index i has exactly one zero among $\alpha_{i0}, \dots, \alpha_{ik}$. For $i \notin L$ the zero occurs in position 0; for $i \in \{a,b\}$ it occurs in position 1; and for $i \in C$ it occurs in position $\phi(i)$. Hence

$$\alpha_i(w_{I,J}) = k \qquad (1 \le i \le n).$$

Moreover,

$$\omega_0(w_{I,J}) = k+1, \qquad \omega_1(w_{I,J}) = n-2, \qquad \omega_j(w_{I,J}) = n-1 \quad (2 \le j \le k).$$

It follows that

$$\begin{aligned} |w_{I,J}| &= (k+1) + 2(n-2) + (n-1)\sum_{j=2}^{k} 2^j \\ &= (k+1) + 2(n-2) + (n-1)(2^{k+1} - 4) \\ &= (n-1)(2^{k+1} - 2) + k - 1 = d - 1. \end{aligned}$$

The sum is empty when $k = 1$, in which case the same identity holds.

Write

$$Q_0 w_{I,J} = \sum_{t \in L} q_t,$$

where q_t is the summand obtained by applying Q_0 to the exterior factor x_t. In terms of the combined exponent h_t, this operation replaces h_t by $h_t + 1$ and leaves every h_i with $i \ne t$ unchanged. If $t \in C$, then the binary digits of h_t in positions 0 and 1 are both equal to 1. Adding 1 therefore changes the position–1 digit from 1 to 0. Consequently,

$$\omega_1(q_t) = n - 3 < n - 1.$$

The exterior factor contributes only to the position–0 digits, so this is also the first weight coordinate of the polynomial factor of q_t. That factor therefore belongs to G_n, and $\pi(q_t) = 0$.

Thus, after applying π, only q_a and q_b can remain; their exterior supports are $J = L \setminus \{a\}$ and $I = L \setminus \{b\}$, respectively.

Kameko's Proposition 5.3 [5] applies because $(\omega_0(w_{I,J}), \omega_1(w_{I,J})) = (k+1, n-2)$ and $\alpha_i(w_{I,J}) = k$ for every i. In the proof of that proposition, the two surviving M_1–terms are identified by deleting, in turn, the two indices at which the position–1 digit vanishes. Their exterior supports are therefore $J = L \setminus \{a\}$ and $I = L \setminus \{b\}$. Since V contains exactly one M_1–basis vector with each fixed exterior support, it follows that

$$\pi(Q_0 w_{I,J}) = \pi(z_I) + \pi(z_J).$$

Applying p_{M_1} gives the asserted identity. □

Proposition 3.5. *With notation as above,*

$$\vartheta(Q_0(N_n^{d-1,*})) = \ker(\varepsilon) \subset V.$$

Proof. Let $z \in N_n^{d-1,*}$ be a monomial. If z has a exterior variables, then $Q_0(z)$ is a sum of monomials with $a-1$ exterior variables. Thus $\mathrm{pr}_k(Q_0 z) = 0$ unless $a = k+1$, equivalently unless $\omega_0(z) = k+1$.

Assume therefore that $\omega_0(z) = k+1$. By Proposition 2.2, $\pi(Q_0 z)$ is a linear combination of elements from $\pi(M_0)$ and pairwise sums $\pi(z_I + z_J)$. After applying p_{M_1}, all M_0–terms vanish and the remaining terms lie in the span of the vectors $e_I + e_J$. Hence

$$\vartheta(Q_0(N_n^{d-1,*})) \subseteq \ker(\varepsilon)$$

by Lemma 3.3.

Conversely, Lemma 3.4 shows that every edge vector of the Johnson graph $J(n,k)$ lies in $\vartheta(Q_0(N_n^{d-1,*}))$. The graph $J(n,k)$ is connected for $1 \le k < n$: given two k–subsets, one replaces the elements of the first which are not in the second, one at a time, by elements of the second which are not in the first. Therefore, if I and J are arbitrary k–subsets and

$$I = I_0, I_1, \dots, I_s = J$$

is a path in $J(n,k)$, then over $\mathbb{F}_2$ one has

$$e_I + e_J = \sum_{t=0}^{s-1} (e_{I_t} + e_{I_{t+1}}).$$

Thus every pairwise sum belongs to $\vartheta(Q_0(N_n^{d-1,*}))$. By Lemma 3.3, the span of all pairwise sums is precisely $\ker(\varepsilon)$, proving the reverse inclusion. □

3.4. **Reduction from $A_+^\sharp$ to the Bockstein.** The following lemma is the point at which one must distinguish carefully between arbitrary positive Steenrod operations and the Bockstein contribution. The proof uses Kameko's observation [5] that reduced powers vanish after projection to Y_n/G_n, but it applies this observation only to the leftmost reduced power in an operation, which is the form needed for the full hit image.

