THE ONE-PERIOD KYLE MODEL HAS ONE EQUILIBRIUM

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Let V and U be independent standard normal random variables. For each Borel-measurable function $\phi\colon \mathbb{R} \to \mathbb{R}$, let $P_\phi\colon \mathbb{R} \to \mathbb{R}$ be a Borel version of the inverse regression $y \mapsto \mathbb{E}[V \mid \phi(V) + U = y]$. We prove the rigidity result

$$\phi(v) \in \arg\max_{x\in\mathbb{R}} \mathbb{E}\Big[(v - P_\phi(x+U))x\Big] \qquad \text{for every } v \in \mathbb{R}$$

if and only if $\phi = \mathrm{id}_{\mathbb{R}}$, the identity function.

The rigidity result implies that the one-period Gaussian Kyle (1985) insider-trading model has a unique Borel-measurable equilibrium strategy, namely Kyle's affine strategy. Building on preliminary results of McLennan, Monteiro, and Tourky (2017), the proof establishes that equilibrium attains a sharp universal upper bound on this noise traders' loss and that any strategy attaining the bound must be affine almost everywhere.

CONTENTS

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1. Introduction. The one-period Gaussian model in Kyle [17] is a noisy signalling model of insider trading. An informed trader observes the asset value

$$V \sim \mathcal{N}(\mu, \tau^2), \qquad \tau > 0\,,$$

and submits demand $\phi(V)$, where $\phi\colon \mathbb{R} \to \mathbb{R}$ is a Borel-measurable *strategy*. Market makers observe total demand

$$Y_\phi = \phi(V) + U, \qquad U \perp V, \qquad U \sim \mathcal{N}(0, \sigma^2), \quad \sigma > 0\,,$$

where U is noise traders' demand, and set the price $P(Y_\phi)$, where $P\colon \mathbb{R} \to \mathbb{R}$ is a Borel-measurable *pricing rule*.

A pair (ϕ, P) is an *equilibrium* if the following two conditions hold:

1. *Profit maximisation.* For every $v \in \mathbb{R}$,

$$\phi(v) \in \arg\max_{x \in \mathbb{R}} \mathbb{E}[(v - P(x + U))x]\;.$$

2. *Market efficiency.*

$$P(Y_\phi) = \mathbb{E}[V \mid Y_\phi] \qquad \text{almost surely}\,.$$

The first condition is expected-profit maximisation by a risk-neutral informed trader; the second is competitive risk-neutral pricing by the inverse regression.

If an equilibrium strategy ϕ is affine, then (V, Y_ϕ) is jointly Gaussian. The equilibrium conditions and Gaussian regression yield *Kyle's affine equilibrium*:

$$\phi(v) = \frac{\sigma}{\tau}(v - \mu) \qquad \text{for every } v \in \mathbb{R}\,, \tag{1.1}$$

$$P(Y_\phi) = \mu + \frac{\tau}{2\sigma} Y_\phi \qquad \text{almost surely}\,. \tag{1.2}$$

This calculation does not rule out non-affine equilibria. We prove that Kyle's affine equilibrium is unique:[1]

THEOREM 1.1. *A pair (ϕ, P) is an equilibrium if and only if* (1.1) *and* (1.2) *hold.*

Empirical analogues of the coefficient $\lambda = \tau/(2\sigma)$ in (1.2), known as *Kyle's lambda*, are widely used to measure price impact and illiquidity [1, 5, 15]. Related measures are used to study adverse selection and whether prices reveal informed trading [13, 14]. Their structural interpretation in the Gaussian Kyle model requires a unique equilibrium prediction. If a non-affine equilibrium existed, it would induce a non-jointly-Gaussian distribution of the observables $\big(P(Y_\phi), Y_\phi\big)$, unlike Kyle's affine equilibrium. Hence, the distribution of observables would depend on equilibrium selection, potentially rendering the model incomplete or partially identified [12, 21]. Theorem 1.1 rules out this ambiguity: the model's prediction for Kyle's lambda is independent of equilibrium selection.

Related uniqueness results concern modified models. Rochet and Vila [19] allow general value distributions but let the insider observe noise demand before choosing demand. In continuous time, Back [2] establishes uniqueness of the pricing rule within a specified class; see Çetin [9]. With uniformly distributed asset value and noise demand and a broad class of trading penalties, Carré, Collin-Dufresne, and Gabriel [8] prove uniqueness, up to their

[1] Equilibrium pricing rules are identified as equivalence classes modulo Lebesgue-almost-everywhere equality. If profit maximisation is imposed only for Lebesgue-almost every v, the equilibrium strategy is likewise unique modulo Lebesgue-null sets; see Appendix B.

notion of indistinguishability, within a specified class of nondecreasing strategies. By contrast, Cho and El Karoui [11] exhibit a unique single-auction equilibrium with a strongly nonlinear pricing rule for a Bernoulli value distribution. The conclusion of Theorem 1.1 depends substantively on the Gaussian specification.

For the frictionless one-period Gaussian model, Boulatov, Kyle, and Livdan [3, 4] introduce a complex-analytic approach to uniqueness, which McLennan, Monteiro, and Tourky [18] develop further. They give an example showing that the first-order conditions alone do not suffice to establish uniqueness, and prove uniqueness when the equilibrium strategy agrees locally with a function admitting a unique analytic continuation. To our knowledge, uniqueness remained open for general Borel-measurable strategies. Our theorem resolves the question without modifying the model or imposing regularity beyond Borel measurability of the strategy and pricing rule.

Our proof has three stages, each combining an economic insight with a brief probabilistic argument.

i. *Market-efficiency bounds.* Market efficiency imposes sharp universal upper and lower bounds on the noise traders' total expected loss over all Borel-measurable insider strategies. Either bound is attained if and only if the strategy agrees Lebesgue-almost everywhere with the corresponding affine extremiser.
ii. *Accounting identity for monotone strategies.* Any nondecreasing strategy with nonnegative conditional expected profit has finite ex ante expected profit equal to the noise traders' total expected loss.
iii. *Saturation at equilibrium.* Pointwise profit maximisation makes the equilibrium strategy strictly increasing and conditional expected profit nonnegative, so the accounting identity applies. Their common value attains the sharp upper bound from (i).

Monotonicity, pointwise profit maximisation, and Gaussian regression then yield uniqueness. The proof uses conditional resampling, a classical Gaussian variance bound, supermodularity, convex analysis, and lattice arguments.

The remainder of the paper has the following organization. The next section records seven auxiliary lemmas, with their provenance noted before the relevant statements. The reader may proceed directly to Section 3, skipping these preliminaries. Section 3 defines noise-trader loss and informed trader profit and records their basic properties. Sections 4–6 then establish, respectively, the sharp universal bounds and their equality cases, the accounting identity, and saturation at equilibrium. Section 7 proves the rigidity theorem and Theorem 1.1.

2. Preliminaries.

This section fixes notation and records seven auxiliary lemmas used below. Appendix A gives proofs for completeness.

Let U and V be independent standard normal random variables. Fix a Borel-measurable function $\phi\colon \mathbb{R}\to\mathbb{R}$ and set

$$Y=\phi(V)+U\,.$$

We assume no further regularity or moment condition on ϕ; in particular, $\phi(V)$ need not be integrable. Throughout, we refer to a version of $y\mapsto \mathbb{E}[V\mid Y=y]$ as the *inverse regression*, and to the random variable $\mathbb{E}[V\mid Y]$ as the *posterior mean*. We denote by $\mathrm{id}_{\mathbb{R}}\colon \mathbb{R}\to\mathbb{R}$ the identity map.

2.1. *Densities of (V,Y) and Y.*

Let

$$\varrho(t):=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\qquad \text{for every } t\in\mathbb{R}\,,$$

denote the standard Gaussian density.

LEMMA 2.1. *The joint law of* (V,Y) *has density*

$$(v,y)\longmapsto \varrho(v)\varrho\bigl(y-\phi(v)\bigr)\,,$$

and the law of Y *has the everywhere positive density*

$$p_Y(y) := \int_{\mathbb{R}} \varrho(v)\varrho\bigl(y-\phi(v)\bigr)\,\mathrm{d}v \qquad \textit{for every } y\in\mathbb{R}\,.$$

2.2. *Continuous version of the inverse regression.* The inverse-regression and Gaussian-smoothing results in this subsection and the next were established in [18], where substantially stronger complex-analytic regularity is proved.

Define $P\colon \mathbb{R}\to\mathbb{R}$ by

$$P(y) := \frac{1}{p_Y(y)}\int_{\mathbb{R}} v\varrho(v)\varrho\bigl(y-\phi(v)\bigr)\,\mathrm{d}v, \qquad \text{for every } y\in\mathbb{R}\,.$$

The integral is absolutely convergent because $\varrho \le (2\pi)^{-1/2}$ and $V\in L^1$. Lemma 2.1 gives $p_Y(y)\in(0,\infty)$. Thus, P is well defined.

LEMMA 2.2. *The function* P *is continuous,*

$$P(Y) = \mathbb{E}[V\mid Y] \qquad \textit{almost surely}\,,$$

and $P(Y)\in L^2$.