Lemma 3.6. *In degree $d = k + 2d_1$ one has*

$$\vartheta\Big(A_+^\sharp(N_n) \cap N_n^{d,*}\Big) = \vartheta(Q_0(N_n^{d-1,*})).$$

Proof. First let $a > 0$ and let $u \in N_n$ be such that $\mathcal{P}^a u \in N_n^{d,*}$. It suffices by linearity to consider a monomial $u = \lambda y$ with $\lambda \in \Lambda_n$ and $y \in Y_n$. By (2.1),

$$\mathcal{P}^a(\lambda y) = \lambda \mathcal{P}^a(y).$$

Since $a > 0$, the element $\mathcal{P}^a(y)$ belongs to $P^+(Y_n)$, hence to G_n by (2.2). Consequently

$$\mathrm{pr}_k(\mathcal{P}^a u) \in \Lambda_n^k \otimes G_n^{2d_1},$$

and this subspace is killed by π. Therefore

$$\vartheta(\mathcal{P}^a u) = 0 \qquad (a > 0). \tag{3.1}$$

The algebra $A^\sharp$ is generated by Q_0 and the reduced powers $\mathcal{P}^a$. Hence every element of $A^\sharp_+(N_n)$ is a sum of terms of the form

$$g_1(g_2 \cdots g_s u),$$

where g_i is one of the generators $Q_0, \mathcal{P}^1, \mathcal{P}^2, \dots$ and $s \geq 1$. If $g_1 = Q_0$, this term lies in $Q_0(N_n)$. If $g_1 = \mathcal{P}^a$ with $a > 0$, then its image under ϑ is zero by (3.1). Therefore

$$\vartheta(A^\sharp_+(N_n) \cap N_n^{d,*}) \subseteq \vartheta(Q_0(N_n) \cap N_n^{d,*}).$$

Since Q_0 raises the topological degree by 1, the right-hand side is $\vartheta(Q_0(N_n^{d-1,*}))$. The opposite inclusion is immediate from $Q_0 \in A^\sharp_+$. □

Theorem 3.7. *Let $n \geq 2$, let $1 \leq k < n$, and let $d = k + 2d_1$ with $d_1 = (n-1)(2^k - 1)$. With V, ϑ, and ε as in Definitions 3.1 and 3.2, one has*

$$\vartheta\Big(A^\sharp_+(N_n) \cap N_n^{d,*}\Big) = \ker(\varepsilon). \tag{3.2}$$

Equivalently, the sequence

$$0 \longrightarrow \vartheta\Big(A^\sharp_+(N_n) \cap N_n^{d,*}\Big) \longrightarrow V \xrightarrow{\varepsilon} \mathbb{F}_2 \longrightarrow 0 \tag{3.3}$$

is exact, and hence

$$V/\vartheta\Big(A^\sharp_+(N_n) \cap N_n^{d,*}\Big) \cong \mathbb{F}_2.$$

If $u \in N_n^{d,*}$ satisfies $\varepsilon(\vartheta(u)) = 1$, then $u \notin A^\sharp_+(N_n)$. In particular, every odd-parity linear combination of the monomials $\sigma(z_k)$ is non-hit in N_n.

Proof. The equality (3.2) follows immediately from Lemma 3.6 and Proposition 3.5. Since ε is visibly nonzero, (3.3) is exact. If u were hit and $\varepsilon(\vartheta(u)) = 1$, then (3.2) would force $\vartheta(u) \in \ker(\varepsilon)$, a contradiction.

Finally, let

$$u_S = \sum_{\sigma \in S} \sigma(z_k)$$

for a non-empty subset $S \subset \mathrm{Mono}(k)$. Since the elements $\pi(\sigma(z_k))$ form the chosen basis of V, one has

$$\vartheta(u_S) = \sum_{\sigma \in S} \pi(\sigma(z_k)),$$

and hence $\varepsilon(\vartheta(u_S)) = |S| \pmod 2$. If $|S|$ is odd, the preceding paragraph shows that u_S is non-hit. □

Remark 3.8. Theorem 3.7 is an exact classification of the local M_1–components of hit elements. It should not be read as the assertion that an even-parity vector of V is itself hit as an element of $N_n^{d,*}$. The precise assertion is that even parity is equivalent to being the image under ϑ of some hit element, whereas odd parity is an obstruction to being hit.

Corollary 3.9. *Let $S \subset \mathrm{Mono}(k)$ have odd cardinality, and set*

$$u_S = \sum_{\sigma \in S} \sigma(z_k) \in N_n^{d,*}.$$

Then the class of u_S is nonzero in

$$QN_n^{d,*} = N_n^{d,*}/(A^\sharp_+(N_n) \cap N_n^{d,*}).$$

Proof. This is exactly the final assertion of Theorem 3.7, restated in the quotient. □

3.5. **Comparison with the classical Peterson hit problem.** The parity functional arises from a structure which is present after reduction modulo τ but is destroyed by classical realization. The following remark records the precise distinction.

Remark 3.10. In the quotient $N_n = M_n/(\tau)$ one has $x_i^2 = 0$, so that

$$N_n = \Lambda(x_1, \dots, x_n) \otimes \mathbb{F}_2[y_1, \dots, y_n].$$

The exterior degree is therefore well defined. The operation Q_0 sends x_i to y_i and lowers exterior degree by one, while every positive reduced power has zero image under ϑ because its polynomial factor lies in $P^+(Y_n) \subset G_n$. Kameko's calculation then shows that the surviving Q_0–terms in the M_1–summand are pairwise sums. The coefficient-sum functional ε vanishes on these sums, and the connectivity of the Johnson graph shows that this is the only linear obstruction.

Classical realization is the specialization

$$A^{*,*}/(\tau+1) \cong A, \qquad M_n/(\tau+1) \cong P_n, \qquad y_i \longmapsto x_i^2.$$

After this specialization the exterior generator x_i and the polynomial generator y_i are coupled by $y_i = x_i^2$. Hence the exterior-degree projection pr_k, and therefore ϑ and ε, do not factor through P_n. In particular, an odd-parity motivic non-hit class may realize to a classical hit element. When $\beta(d) > n$, Wood's theorem indeed forces every element of P_n^d to be hit. Thus the local parity theorem is compatible with, rather than contrary to, the classical Peterson theorem.

3.6. **A transfer question.** Singer's rank–n algebraic transfer has source the $GL_n(\mathbb{F}_2)$–coinvariants of the A–annihilated part of $H_*(BV_n; \mathbb{F}_2)$; the graded dual of this source is the invariant subspace $(QP_n)^{GL_n(\mathbb{F}_2)}$ of the classical hit quotient [8]. In the motivic setting, Gregersen and Rognes construct small and large motivic Singer constructions. They prove that the associated evaluation homomorphisms are Ext–equivalences [3]. This provides relevant Adams-theoretic structure, but it does not identify a transfer whose source is QM_n or QN_n. There is also an invariance issue. The splitting $U_0 \oplus V$ is defined using Kameko's order-sensitive monomial basis, and the projection ϑ is not asserted to be $GL_n(\mathbb{F}_2)$–equivariant. In particular, Theorem 3.7 does not assert that an individual odd-parity class is fixed by $GL_n(\mathbb{F}_2)$.

For this question, take the base field to be $\mathbb{C}$. Write $H^{*,*} = H^{*,*}(\operatorname{Spec}\mathbb{C}; \mathbb{F}_2)$, and let $\mathcal{A}_{\mathrm{mot}}$ denote the mod 2 motivic Steenrod algebra over $\mathbb{C}$.

Question 3.11. Can one construct a natural motivic algebraic transfer, defined on a suitable invariant subquotient of the motivic hit quotient, with target

$$\operatorname{Ext}^{n,*,*}_{\mathcal{A}_{\mathrm{mot}}}\left(H^{*,*}, H^{*,*}\right)?$$

Can such a transfer be made compatible with classical realization and with the motivic Singer constructions of Gregersen and Rognes? If so, does the $GL_n(\mathbb{F}_2)$–submodule generated by the odd-parity classes of Theorem 3.7 contain an invariant class with nonzero transfer image?