For each $y\in\mathbb{R}$, define the probability measure

$$\pi_y(\mathrm{d}v) := \frac{\varrho(v)\varrho\bigl(y-\phi(v)\bigr)}{p_Y(y)}\,\mathrm{d}v\,.$$

By Lemma 2.1 and Bayes' formula, $y\mapsto\pi_y$ is a version of the conditional law of V given $Y=y$. Moreover,

$$P(y) = \int_{\mathbb{R}} v\,\pi_y(\mathrm{d}v), \qquad \text{for every } y\in\mathbb{R}\,.$$

2.3. *Gaussian smoothing of the inverse regression.* The following lemma is the principal regularity input used in the remainder of the paper.

Define the Gaussian smoothing of the continuous version by

$$F(x) := \mathbb{E}[P(x+U)], \qquad \text{for every } x\in\mathbb{R}\,. \tag{2.1}$$

LEMMA 2.3. *The expectation in* (2.1) *is finite for every* $x\in\mathbb{R}$*, and* $F\colon\mathbb{R}\to\mathbb{R}$ *is continuously differentiable. For every* $x\in\mathbb{R}$*,*

$$F(x) = \int_{\mathbb{R}} P(y)\varrho(y-x)\,\mathrm{d}y\,, \tag{2.2}$$

and

$$\begin{aligned} F'(x) &= \int_{\mathbb{R}} P(y)(y-x)\varrho(y-x)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} uP(x+u)\varrho(u)\,\mathrm{d}u\,. \end{aligned} \tag{2.3}$$

The integrals in (2.2) *and* (2.3) *are absolutely convergent. For every Borel version* $\widetilde{P}$ *of* $\mathbb{E}[V\mid Y]$ *and every* $x\in\mathbb{R}$*,*

$$\widetilde{P}(x+U)\in L^1, \qquad \mathbb{E}[\widetilde{P}(x+U)] = F(x)\,.$$

Moreover,

$$F(\phi(V)) = \mathbb{E}[\mathbb{E}[V\mid Y]\mid V] \qquad \textit{almost surely}\,.$$

2.4. *Consequences of pointwise maximisation.* The next lemma supplies the monotonicity of ϕ used in Proposition 5.1, together with the first-order and convex-analytic identities used in Proposition 6.1. Its first-order and monotonicity conclusions were established in [18]. We formulate them in terms of the value function W. The convex-analytic facts used in the proof are standard and may be found in Rockafellar [20].

Define

$$W(v) := \sup_{x\in\mathbb{R}} x\big(v - F(x)\big), \qquad \text{for every } v \in \mathbb{R}\,,$$

where the value $+\infty$ is allowed. The function W is nonnegative because $x = 0$ is admissible, and convex because it is a pointwise supremum of affine functions of v.

LEMMA 2.4. *If*

$$\phi(v) \in \arg\max_{x\in\mathbb{R}} x\big(v - F(x)\big) \qquad \textit{for every } v \in \mathbb{R}\,,$$

then W is real-valued and, for every $v \in \mathbb{R}$,

$$F(\phi(v)) + \phi(v)F'(\phi(v)) = v \tag{2.4}$$

$$W(v) = \phi(v)^2 F'(\phi(v))\,. \tag{2.5}$$

Moreover, W is locally absolutely continuous and

$$W'(v) = \phi(v) \qquad \textit{for Lebesgue-almost every } v \in \mathbb{R}\,. \tag{2.6}$$

The map ϕ is strictly increasing.

2.5. *Monotonicity of the inverse regression and its smoothing.* For a probability measure π on $\mathbb{R}$ and $f \in L^1(\pi)$, write

$$\mathbb{E}_\pi[f] := \int_{\mathbb{R}} f \, \mathrm{d}\pi\,.$$

For $f, g \in L^2(\pi)$, write

$$\mathrm{Cov}_\pi(f,g) := \mathbb{E}_\pi[fg] - \mathbb{E}_\pi[f]\,\mathbb{E}_\pi[g]\,.$$

Under the hypothesis of Lemma 2.4, the strategy ϕ is strictly increasing. The next lemma shows that, under the same hypothesis, the continuous inverse regression P and its Gaussian smoothing F are also strictly increasing; in particular, it supplies the strict positivity of F' used in Proposition 6.1. Its proof uses the score–covariance identity for exponential families [6].

LEMMA 2.5. *Under the hypothesis of Lemma 2.4, for every $y \in \mathbb{R}$, the functions $\mathrm{id}_{\mathbb{R}}$ and ϕ lie in $L^2(\pi_y)$. The function P is continuously differentiable, with*

$$P'(y) = \mathrm{Cov}_{\pi_y}(\mathrm{id}_{\mathbb{R}}, \phi) > 0, \qquad \textit{for every } y \in \mathbb{R}\,. \tag{2.7}$$

Consequently,

$$F'(x) > 0, \qquad \textit{for every } x \in \mathbb{R}\,. \tag{2.8}$$

Thus, P and F are strictly increasing.

2.6. *Conditional resampling.* The following lemma records the conditional-resampling and exchangeable-pair identities. It is a textbook application of the transfer theorem in Kallenberg [16, Theorem 8.17]. The construction yields an exchangeable pair in the sense of Chatterjee [10].

LEMMA 2.6. *Set $V^{(0)} := V$. On an extension of the underlying probability space, there exists a random variable $V^{(1)}$ such that $V^{(0)}$ and $V^{(1)}$ are conditionally independent and identically distributed given Y. Moreover, the following hold:*

(i) *the triples $(V^{(0)}, V^{(1)}, Y)$ and $(V^{(1)}, V^{(0)}, Y)$ have the same law;*
(ii) *the pairs $(V^{(1)}, Y)$ and (V, Y) have the same law;*
(iii) $\mathbb{E}[V^{(1)} \mid Y, V^{(0)}] = P(Y)$ *almost surely;*
(iv) $\mathbb{E}[V^{(1)} \mid V^{(0)}] = F\big(\phi(V^{(0)})\big)$ *almost surely.*

2.7. *A lower Gaussian variance bound.* The following classical inequality is the bounded Lipschitz case of the lower Gaussian variance bound of Cacoullos [7]. This is the only form needed in the proof of Proposition 6.1.

LEMMA 2.7 (Lower Gaussian variance bound). *Let V be a standard normal random variable, and let $f\colon \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz. Then*

$$\big(\mathbb{E}[f'(V)]\big)^2 \le \operatorname{Var}\big(f(V)\big)\,,$$

where f' denotes the almost-everywhere derivative of f.

3. Noise-trader loss and informed trader profit. We work in the normalised setting and restate the notation needed below. Let U and V be independent standard normal random variables, let $\phi\colon \mathbb{R} \to \mathbb{R}$ be Borel measurable, and set

$$Y = \phi(V) + U\,.$$

Writing ϱ for the standard normal density, define $P\colon \mathbb{R} \to \mathbb{R}$ by

$$P(y) := \frac{\displaystyle\int_{\mathbb{R}} v\varrho(v)\varrho\big(y - \phi(v)\big)\, \mathrm{d}v}{\displaystyle\int_{\mathbb{R}} \varrho(v)\varrho\big(y - \phi(v)\big)\, \mathrm{d}v}, \qquad y \in \mathbb{R}\,.$$

By Lemma 2.2, P is a continuous inverse regression associated with ϕ, and

$$P(Y) = \mathbb{E}[V \mid Y] \qquad \text{almost surely}\,.$$

Define its Gaussian smoothing $F\colon \mathbb{R} \to \mathbb{R}$ by

$$F(x) := \mathbb{E}[P(x + U)], \qquad x \in \mathbb{R}\,.$$

The properties of F used below are in Lemma 2.3. All remaining notation is as in Section 2.

3.1. *Noise-trader loss and covariance.* The *noise traders' expected loss* is

$$\Gamma := \mathbb{E}\big[U\big(P(Y) - V\big)\big]\,.$$

LEMMA 3.1. *The loss Γ is well defined and finite. For every Borel version $\widetilde{P}$ of the inverse regression $y \mapsto \mathbb{E}[V \mid Y = y]$,*

$$\Gamma = \mathbb{E}\Big[U\big(\widetilde{P}(Y) - V\big)\Big] = \mathbb{E}[U\widetilde{P}(Y)] = \operatorname{Cov}\big(U, \mathbb{E}[V \mid Y]\big) = \mathbb{E}[F'(\phi(V))]\,.$$

PROOF. By Lemma 2.2, $P(Y) \in L^2$. Since $U, V \in L^2$, the Cauchy–Schwarz inequality gives $UP(Y), UV \in L^1$, so Γ is well defined and finite. Since $\mathbb{E}[UV] = 0$ and $P(Y) = \mathbb{E}[V \mid Y]$ almost surely,

$$\Gamma = \mathbb{E}[UP(Y)] = \operatorname{Cov}\big(U, \mathbb{E}[V \mid Y]\big)\,.$$

Conditioning on V, using the independence of U and V, and applying (2.3) give

$$\mathbb{E}[UP(Y) \mid V] = \int_{\mathbb{R}} uP\big(\phi(V) + u\big)\varrho(u)\,\mathrm{d}u = F'(\phi(V)) \qquad \text{almost surely}\,.$$

Since $UP(Y)$ is integrable, so is $F'(\phi(V))$. Taking expectations gives

$$\Gamma = \mathbb{E}[F'(\phi(V))]\,.$$

Finally, if $\widetilde{P}$ is any Borel version of the inverse regression, then

$$\widetilde{P}(Y) = P(Y) \qquad \text{almost surely}\,.$$

Hence

$$U\big(\widetilde{P}(Y) - V\big) = U\big(P(Y) - V\big) \qquad \text{and} \qquad U\widetilde{P}(Y) = UP(Y) \qquad \text{almost surely}\,,$$

which proves the remaining equalities. Q.E.D.

3.2. *Informed-trader profit.* The *informed trader's conditional expected profit* is

$$\Pi(v) := \phi(v)\big(v - F(\phi(v))\big), \qquad \text{for every } v \in \mathbb{R}\,.$$

LEMMA 3.2. *The function $\Pi\colon \mathbb{R} \to \mathbb{R}$ is finite-valued and Borel measurable. For every Borel version $\widetilde{P}$ of the inverse regression and every $v \in \mathbb{R}$,*

$$\Pi(v) = \mathbb{E}\Big[\big(v - \widetilde{P}(\phi(v) + U)\big)\phi(v)\Big]\,.$$

PROOF. By Lemma 2.3, F is finite-valued and continuous. Since ϕ is Borel measurable, Π is finite-valued and Borel measurable.

For every Borel version $\widetilde{P}$ of the inverse regression, Lemma 2.3 gives, for every $v \in \mathbb{R}$,

$$\widetilde{P}(\phi(v) + U) \in L^1, \qquad \mathbb{E}[\widetilde{P}(\phi(v) + U)] = F(\phi(v))\,.$$

Therefore,

$$\mathbb{E}\Big[\big(v - \widetilde{P}(\phi(v) + U)\big)\phi(v)\Big] = \phi(v)\big(v - F(\phi(v))\big) = \Pi(v)\,.$$

Q.E.D.

The random variable $\Pi(V)$ need not be integrable, so the informed trader's ex ante expected profit $\mathbb{E}[\Pi(V)]$ need not exist.

4. Sharp universal bounds on noise-trader loss. Retain the notation of Section 3. The following proposition gives sharp universal bounds on noise-trader loss and characterises the equality cases.

PROPOSITION 4.1 (Covariance bound and equality cases). *The quantity Γ satisfies*

$$-\frac{1}{2} \le \Gamma \le \frac{1}{2}\,. \tag{4.1}$$

Moreover, $\Gamma = \pm\frac{1}{2}$ if and only if there exists $a \in \mathbb{R}$ such that

$$\phi(v) = \pm v + a \qquad \textit{for Lebesgue-almost every } v \in \mathbb{R}\,,$$

with the same choice of sign.

PROOF. Set $M = \mathbb{E}[V \mid Y]$. Conditional Jensen's inequality and the tower property give

$$\mathbb{E}[M^2] \le \mathbb{E}\big[\mathbb{E}[V^2 \mid Y]\big] = \mathbb{E}[V^2] = 1\,.$$

Thus, $U, M \in L^2$, and Cauchy–Schwarz gives $UM \in L^1$. By Lemma 3.1,

$$\Gamma = \operatorname{Cov}(U, M) = \mathbb{E}[UM]\,,$$

where the second equality uses $\mathbb{E}[U] = 0$.

Define

$$X := 2M - V\,.$$

Because $M, V \in L^2$, we have $X \in L^2$. For $s \in \{-1, 1\}$,

$$\begin{aligned} 0 &\le \mathbb{E}[(X - sU)^2] \\ &= \mathbb{E}[X^2] + \mathbb{E}[U^2] - 2s\mathbb{E}[XU] \\ &= 1 + \mathbb{E}[X^2] - 2s\mathbb{E}[XU]\,. \end{aligned}$$

By the preceding identity and the independence of U and V,

$$\mathbb{E}[XU] = 2\mathbb{E}[MU] - \mathbb{E}[VU] = 2\Gamma\,.$$

Therefore,

$$0 \le \mathbb{E}[(X - sU)^2] = 1 + \mathbb{E}[X^2] - 4s\Gamma\,.$$

We have

$$\mathbb{E}[X^2] = 4\mathbb{E}[M^2] - 4\mathbb{E}[VM] + \mathbb{E}[V^2]\,.$$

Because $V, M \in L^2$, Cauchy–Schwarz gives $VM \in L^1$. Moreover, because $M = \mathbb{E}[V \mid Y]$, the pull-out property gives

$$\mathbb{E}[VM \mid Y] = M\mathbb{E}[V \mid Y] = M^2 \qquad \text{almost surely}\,,$$

and therefore

$$\mathbb{E}[VM] = \mathbb{E}[M^2]\,.$$

Consequently,

$$\mathbb{E}[X^2] = \mathbb{E}[V^2] = 1\,.$$

Thus, for $s \in \{-1, 1\}$,

$$0 \le \mathbb{E}[(X - sU)^2] = 2 - 4s\Gamma\,.$$

Taking $s = 1$ and $s = -1$ proves (4.1).

Fix $s \in \{-1, 1\}$ and suppose that

$$\Gamma = \frac{s}{2}\,.$$

The preceding display gives

$$\mathbb{E}[(X - sU)^2] = 0\,,$$

so $X = sU$ almost surely. Because $X = 2M - V$ and $M = P(Y)$ almost surely by Lemma 2.2,

$$2P(Y) = V + sU \qquad \text{almost surely}\,.$$

Because $s^2 = 1$ and $Y = \phi(V) + U$, this implies

$$Y - 2sP(Y) = \phi(V) - sV \qquad \text{almost surely}\,.$$

Because (V, U) has an everywhere positive density, its law and Lebesgue measure on $\mathbb{R}^2$ have the same null sets. Consequently,

$$\phi(v) + u - 2sP\big(\phi(v) + u\big) = \phi(v) - sv$$

for Lebesgue-almost every $(v, u) \in \mathbb{R}^2$.

Let $\mathcal{L}^1$ and $\mathcal{L}^2$ denote Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Because ϕ and P are Borel measurable, the exceptional set

$$N := \big\{(v, u) \in \mathbb{R}^2 : \phi(v) + u - 2sP\big(\phi(v) + u\big) \neq \phi(v) - sv\big\}$$

is Borel measurable. The preceding almost-everywhere identity gives

$$\mathcal{L}^2(N) = 0\,.$$

For $v \in \mathbb{R}$, let $N_v := \{u \in \mathbb{R} : (v, u) \in N\}$ be the vertical section of N at v. Tonelli's theorem gives

$$0 = \mathcal{L}^2(N) = \int_{\mathbb{R}} \mathcal{L}^1(N_v)\, \mathrm{d}v\,.$$

Because the integrand is nonnegative, there exists a set $A \subseteq \mathbb{R}$ of full Lebesgue measure such that

$$\mathcal{L}^1(N_v) = 0 \qquad \text{for every } v \in A\,.$$

Fix $v \in A$ and define

$$C_v := \{y \in \mathbb{R} : y - 2sP(y) \neq \phi(v) - sv\}\,.$$

We have

$$C_v = \phi(v) + N_v := \{\phi(v) + u : u \in N_v\}\,.$$

Indeed, writing $y = \phi(v) + u$, the inequality defining C_v is equivalent to $(v, u) \in N$. Because Lebesgue measure is invariant under translations,

$$\mathcal{L}^1(C_v) = \mathcal{L}^1(N_v) = 0\,.$$

Consequently, for every $v \in A$,

$$y - 2sP(y) = \phi(v) - sv \qquad \text{for Lebesgue-almost every } y \in \mathbb{R}\,.$$

Let $v_1, v_2 \in A$. Both C_{v_1} and C_{v_2} are null, so their union is null. Choose

$$y \in \mathbb{R} \setminus \big(C_{v_1} \cup C_{v_2}\big)\,.$$

We have

$$\phi(v_1) - sv_1 = y - 2sP(y) = \phi(v_2) - sv_2\,.$$

Thus, $\phi(v) - sv$ is constant on A. Because A has full Lebesgue measure, there exists $a \in \mathbb{R}$ such that

$$\phi(v) = sv + a \qquad \text{for Lebesgue-almost every } v \in \mathbb{R}\,.$$

Conversely, suppose that, for some $s \in \{-1, 1\}$ and $a \in \mathbb{R}$,

$$\phi(v) = sv + a \qquad \text{for Lebesgue-almost every } v \in \mathbb{R}\,.$$

Because the law of V is absolutely continuous,

$$Y = sV + U + a \qquad \text{almost surely}\,.$$

The pair (V, Y) is jointly Gaussian, with

$$\mathbb{E}[Y] = a, \qquad \mathrm{Cov}(V, Y) = s, \qquad \mathrm{Var}(Y) = 2\,.$$

The Gaussian regression formula gives

$$M = \mathbb{E}[V \mid Y] = \frac{s}{2}(Y - a) = \frac{V + sU}{2} \qquad \text{almost surely}\,.$$

It follows that

$$\Gamma = \mathbb{E}[UM] = \frac{1}{2}\big(\mathbb{E}[UV] + s\mathbb{E}[U^2]\big) = \frac{s}{2}\,.$$

This proves the equality characterisation for both choices of sign. Q.E.D.