The formulation deliberately concerns the invariant submodule generated by the local classes, rather than assigning a transfer value to every odd-parity sum. Establishing invariance under $GL_n(\mathbb{F}_2)$ is logically prior to evaluating any algebraic transfer.

4. Base fields and systematic motivic Peterson counterexamples

This section first identifies base-field hypotheses under which the local parity argument applies without alteration. It then uses naturality under field extension to descend the resulting non-hit classes to arbitrary fields of characteristic different from 2. The final part proves the binary criterion for the general family $k = n - m$.

For every field K with $\mathrm{char}(K) \neq 2$, write

$$M_n(K) = H^{*,*}(BV_n/K; \mathbb{F}_2),$$

let $A_K^{*,*}$ be the mod 2 motivic Steenrod algebra over K, and let $A_{K,+}^{*,*}$ be its homogeneous positive-degree part. We use the notation

$$QM_n^{d,*}(K) = M_n(K)_{d,*}/\Big(A_{K,+}^{*,*}(M_n(K)) \cap M_n(K)_{d,*}\Big).$$

4.1. **Algebraically closed fields of characteristic different from 2.** Assume now that K is algebraically closed and $\mathrm{char}(K) \neq 2$. The norm-residue theorem [10] implies

$$H^{*,*}(\operatorname{Spec} K; \mathbb{F}_2) \cong \mathbb{F}_2[\tau],$$

and the class $\rho = [-1] \in H^{1,1}(\operatorname{Spec} K; \mathbb{F}_2)$ vanishes. Voevodsky's construction in characteristic zero and its positive-characteristic extension by Hoyois–Kelly–Østvær supply the mod 2 motivic Steenrod operations whenever the characteristic is different from 2 [4, 9]. In the present algebraically closed case, the standard presentation of the motivic cohomology of BV_n becomes

$$M_n(K) \cong \mathbb{F}_2[\tau, x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_1^2 + \tau y_1, \ldots, x_n^2 + \tau y_n),$$

with the same bidegrees and the same formulas for Q_0 and the reduced powers as over $\mathbb{C}$; compare the general $B\mu_2$ presentation in [3]. This includes the characteristic-zero description used by Kameko and Yagita [5, 14].

Let

$$q_\tau : M_n(K) \longrightarrow N_n(K) = M_n(K)/(\tau)$$

be the quotient map, and put

$$A_K^\sharp = A_K^{*,*}/(\tau).$$

We denote the homogeneous positive-degree part of $A_K^\sharp$ by $A_{K,+}^\sharp$, and write

$$QN_n^{d,*}(K) = N_n(K)^{d,*}/\Big(A_{K,+}^\sharp(N_n(K)) \cap N_n(K)^{d,*}\Big).$$

Lemma 4.1. *If $u \in M_n(K)_{d,*}$ is hit, then $q_\tau(u)$ is hit in $N_n(K)$, that is,*

$$q_\tau(u) \in A_{K,+}^\sharp(N_n(K)) \cap N_n(K)^{d,*}.$$

Consequently, if $q_\tau(u)$ is nonzero in $QN_n^{d,*}(K)$, then u is nonzero in $QM_n^{d,*}(K)$.

Proof. Write

$$u = \sum_i a_i(v_i)$$

with $a_i \in A_{K,+}^{*,*}$ and $v_i \in M_n(K)$. Passing modulo (τ) gives

$$q_\tau(u) = \sum_i \overline{a}_i\Big(q_\tau(v_i)\Big),$$

where $\overline{a}_i$ is the image of a_i in $A_K^\sharp$. Every nonzero $\overline{a}_i$ has positive degree, while the terms for which $\overline{a}_i = 0$ disappear. Hence $q_\tau(u)$ lies in $A_{K,+}^\sharp(N_n(K))$. The second assertion is the contrapositive. □

Proposition 4.2. *Let K be an algebraically closed field of characteristic different from 2. The local parity Theorem 3.7 and Corollary 3.9 hold over K. In particular, if $S \subset \mathrm{Mono}(k)$ has odd cardinality, then*

$$u_S = \sum_{\sigma \in S} \sigma(z_k) \in M_n(K)_{d,*}$$

represents a nonzero class in $QM_n^{d,*}(K)$.

Proof. The proof of Theorem 3.7 uses only the presentation

$$N_n(K) = \Lambda(x_1, \ldots, x_n) \otimes \mathbb{F}_2[y_1, \ldots, y_n],$$

the formulas for Q_0 and the reduced powers, the inclusion $P^+(Y_n) \subset G_n$, and Kameko's monomial basis calculation. All these data are identical over K and over $\mathbb{C}$. Hence the same proof gives

$$\vartheta_K\Big(A_{K,+}^\sharp(N_n(K)) \cap N_n(K)^{d,*}\Big) = \ker(\varepsilon_K).$$

Every odd-parity sum is therefore nonzero in $QN_n^{d,*}(K)$, and Lemma 4.1 lifts this non-vanishing to $QM_n^{d,*}(K)$. $\square$

4.2. **Descent of non-hit classes along field extensions.** The local decomposition need not retain the same form over a non-algebraically-closed field. The non-hit conclusion nevertheless descends because motivic Steenrod operations are natural under extension of the base field.

Lemma 4.3. *Let L/K be an extension of fields of characteristic different from 2, and let*

$$b_{L/K} : M_n(K) \longrightarrow M_n(L)$$

be the base-change homomorphism. If $u \in M_n(K)$ is hit over K, then $b_{L/K}(u)$ is hit over L.

Proof. The motivic Steenrod operations commute with pullback. Thus, from an expression

$$u = \sum_i a_i(v_i), \qquad a_i \in A^{*,*}_{K,+}, \quad v_i \in M_n(K),$$

one obtains

$$b_{L/K}(u) = \sum_i (a_i)_L\Big(b_{L/K}(v_i)\Big),$$

where $(a_i)_L$ is the operation induced over L. Each nonzero $(a_i)_L$ has positive degree. Hence the base-changed element is hit. $\square$

Proposition 4.4. *Let K be any field of characteristic different from 2. If $S \subset \mathrm{Mono}(k)$ has odd cardinality, then*

$$u_S = \sum_{\sigma \in S} \sigma(z_k) \in M_n(K)_{d,*}$$

represents a nonzero class in $QM_n^{d,*}(K)$.

Proof. Let $\overline{K}$ be an algebraic closure of K. Under base change from K to $\overline{K}$, the universal classes x_i and y_i map to the corresponding universal classes, and therefore u_S maps to the same odd-parity sum in $M_n(\overline{K})$. By Proposition 4.2, this base-changed element is not hit. If u_S were hit over K, Lemma 4.3 would make its image hit over $\overline{K}$, a contradiction. $\square$

Remark 4.5. Over a general field K of characteristic different from 2, the coefficient ring $H^{*,*}(\mathrm{Spec}\, K; \mathbb{F}_2)$ need not be $\mathbb{F}_2[\tau]$, and the class $\rho = [-1] \in H^{1,1}(\mathrm{Spec}\, K; \mathbb{F}_2)$ need not vanish. The standard presentation has relations

$$x_i^2 = \tau y_i + \rho x_i;$$

see [3]. After quotienting by (τ) one obtains $x_i^2 = \rho x_i$, rather than an exterior algebra unless $\rho = 0$. Moreover, even when $\rho = 0$, nontrivial coefficient classes remain over many fields. Through the norm-residue theorem, these classes are governed by Milnor K–theory modulo 2 and Galois cohomology [10]. Thus the full local parity classification is not asserted here over an arbitrary field, including a finite field. What does hold over every such field is the non-hit conclusion of Proposition 4.4, obtained by passage to an algebraic closure. Characteristic 2 is excluded because the present argument uses the mod 2 Steenrod algebra with 2 invertible on the base and the corresponding $B\mu_2$ presentation.

4.3. **Binary arithmetic for the family $k = n - m$.** We now determine exactly when the specialization $n = 2^r + 1$ forces $\beta(d) > n$ for a fixed distance $m = n - k$. The only arithmetic input is the following complement identity.