5. Accounting identity for monotone strategies. Retain the notation of Section 3. This section establishes an accounting identity between the noise traders' expected loss and the informed trader's ex ante expected profit.

The proof uses a conditionally independent replica of V to express 2Γ as the expectation of a symmetric supermodularity increment of the bilinear map $(x, v) \mapsto xv$. This increment is nonnegative when ϕ is nondecreasing. Lattice truncation and two applications of monotone convergence avoid any a priori integrability assumption on either $\phi(V)$ or $\Pi(V)$.

PROPOSITION 5.1 (Accounting identity). *If ϕ is nondecreasing and*

$$\Pi(V) \geq 0 \qquad \textit{almost surely}\,,$$

then $\Pi(V) \in L^1$ and

$$\Gamma = \mathbb{E}[\Pi(V)]\,.$$

PROOF. Let $V^{(0)} := V$, and let $V^{(1)}$ be the conditionally independent replica supplied by Lemma 2.6. For $i \in \{0, 1\}$, define

$$U^{(i)} := Y - \phi(V^{(i)})\,.$$

By definition, $U^{(0)} = U$. By Lemma 2.6 (ii),

$$(V^{(1)}, Y) \overset{d}{=} (V^{(0)}, Y)\,.$$

Therefore,

$$(V^{(1)}, U^{(1)}) \overset{d}{=} (V^{(0)}, U^{(0)})\,.$$

In particular, $V^{(1)}, U^{(1)} \in L^2$. Hence, by Cauchy–Schwarz, all four products

$$U^{(0)}V^{(1)}, \qquad U^{(1)}V^{(0)}, \qquad U^{(1)}V^{(1)}, \qquad U^{(0)}V^{(0)}$$

belong to L^1.

Part I. Symmetric representation of Γ. Because $U^{(0)} = U$ and $V^{(0)} = V$ are independent and centred,

$$\mathbb{E}[U^{(0)}V^{(0)}] = 0\,.$$

By Lemma 3.1,

$$\Gamma = \mathbb{E}[U^{(0)}P(Y)]\,.$$

By Lemma 2.6 (i),

$$\mathbb{E}[U^{(1)}V^{(0)}] = \mathbb{E}[U^{(0)}V^{(1)}]\,.$$

Moreover,

$$U^{(0)} = Y - \phi(V^{(0)})$$

is measurable with respect to the sigma-algebra generated by $(Y, V^{(0)})$. Because $U^{(0)}V^{(1)} \in L^1$, the tower and pull-out properties give

$$\begin{aligned}
\mathbb{E}[U^{(1)}V^{(0)}] &= \mathbb{E}[U^{(0)}V^{(1)}] \\
&= \mathbb{E}\Big[\mathbb{E}[U^{(0)}V^{(1)} \mid Y, V^{(0)}]\Big] \\
&= \mathbb{E}\Big[U^{(0)}\mathbb{E}[V^{(1)} \mid Y, V^{(0)}]\Big] \\
&= \mathbb{E}[U^{(0)}P(Y)] \\
&= \Gamma\,,
\end{aligned}$$

where the penultimate equality follows from Lemma 2.6 (iii).

The equality in law of $(V^{(1)}, U^{(1)})$ and $(V^{(0)}, U^{(0)})$ gives

$$\mathbb{E}[U^{(1)}V^{(1)}] = \mathbb{E}[U^{(0)}V^{(0)}] = 0\,.$$

Because

$$U^{(1)} - U^{(0)} = \phi(V^{(0)}) - \phi(V^{(1)})\,,$$

we obtain

$$\begin{aligned}
2\Gamma &= \mathbb{E}[U^{(1)}V^{(0)}] + \mathbb{E}[U^{(0)}V^{(1)}] - \mathbb{E}[U^{(1)}V^{(1)}] - \mathbb{E}[U^{(0)}V^{(0)}] \\
&= \mathbb{E}\Big[\big(U^{(1)} - U^{(0)}\big)\big(V^{(0)} - V^{(1)}\big)\Big] \\
&= \mathbb{E}\Big[\big(\phi(V^{(0)}) - \phi(V^{(1)})\big)\big(V^{(0)} - V^{(1)}\big)\Big]\,.
\end{aligned} \tag{5.1}$$

Part II. Truncation. For every integer $n \geq 1$, define

$$\phi_n(v) := (-n) \vee \big(\phi(v) \wedge n\big)\,.$$

By Lemma 2.6 (iv),

$$\mathbb{E}[V^{(1)} \mid V^{(0)}] = F\big(\phi(V^{(0)})\big) \qquad \text{almost surely}\,.$$

Because $V^{(1)} \in L^1$, it follows that

$$F\big(\phi(V^{(0)})\big) \in L^1\,.$$

Because ϕ_n is bounded, all terms in the following calculation are integrable. Expanding the product and using Lemma 2.6 (i) give

$$\frac{1}{2}\mathbb{E}\Big[\big(\phi_n(V^{(0)}) - \phi_n(V^{(1)})\big)\big(V^{(0)} - V^{(1)}\big)\Big]$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\phi_n(V^{(0)})\big(V^{(0)} - V^{(1)}\big)\Big] \\
&= \mathbb{E}\Big[\phi_n(V^{(0)})\left(V^{(0)} - \mathbb{E}[V^{(1)} \mid V^{(0)}]\right)\Big] \\
&= \mathbb{E}\Big[\phi_n(V^{(0)})\big(V^{(0)} - F(\phi(V^{(0)}))\big)\Big] \,.
\end{aligned}
\tag{5.2}
$$

Part III. First monotone convergence. By hypothesis, ϕ is nondecreasing. Because the truncation map is nondecreasing, each ϕ_n is also nondecreasing. Hence, for every $n \geq 1$,

$$
\begin{aligned}
&\big(\phi_n(V^{(0)}) - \phi_n(V^{(1)})\big)\big(V^{(0)} - V^{(1)}\big) \\
&\qquad = \big|\phi_n(V^{(0)}) - \phi_n(V^{(1)})\big|\,\big|V^{(0)} - V^{(1)}\big| \geq 0\,.
\end{aligned}
$$

For each fixed outcome,

$$\big|\phi_n(V^{(0)}) - \phi_n(V^{(1)})\big|$$

is the Lebesgue length of the intersection of $[-n, n]$ with the interval having endpoints $\phi(V^{(0)})$ and $\phi(V^{(1)})$. As n increases, the intervals $[-n, n]$ expand to $\mathbb{R}$. Therefore,

$$\big|\phi_n(V^{(0)}) - \phi_n(V^{(1)})\big| \uparrow \big|\phi(V^{(0)}) - \phi(V^{(1)})\big|$$

almost surely. Because $|V^{(0)} - V^{(1)}|$ is nonnegative and does not depend on n, it follows that

$$
\begin{aligned}
0 &\leq \big(\phi_n(V^{(0)}) - \phi_n(V^{(1)})\big)\big(V^{(0)} - V^{(1)}\big) \\
&\uparrow \big(\phi(V^{(0)}) - \phi(V^{(1)})\big)\big(V^{(0)} - V^{(1)}\big) \qquad \text{almost surely}\,.
\end{aligned}
$$

Hence the monotone convergence theorem and (5.1) give

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2}\mathbb{E}\Big[\big(\phi_n(V^{(0)}) - \phi_n(V^{(1)})\big)\big(V^{(0)} - V^{(1)}\big)\Big] \\
&\qquad = \frac{1}{2}\mathbb{E}\Big[\big(\phi(V^{(0)}) - \phi(V^{(1)})\big)\big(V^{(0)} - V^{(1)}\big)\Big] = \Gamma\,.
\end{aligned}
\tag{5.3}
$$

In particular, this limit is finite.

Part IV. Matching monotone convergence. Because $V^{(0)} = V$, the hypothesis gives

$$\Pi(V^{(0)}) \geq 0 \qquad \text{almost surely}\,.$$

By the definition of Π,

$$\Pi(V^{(0)}) = \phi(V^{(0)})\big(V^{(0)} - F(\phi(V^{(0)}))\big)\,.$$

On a full-probability event on which $\Pi(V^{(0)}) \geq 0$, consider the following three cases. If $\phi(V^{(0)}) > 0$, then

$$V^{(0)} - F(\phi(V^{(0)})) \geq 0$$

and

$$\phi_n(V^{(0)}) \uparrow \phi(V^{(0)})\,.$$

If $\phi(V^{(0)}) < 0$, then

$$V^{(0)} - F(\phi(V^{(0)})) \leq 0$$

and

$$\phi_n(V^{(0)}) \downarrow \phi(V^{(0)}) .$$

Multiplication by the nonpositive quantity $V^{(0)} - F(\phi(V^{(0)}))$ reverses the latter order. If $\phi(V^{(0)}) = 0$, then both $\phi_n(V^{(0)})$ and $\Pi(V^{(0)})$ are zero. Consequently,

$$0 \le \phi_n(V^{(0)})\big(V^{(0)} - F(\phi(V^{(0)}))\big) \uparrow \Pi(V^{(0)}) \qquad \text{almost surely} .$$

The monotone convergence theorem, with $\mathbb{E}[\Pi(V^{(0)})]$ initially understood as an extended expectation in $[0, \infty]$, therefore gives

$$\begin{aligned}
\mathbb{E}[\Pi(V)] &= \mathbb{E}[\Pi(V^{(0)})] \\
&= \lim_{n\to\infty} \mathbb{E}\Big[\phi_n(V^{(0)})\big(V^{(0)} - F(\phi(V^{(0)}))\big)\Big] \\
&= \frac{1}{2}\lim_{n\to\infty} \mathbb{E}\Big[\big(\phi_n(V^{(0)}) - \phi_n(V^{(1)})\big)\big(V^{(0)} - V^{(1)}\big)\Big] \\
&= \Gamma .
\end{aligned}$$

Here the first equality uses $V^{(0)} = V$, the second follows from the monotone convergence theorem, the third follows from (5.2), and the last follows from (5.3). Because the limit in Part III is finite and $\Pi(V) \ge 0$ almost surely,

$$\mathbb{E}[|\Pi(V)|] = \mathbb{E}[\Pi(V)] = \Gamma < \infty .$$

Thus, $\Pi(V) \in L^1$. Q.E.D.

6. Noise-trader loss saturation. Retain the notation of Section 3. The following proposition shows that, at equilibrium, the noise traders' expected loss attains its sharp upper bound. Throughout this section, P denotes the continuous inverse regression associated with ϕ. By the version clause of Lemma 2.3, (ϕ, P) is an equilibrium if and only if $(\phi, \widetilde{P})$ is an equilibrium for any other Borel version $\widetilde{P}$ of the inverse regression. By Lemma 3.1, the value of Γ is also independent of the chosen version.

PROPOSITION 6.1 (Covariance saturation). *If (ϕ, P) is an equilibrium in the normalised model, then*

$$\Gamma = \frac{1}{2} .$$

PROOF. By Lemma 2.3, for every $x \in \mathbb{R}$,

$$P(x+U) \in L^1 , \qquad \mathbb{E}[P(x+U)] = F(x) .$$

Hence, for every $v, x \in \mathbb{R}$,

$$\mathbb{E}\big[(v - P(x+U))x\big] = x\big(v - F(x)\big) .$$

Profit maximisation therefore gives

$$\phi(v) \in \arg\max_{x\in\mathbb{R}} x\big(v - F(x)\big) \qquad \text{for every } v \in \mathbb{R} .$$

Lemma 2.4 shows that ϕ is strictly increasing. It also shows that W is real-valued and locally absolutely continuous, with

$$W'(v) = \phi(v) \qquad \text{for Lebesgue-almost every } v \in \mathbb{R} ,$$

and

$$W(v) = \phi(v)^2 F'(\phi(v)) \qquad \text{for every } v \in \mathbb{R}\,.$$

By the definition of Π and pointwise maximisation,

$$\Pi(v) = \phi(v)\big(v - F(\phi(v))\big) = W(v) \qquad \text{for every } v \in \mathbb{R}\,.$$

Because ϕ is nondecreasing and

$$\Pi(v) = W(v) \geq 0 \qquad \text{for every } v \in \mathbb{R}\,,$$

Proposition 5.1 applies. It gives $\Pi(V) \in L^1$ and

$$\Gamma = \mathbb{E}[\Pi(V)] = \mathbb{E}[W(V)]\,.$$

In particular,

$$W(V) \in L^1 \qquad \text{and} \qquad \Gamma < \infty\,.$$

Lemma 2.5 gives

$$F'(x) > 0 \qquad \text{for every } x \in \mathbb{R}\,.$$

Define

$$q(v) := F'(\phi(v)) \qquad \text{for every } v \in \mathbb{R}\,.$$

Then $q\colon \mathbb{R} \to (0,\infty)$ is Borel measurable, and

$$W(v) = \phi(v)^2 q(v) \qquad \text{for every } v \in \mathbb{R}\,.$$

By Lemma 3.1,

$$q(V) \in L^1 \qquad \text{and} \qquad \Gamma = \mathbb{E}[q(V)]\,.$$

Because $q(V) > 0$ almost surely, $\Gamma > 0$.

Set

$$c := F(0)\,.$$

Because F is strictly increasing,

$$x\big(c - F(x)\big) < 0 \qquad \text{for every } x \neq 0\,,$$

whereas the expression vanishes at $x = 0$. Thus, 0 is the unique maximiser of

$$x \longmapsto x\big(c - F(x)\big)\,.$$

The pointwise maximisation condition at $v = c$ therefore gives

$$\phi(c) = 0\,.$$

Because ϕ is strictly increasing,

$$\phi(v) < 0 \qquad \text{for every } v < c\,,$$

and

$$\phi(v) > 0 \qquad \text{for every } v > c\,.$$

The identity $W = \phi^2 q$ gives

$$W(c) = 0$$

and

$$W(v) > 0 \qquad \text{for every } v \neq c\,.$$

Define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(v) := \begin{cases} -\sqrt{2W(v)}\,, & v < c\,, \\ 0\,, & v = c\,, \\ \sqrt{2W(v)}\,, & v > c\,. \end{cases}$$

Let J be a nonempty compact interval contained in either $(-\infty, c)$ or (c, ∞), and let $s \in \{-1, 1\}$ be the constant sign of ϕ on J. We have

$$W(v) = \phi(v)^2 q(v) > 0 \qquad \text{for every } v \in J\,.$$

Because W is absolutely continuous on J and $2W(J)$ is a compact subset of $(0,\infty)$, on which the square root is Lipschitz, the function $h = s\sqrt{2W}$ is absolutely continuous on J. Using $W' = \phi$ almost everywhere, the chain rule gives

$$\begin{aligned} h'(v) &= \frac{s\phi(v)}{\sqrt{2W(v)}} \\ &= \frac{|\phi(v)|}{\sqrt{2W(v)}} \\ &= \frac{1}{\sqrt{2q(v)}} \end{aligned}$$

for Lebesgue-almost every $v \in J$.

Define

$$r(v) := \frac{1}{\sqrt{2q(v)}} \qquad \text{for every } v \in \mathbb{R}\,.$$

Then r is Borel measurable and everywhere positive. We next show that h is locally absolutely continuous, including at c, and that r is a representative of its almost-everywhere derivative. Let $K = [a, b]$ be a compact interval containing c. Because ϕ is increasing and F' is continuous and everywhere positive,

$$m_K := \min_{x \in [\phi(a), \phi(b)]} F'(x) > 0\,.$$

Thus,

$$q(v) \geq m_K \qquad \text{and} \qquad r(v) \leq \frac{1}{\sqrt{2m_K}} \qquad \text{for every } v \in K\,.$$

The preceding absolute-continuity and derivative calculation show that h is $(2m_K)^{-1/2}$-Lipschitz on each compact subinterval of $K \setminus \{c\}$. Moreover, W is continuous and $W(c) = 0$, so h is continuous at c. Letting an endpoint tend to c, and then using the triangle inequality for points on opposite sides of c, shows that h is $(2m_K)^{-1/2}$-Lipschitz on K. Hence h is locally absolutely continuous on $\mathbb{R}$, and

$$h'(v) = r(v)$$

for Lebesgue-almost every $v \in \mathbb{R}$. Because $r > 0$, the function h is strictly increasing.

Moreover,

$$h(v)^2 = 2W(v) \qquad \text{for every } v \in \mathbb{R}\,,$$

and

$$\frac{1}{r(v)^2} = 2q(v) \qquad \text{for every } v \in \mathbb{R}\,.$$

Consequently,

$$\mathbb{E}[h(V)^2] = 2\mathbb{E}[W(V)] = 2\Gamma < \infty$$

and

$$\mathbb{E}\left[\frac{1}{r(V)^2}\right] = 2\mathbb{E}[q(V)] = 2\Gamma < \infty\,.$$

For every integer $N > |c|$, define

$$T_N(v) := (-N) \vee (v \wedge N), \qquad h_N := h \circ T_N\,.$$

The function h_N is bounded and Lipschitz, and

$$h_N'(v) = r(v)\mathbf{1}_{(-N,N)}(v)$$

for Lebesgue-almost every $v \in \mathbb{R}$. Lemma 2.7 therefore gives

$$\big(\mathbb{E}\big[r(V)\mathbf{1}_{\{|V|<N\}}\big]\big)^2 \leq \mathrm{Var}\big(h_N(V)\big)\,.$$

Because h is increasing and $h(c) = 0$, we have $|h_N(v)| \leq |h(v)|$ for every $v \in \mathbb{R}$ and every $N > |c|$. Thus, $h_N(V) \to h(V)$ in L^2, and hence $\mathrm{Var}(h_N(V)) \to \mathrm{Var}(h(V))$. Letting $N \to \infty$ and using monotone convergence on the left gives

$$\big(\mathbb{E}[r(V)]\big)^2 \leq \mathrm{Var}\big(h(V)\big)\,.$$

In particular, $r(V) \in L^1$.