Lemma 4.6. *Let $N \geq 1$ and let X be an integer satisfying $1 \leq X < 2^N$. Then*

$$\alpha(2^N - X) = N - \alpha(X - 1).$$

Proof. Write $X-1$ using exactly N binary digits, allowing leading zeros. Since

$$2^N - X = (2^N - 1) - (X-1)$$

and $2^N - 1$ is represented by a string of N ones, the subtraction is digitwise and involves no borrowing: every binary digit of $X-1$ is replaced by its complement. Hence the number of ones in $2^N - X$ is N minus the number of ones in $X-1$. $\square$

Theorem 4.7. *Let $m \geq 3$ and let $r \geq 1$ satisfy $2^r + 1 > m$. Set*

$$n = 2^r + 1, \qquad k = n - m, \qquad d = (n-1)(2^{k+1} - 2) + k.$$

Then $1 \leq k < n$, and

$$\beta(d) > n \quad \iff \quad r \geq m + \alpha(m-3).$$

Proof. The inequality $2^r + 1 > m$ gives $k = n - m \geq 1$, while $m \geq 3$ gives $k < n$. By Proposition 2.1, it remains to determine exactly when $\alpha(d+n) > n$. Substituting $k = n - m$ gives

$$\begin{aligned} d + n &= (n-1)(2^{k+1} - 2) + k + n \\ &= (n-1)2^{n-m+1} - 2(n-1) + (n-m) + n \\ &= (n-1)2^{n-m+1} - (m-2). \end{aligned}$$

Because $n - 1 = 2^r$, this becomes

$$d + n = 2^{r+n-m+1} - (m-2).$$

Put

$$N = r + n - m + 1, \qquad X = m - 2.$$

The inequalities $m \geq 3$ and $n = 2^r + 1 > m$ give

$$1 \leq X = m - 2 < n - 2 = 2^r - 1 < 2^N.$$

Lemma 4.6 therefore yields

$$\begin{aligned} \alpha(d+n) &= N - \alpha(X-1) \\ &= r + n - m + 1 - \alpha(m-3). \end{aligned}$$

Consequently,

$$\begin{aligned} \alpha(d+n) > n &\iff r - m + 1 - \alpha(m-3) > 0 \\ &\iff r \geq m + \alpha(m-3), \end{aligned}$$

where the last equivalence uses the integrality of r. Combining this equivalence with Proposition 2.1 proves the assertion. $\square$

Theorem 4.8. *Let K be a field of characteristic different from 2. Let $m \geq 3$ and $r \geq m + \alpha(m-3)$, and set*

$$n = 2^r + 1, \qquad k = n - m, \qquad d = (n-1)(2^{k+1} - 2) + k.$$

Then $\beta(d) > n$ and $QM_n^{d,*}(K) \neq 0$. More precisely, for every nonempty subset $S \subset \mathrm{Mono}(k)$ of odd cardinality, the element

$$u_S = \sum_{\sigma \in S} \sigma(z_k) \in M_n(K)_{d,*}$$

represents a nonzero class in $QM_n^{d,*}(K)$.

Proof. The inequality $\beta(d) > n$ is Theorem 4.7. The non-vanishing of every odd-parity sum follows from Proposition 4.4. Taking S to be a singleton gives $QM_n^{d,*}(K) \neq 0$. $\square$

Corollary 4.9. *For every fixed integer $m \geq 3$ and every field K of characteristic different from 2, there are infinitely many pairs (n, d) satisfying*

$$\beta(d) > n \qquad \text{and} \qquad QM_n^{d,*}(K) \neq 0,$$

with $n = 2^r + 1$ and $k = n - m$.

Proof. Every integer $r \geq m+\alpha(m-3)$ gives such a pair by Theorem 4.8, and there are infinitely many such integers r. □

Remark 4.10. For $m = 3$, one has $\alpha(m-3) = \alpha(0) = 0$, so the condition is $r \geq 3$. Moreover, $n-2 = 2^r - 1$ has exactly r ones in its binary expansion. Thus this case recovers the subfamily $n = 2^r + 1$ of Kameko's original $k = n - 3$ family [5]. For $m = 4$, one has $\alpha(m-3) = 1$, and the condition is $r \geq 5$, recovering the family $k = n - 4$ established by the special calculation $d + n = 2^{r+n-3} - 2$. For every $m \geq 5$, Theorem 4.8 supplies an additional fixed-distance family.

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