Moreover, Cauchy–Schwarz and the preceding moment bound give

$$\mathbb{E}\left[\frac{1}{r(V)}\right] \leq \sqrt{\mathbb{E}\left[\frac{1}{r(V)^2}\right]} < \infty\,.$$

Combining the preceding variance bound with Cauchy–Schwarz now gives

$$\begin{aligned} 1 &= \left(\mathbb{E}\left[\sqrt{r(V)}\frac{1}{\sqrt{r(V)}}\right]\right)^2 \\ &\leq \mathbb{E}[r(V)]\,\mathbb{E}\left[\frac{1}{r(V)}\right] \\ &\leq \sqrt{\mathrm{Var}\big(h(V)\big)}\sqrt{\mathbb{E}\left[\frac{1}{r(V)^2}\right]} \\ &\leq \sqrt{\mathbb{E}[h(V)^2]\mathbb{E}\left[\frac{1}{r(V)^2}\right]} \\ &= 2\sqrt{\mathbb{E}[W(V)]\,\mathbb{E}[q(V)]} = 2\Gamma\,. \end{aligned}$$

Thus, $\Gamma \geq 1/2$. The reverse inequality follows from (4.1). Therefore,

$$\Gamma = \frac{1}{2}\,.$$

Q.E.D.

7. Rigidity and equilibrium uniqueness. The well-known Gaussian regression formula shows that the identity map satisfies the pointwise maximisation condition below. The content of the next theorem is the converse: among all Borel-measurable maps, this condition forces the identity.

THEOREM 7.1 (Rigidity). *Let V and U be independent standard normal random variables. For every Borel-measurable map $\phi\colon \mathbb{R}\to\mathbb{R}$, let $P_\phi\colon \mathbb{R}\to\mathbb{R}$ be a Borel version of the inverse regression*

$$P_\phi(y) = \mathbb{E}[V \mid \phi(V) + U = y]\,,$$

and let

$$F_\phi(x) := \mathbb{E}[P_\phi(x+U)]\,.$$

The pointwise maximisation condition

$$\phi(v) \in \arg\max_{x\in\mathbb{R}} \bigl(xv - xF_\phi(x)\bigr) \qquad \text{for every } v\in\mathbb{R}$$

holds if and only if $\phi = \mathrm{id}_{\mathbb{R}}$.

PROOF. Suppose first that the pointwise maximisation condition holds. Let P be the continuous inverse regression associated with ϕ. By the version clause of Lemma 2.3,

$$\mathbb{E}[P(x+U)] = F_\phi(x) \qquad \text{for every } x\in\mathbb{R}\,.$$

Thus, the pointwise maximisation condition and market efficiency show that (ϕ, P) is an equilibrium of the standardised model. By Lemma 2.4, the map ϕ is strictly increasing, while Propositions 6.1 and 4.1 give

$$\phi(v) = v + a \qquad \text{for Lebesgue-almost every } v\in\mathbb{R}$$

for some $a\in\mathbb{R}$. Because ϕ is increasing and agrees almost everywhere with the continuous function $v\mapsto v+a$, this equality holds for every $v\in\mathbb{R}$.

Gaussian regression now gives

$$F_\phi(x) = \frac{x-a}{2} \qquad \text{for every } x\in\mathbb{R}\,.$$

Hence the unique maximiser of

$$x\bigl(v - F_\phi(x)\bigr) = xv - \frac{x^2}{2} + \frac{a}{2}x$$

is $v + a/2$. Pointwise maximisation therefore gives $v + a = v + a/2$ for every v, so $a = 0$ and $\phi = \mathrm{id}_{\mathbb{R}}$.

Conversely, if $\phi = \mathrm{id}_{\mathbb{R}}$, then Gaussian regression gives $F_\phi(x) = x/2$, and $x(v - x/2)$ is uniquely maximised at $x = v$. Q.E.D.

We now prove the main theorem. It is a translation and rescaling of the rigidity theorem.

PROOF OF THEOREM 1.1. Define

$$\widehat{V} := \frac{V-\mu}{\tau}, \qquad \widehat{U} := \frac{U}{\sigma}, \qquad \widehat{\phi}(t) := \frac{\phi(\mu+\tau t)}{\sigma}, \qquad \widehat{P}(y) := \frac{P(\sigma y)-\mu}{\tau}\,.$$

Then $\widehat{V}$ and $\widehat{U}$ are independent standard normal random variables, and

$$\widehat{\phi}(\widehat{V}) + \widehat{U} = \frac{Y_\phi}{\sigma}\,.$$

A direct change of variables shows that (ϕ, P) is an equilibrium if and only if $(\widehat{\phi}, \widehat{P})$ is an equilibrium of the standardised model: market efficiency is preserved, and expected profits are multiplied by the positive constant $\sigma\tau$.

By Theorem 7.1 and Gaussian regression, this holds if and only if

$$\widehat{\phi} = \mathrm{id}_{\mathbb{R}} \qquad \text{and} \qquad \widehat{P}\big(\widehat{\phi}(\widehat{V}) + \widehat{U}\big) = \frac{\widehat{\phi}(\widehat{V}) + \widehat{U}}{2} \quad \text{almost surely}\,.$$

Undoing the normalisation gives

$$\phi(v) = \frac{\sigma}{\tau}(v - \mu) \qquad \text{for every } v \in \mathbb{R}$$

and

$$P(Y_\phi) = \mu + \frac{\tau}{2\sigma} Y_\phi \qquad \text{almost surely}\,.$$

Q.E.D.

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APPENDIX A: PROOFS OF THE PRELIMINARY LEMMAS

PROOF OF LEMMA 2.1. Because U and V are independent, the pair (V, U) has density $\varrho(v)\varrho(u)$. For every bounded Borel-measurable function $g\colon \mathbb{R}^2 \to \mathbb{R}$, substitution of $y = \phi(v) + u$ at fixed v gives

$$\begin{aligned}
\mathbb{E}[g(V, Y)] &= \int_{\mathbb{R}} \int_{\mathbb{R}} g\big(v, \phi(v) + u\big)\varrho(v)\varrho(u)\; \mathrm{d}u\; \mathrm{d}v \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} g(v, y)\varrho(v)\varrho\big(y - \phi(v)\big)\; \mathrm{d}y\; \mathrm{d}v\,.
\end{aligned}$$

This identifies the joint density. Integrating out v and applying Tonelli's theorem gives the density p_Y of Y. Moreover, the integrand defining p_Y is everywhere positive and bounded by $(2\pi)^{-1/2}\varrho(v)$. Thus, for every $y \in \mathbb{R}$,

$$0 < p_Y(y) \le \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varrho(v)\; \mathrm{d}v = \frac{1}{\sqrt{2\pi}}\,;$$

in particular, p_Y is finite and everywhere positive. Q.E.D.

PROOF OF LEMMA 2.2. Define, locally in this proof,

$$n(y) := \int_{\mathbb{R}} v\varrho(v)\varrho\big(y - \phi(v)\big)\; \mathrm{d}v, \qquad q(y) := \int_{\mathbb{R}} |v|\varrho(v)\varrho\big(y - \phi(v)\big)\; \mathrm{d}v\,.$$

The integrands of n and q are dominated by $(2\pi)^{-1/2}|v|\varrho(v)$, and that of p_Y by $(2\pi)^{-1/2}\varrho(v)$, uniformly in y. Because ϱ is continuous, dominated convergence shows that n, q, and p_Y are continuous. Because p_Y is everywhere positive, $P = n/p_Y$ is continuous.

Let $g\colon \mathbb{R} \to \mathbb{R}$ be bounded and Borel measurable. Because the joint law of (V, Y) has the density in Lemma 2.1 and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |vg(y)|\varrho(v)\varrho\big(y - \phi(v)\big)\; \mathrm{d}y\; \mathrm{d}v \le \sup|g|\, \mathbb{E}[|V|] < \infty\,,$$

Fubini's theorem gives

$$\mathbb{E}[Vg(Y)] = \int_{\mathbb{R}} g(y)n(y)\,\mathrm{d}y = \int_{\mathbb{R}} g(y)P(y)p_Y(y)\,\mathrm{d}y = \mathbb{E}\big[g(Y)P(Y)\big]\,.$$

Moreover, Tonelli's theorem gives $\int_{\mathbb{R}} q(y)\,\mathrm{d}y = \mathbb{E}[|V|]$, so

$$\mathbb{E}\big[|P(Y)|\big] = \int_{\mathbb{R}} |n(y)|\,\mathrm{d}y \le \int_{\mathbb{R}} q(y)\,\mathrm{d}y < \infty\,.$$

Thus, $P(Y) = \mathbb{E}[V \mid Y]$ almost surely. Conditional Jensen's inequality gives

$$\mathbb{E}[P(Y)^2] \le \mathbb{E}[V^2] = 1\,.$$

Q.E.D.

PROOF OF LEMMA 2.3. We first prove that $P(x+U) \in L^1$ for every $x \in \mathbb{R}$. Define, locally in this proof,

$$q(y) := \int_{\mathbb{R}} |v|\varrho(v)\varrho\big(y-\phi(v)\big)\,\mathrm{d}v \qquad \text{for every } y \in \mathbb{R}\,.$$

Let ν be the law of $\phi(V)$, so that

$$p_Y(y) = \int_{\mathbb{R}} \varrho(y-a)\,\nu(\mathrm{d}a)\,.$$

For fixed $x \in \mathbb{R}$, define

$$K_x(a) := \int_{\mathbb{R}} \frac{\varrho(y-x)\varrho(y-a)}{p_Y(y)}\,\mathrm{d}y \qquad \text{for every } a \in \mathbb{R}\,.$$

Because $|P| \le q/p_Y$, Tonelli's theorem gives

$$\begin{aligned}\mathbb{E}\big[|P(x+U)|\big] &= \int_{\mathbb{R}} |P(y)|\varrho(y-x)\,\mathrm{d}y \\ &\le \int_{\mathbb{R}} \frac{q(y)}{p_Y(y)}\varrho(y-x)\,\mathrm{d}y \\ &= \mathbb{E}\big[|V|K_x(\phi(V))\big]\,.\end{aligned}$$

We claim that K_x is bounded outside a ν-null set. Suppose first that $\nu((x,\infty)) > 0$. Choose $x < \alpha < \beta$ such that $r = \nu([\alpha,\beta]) > 0$, and put $m = (\alpha+\beta)/2$. For $y \ge m$ and $a \in [\alpha,\beta]$, we have $|y-a| \le |y-\alpha|$, so $p_Y(y) \ge r\varrho(y-\alpha)$. Similarly, $p_Y(y) \ge r\varrho(y-\beta)$ for $y < m$. Thus, for every $a \ge x$,

$$K_x(a) \le \frac{1}{r}\sum_{d\in\{\alpha,\beta\}} \int_{\mathbb{R}} \frac{\varrho(y-x)\varrho(y-a)}{\varrho(y-d)}\,\mathrm{d}y\,.$$

Completing the square gives

$$\int_{\mathbb{R}} \frac{\varrho(y-x)\varrho(y-a)}{\varrho(y-d)}\,\mathrm{d}y = e^{(x-d)(a-d)}\,.$$

Because $d > x$ and $a \ge x$, the exponent is at most $(x-d)^2$. Thus, K_x is bounded on $[x,\infty)$. The symmetric argument gives boundedness on $(-\infty,x]$ whenever $\nu((-\infty,x)) > 0$. If neither open half-line has positive ν-measure, then $\nu = \delta_x$ and $K_x(x) = 1$. This proves the claim. Consequently, $K_x(\phi(V))$ is almost surely bounded, and

$$\mathbb{E}\big[|P(x+U)|\big] \le \mathbb{E}\big[|V|K_x(\phi(V))\big] < \infty\,.$$

Because $x+U$ has density $y \mapsto \varrho(y-x)$, the definition (2.1) reads

$$F(x) = \int_{\mathbb{R}} P(y)\varrho(y-x)\,\mathrm{d}y\,.$$

This proves (2.2).

If $\widetilde{P}$ is as in the statement, then $\widetilde{P}(Y) = P(Y)$ almost surely, so $\widetilde{P} = P$ almost everywhere under the law of Y. Because $p_Y > 0$ everywhere by Lemma 2.1, the equality holds Lebesgue-almost everywhere. The law of $x+U$ is absolutely continuous, so $\widetilde{P}(x+U) = P(x+U)$ almost surely. In particular, $\widetilde{P}(x+U) \in L^1$ and $\mathbb{E}[\widetilde{P}(x+U)] = F(x)$.

It remains to differentiate the convolution. Fix $R > 0$ and choose $A > R$. There is a constant $C = C(R,A)$ such that

$$(1+|y|)e^{R|y|} \le C\big(e^{Ay} + e^{-Ay}\big) \qquad \text{for every } y \in \mathbb{R}\,.$$

Because $\varrho(y-x) \le \varrho(y)e^{|x||y|}$ and $e^{\pm Ay}\varrho(y) = e^{A^2/2}\varrho(y \mp A)$, uniformly for $|x| \le R$, both $|P(y)|\varrho(y-x)$ and $|P(y)(y-x)|\varrho(y-x)$ are bounded by a constant multiple of

$$|P(y)|\big(\varrho(y-A) + \varrho(y+A)\big)\,.$$

This function is integrable by the shifted integrability proved above, applied at $x = A$ and $x = -A$. Differentiation under the integral sign is therefore valid locally uniformly in x. This gives the first formula in (2.3) and proves that $F \in C^1(\mathbb{R})$; the second formula follows by substituting $y = x+u$.

Finally, because $P(Y) = \mathbb{E}[V \mid Y]$ almost surely and $Y = \phi(V) + U$, independence of U and V gives

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[V \mid Y] \mid V] &= \mathbb{E}[P(Y) \mid V] \\
&= \mathbb{E}[P(\phi(V)+U) \mid V] \\
&= F(\phi(V)) \qquad \text{almost surely}\,.
\end{aligned}$$

Q.E.D.

PROOF OF LEMMA 2.4. By hypothesis, $\phi(v)$ attains the supremum defining $W(v)$; in particular, W is real-valued. For every $v, w \in \mathbb{R}$,

$$W(w) \ge W(v) + \phi(v)(w-v)\,. \tag{A.1}$$

For fixed v, the function $x \mapsto x(v - F(x))$ is continuously differentiable by Lemma 2.3 and is maximised at $x = \phi(v)$. Its derivative vanishes there, proving (2.4). Substitution into $W(v) = \phi(v)(v - F(\phi(v)))$ gives (2.5).

A finite convex function on $\mathbb{R}$ is locally absolutely continuous and differentiable almost everywhere [20, Sections 23–25]. Inequality (A.1) states that $\phi(v)$ is a subgradient of W at v. At every point where W is differentiable, the subgradient is unique and equals $W'(v)$. Therefore, (2.6) holds.

Let $v_1 < v_2$. Applying (A.1) in both directions and adding the two inequalities gives

$$\big(\phi(v_2) - \phi(v_1)\big)(v_2 - v_1) \ge 0\,.$$

Thus, ϕ is nondecreasing. If $\phi(v_1) = \phi(v_2)$, then (2.4) gives $v_1 = v_2$, a contradiction. Therefore, ϕ is strictly increasing. Q.E.D.

PROOF OF LEMMA 2.5. Define, locally in this proof,

$$n(y) := \int_{\mathbb{R}} v\varrho(v)\varrho\big(y - \phi(v)\big)\,\mathrm{d}v \qquad \text{for every } y \in \mathbb{R}\,,$$

so that $P = n/p_Y$ by the definition of the canonical version of the inverse regression. On compact y-sets, differentiation under the integral sign is justified because ϱ and $|t|\varrho(t)$ are bounded, while $|v|\varrho(v)$ is integrable. Thus, $p_Y, n \in C^1(\mathbb{R})$. Because p_Y is everywhere positive by Lemma 2.1, the function $P = n/p_Y$ is continuously differentiable.

The required second moments under π_y are locally uniformly bounded. Indeed, because $\varrho \le (2\pi)^{-1/2}$,

$$\int_{\mathbb{R}} v^2 \varrho(v)\varrho\big(y-\phi(v)\big)\,\mathrm{d}v \le (2\pi)^{-1/2} \int_{\mathbb{R}} v^2 \varrho(v)\,\mathrm{d}v = (2\pi)^{-1/2}\,.$$

Moreover, if $|y| \le M$ and $t = y - \phi(v)$, then

$$\phi(v)^2 \varrho\big(y-\phi(v)\big) = (y-t)^2\varrho(t) \le 2M^2\varrho(t) + 2t^2\varrho(t)\,.$$

The right-hand side is bounded uniformly in t. Because p_Y is continuous and everywhere positive, it is bounded away from zero on compact sets. It follows that $\mathrm{id}_{\mathbb{R}}, \phi \in L^2(\pi_y)$ for every $y \in \mathbb{R}$.

Because

$$\partial_y \varrho\big(y-\phi(v)\big) = \big(\phi(v) - y\big)\,\varrho\big(y-\phi(v)\big)\,,$$

we obtain

$$\frac{p_Y'(y)}{p_Y(y)} = \mathbb{E}_{\pi_y}[\phi - y], \qquad \frac{n'(y)}{p_Y(y)} = \mathbb{E}_{\pi_y}\big[\mathrm{id}_{\mathbb{R}}\,(\phi - y)\big]\,.$$

The quotient rule gives

$$P'(y) = \mathbb{E}_{\pi_y}\big[\mathrm{id}_{\mathbb{R}}\,(\phi - y)\big] - \mathbb{E}_{\pi_y}[\mathrm{id}_{\mathbb{R}}]\,\mathbb{E}_{\pi_y}[\phi - y] = \mathrm{Cov}_{\pi_y}(\mathrm{id}_{\mathbb{R}}, \phi)\,.$$

The last equality holds because adding a constant to the second argument leaves the covariance unchanged. Symmetrising the covariance gives

$$P'(y) = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} (v-w)\big(\phi(v) - \phi(w)\big)\,\pi_y(\mathrm{d}v)\,\pi_y(\mathrm{d}w)\,.$$

The density of π_y is everywhere positive. Because ϕ is strictly increasing by Lemma 2.4, the integrand is strictly positive off the diagonal, while the diagonal has zero $(\pi_y \otimes \pi_y)$-measure. This proves (2.7).

Let U_0, U_1 be independent standard normal random variables. By (2.3) of Lemma 2.3,

$$F'(x) = \mathbb{E}[U_0 P(x+U_0)]\,.$$

The absolute convergence in Lemma 2.3 gives

$$U_i P(x+U_i) \in L^1 \qquad \text{for every } i \in \{0,1\}\,.$$

The same lemma gives $P(x+U_i) \in L^1$. Hence, by independence, the cross terms $U_i P(x+U_j)$ are integrable and have expectation zero whenever $i \neq j$. Symmetrisation therefore gives

$$2F'(x) = \mathbb{E}\big[(U_0 - U_1)\big(P(x+U_0) - P(x+U_1)\big)\big]\,.$$

Because P is strictly increasing, the integrand is nonnegative and is strictly positive whenever $U_0 \neq U_1$. Because $\mathbb{P}(U_0 \neq U_1) = 1$, this proves (2.8).

Thus, P and F are strictly increasing. Q.E.D.

PROOF OF LEMMA 2.6. Write $(\Omega,\mathcal{F},\mathbb{P})$ for the original probability space, and let $y\mapsto\pi_y$ be the conditional-law kernel defined above. The transfer theorem [16, Theorem 8.17] guarantees the existence of the required conditional replica. In the present setting, that consequence can be realised directly from the kernel π.

Set

$$\overline{\Omega} := \Omega\times\mathbb{R}, \qquad \overline{\mathcal{F}} := \mathcal{F}\otimes\mathcal{B}(\mathbb{R}),$$

and define a probability measure on this product space by

$$\overline{\mathbb{P}}(A) := \int_\Omega\int_\mathbb{R} \mathbf{1}_A(\omega,v_1)\,\pi_{Y(\omega)}(\mathrm{d}v_1)\,\mathbb{P}(\mathrm{d}\omega), \qquad A\in\overline{\mathcal{F}}.$$

We identify each original random variable with its lift to $\overline{\Omega}$, set $V^{(0)} := V$, and define

$$V^{(1)}(\omega,v_1) := v_1\,.$$

By construction, the conditional law of $V^{(1)}$ given the lifted sigma-algebra $\mathcal{F}$ is π_Y. In particular,

$$\mathcal{L}(V^{(1)}\mid Y,V^{(0)}) = \pi_Y \qquad \text{almost surely}\,.$$

The conditional law of $V^{(0)}=V$ given Y is also π_Y. Hence, for bounded Borel functions $f,g\colon\mathbb{R}\to\mathbb{R}$,

$$\begin{aligned}\mathbb{E}[f(V^{(0)})g(V^{(1)})\mid Y] &= \mathbb{E}\left[f(V^{(0)})\int_\mathbb{R} g(v)\,\pi_Y(\mathrm{d}v)\,\middle|\, Y\right]\\ &= \left(\int_\mathbb{R} f(v)\,\pi_Y(\mathrm{d}v)\right)\left(\int_\mathbb{R} g(v)\,\pi_Y(\mathrm{d}v)\right) \qquad \text{almost surely}\,.\end{aligned}$$

Thus, conditionally on Y, the variables $V^{(0)}$ and $V^{(1)}$ are independent and both have law π_Y. Equivalently,

$$\mathcal{L}\big((V^{(0)},V^{(1)})\mid Y\big) = \pi_Y\otimes\pi_Y \qquad \text{almost surely}\,.$$

The product kernel is invariant under interchange of its coordinates, so

$$(V^{(0)},V^{(1)},Y) \overset{d}{=} (V^{(1)},V^{(0)},Y)\,.$$

This proves (i). Integrating out $V^{(0)}$ in the conditional product law gives

$$(V^{(1)},Y) \overset{d}{=} (V,Y)\,,$$

which proves (ii). In particular, $V^{(1)}\overset{d}{=}V$, so $V^{(1)}\in L^2$.

For (iii), the conditional law identified above gives

$$\mathbb{E}[V^{(1)}\mid Y,V^{(0)}] = \int_\mathbb{R} v\,\pi_Y(\mathrm{d}v) = P(Y) \qquad \text{almost surely}\,.$$

Finally, the tower property and (iii) give

$$\begin{aligned}\mathbb{E}[V^{(1)}\mid V^{(0)}] &= \mathbb{E}\Big[\mathbb{E}[V^{(1)}\mid Y,V^{(0)}]\,\Big|\, V^{(0)}\Big]\\ &= \mathbb{E}[P(Y)\mid V^{(0)}]\\ &= F\big(\phi(V^{(0)})\big) \qquad \text{almost surely}\,.\end{aligned}$$

The last equality is the final assertion of Lemma 2.3. This proves (iv). Q.E.D.

PROOF OF LEMMA 2.7. Let ϱ denote the standard Gaussian density. Because f is Lipschitz, it is locally absolutely continuous, its almost-everywhere derivative is bounded, and

$$\int_{\mathbb{R}} |f'(v)|\varrho(v)\,\mathrm{d}v < \infty\,.$$

For every $R > 0$, integration by parts gives

$$\int_{-R}^{R} f'(v)\varrho(v)\,\mathrm{d}v = \big[f(v)\varrho(v)\big]_{-R}^{R} - \int_{-R}^{R} f(v)\varrho'(v)\,\mathrm{d}v\,.$$

Because f is bounded and $\varrho(v) \to 0$ as $|v| \to \infty$, the boundary term tends to zero. Moreover, $\varrho'(v) = -v\varrho(v)$, and $v \mapsto |vf(v)|\varrho(v)$ is integrable. Dominated convergence therefore gives

$$\mathbb{E}[f'(V)] = \mathbb{E}[Vf(V)]\,.$$

Because $\mathbb{E}[V] = 0$,

$$\mathbb{E}[Vf(V)] = \mathbb{E}\big[V\big(f(V) - \mathbb{E}[f(V)]\big)\big]\,.$$

The Cauchy–Schwarz inequality and $\mathbb{E}[V^2] = 1$ now give

$$\big|\mathbb{E}[f'(V)]\big| \le \sqrt{\mathrm{Var}\big(f(V)\big)}\,.$$

Squaring proves the result. Q.E.D.

APPENDIX B: FROM ALMOST-EVERYWHERE TO POINTWISE MAXIMISATION

We return to the setting of Section 1, where $V \sim \mathcal{N}(\mu, \tau^2)$ and $U \sim \mathcal{N}(0, \sigma^2)$ are independent. We say that (ϕ, P) is a *quasi-equilibrium* if the following conditions hold:

1. (almost everywhere profit maximisation) for Lebesgue-almost every $v \in \mathbb{R}$,

$$\mathbb{E}\big[\big(v - P(\phi(v) + U)\big)\phi(v)\big] \ge \mathbb{E}\big[\big(v - P(x + U)\big)x\big] \qquad \text{for every } x \in \mathbb{R}\,;$$

2. (market efficiency) $P(Y_\phi) = \mathbb{E}[V \mid Y_\phi]$ almost surely.

LEMMA B.1. *If (ϕ, P) is a quasi-equilibrium, then*

$$\phi(v) = \frac{\sigma}{\tau}(v - \mu) \qquad \textit{for Lebesgue-almost every } v \in \mathbb{R}\,, \tag{B.1}$$

and

$$P(Y_\phi) = \mu + \lambda Y_\phi \qquad \textit{almost surely}\,, \tag{B.2}$$

where

$$\lambda = \frac{\mathrm{Cov}(V, Y_\phi)}{\mathrm{Var}(Y_\phi)} = \frac{\tau}{2\sigma}\,.$$

PROOF OF LEMMA B.1. Let (ϕ, P) be a quasi-equilibrium. Define

$$\widehat{V} := \frac{V - \mu}{\tau}, \qquad \widehat{U} := \frac{U}{\sigma}, \qquad \widehat{\phi}(t) := \frac{\phi(\mu + \tau t)}{\sigma}, \qquad \widehat{P}(y) := \frac{P(\sigma y) - \mu}{\tau}\,.$$

Then $\widehat{V}$ and $\widehat{U}$ are independent standard normal random variables, and market efficiency gives

$$\widehat{P}\big(\widehat{\phi}(\widehat{V}) + \widehat{U}\big) = \mathbb{E}[\widehat{V} \mid \widehat{\phi}(\widehat{V}) + \widehat{U}] \qquad \text{almost surely}\,.$$

Define

$$\widehat{F}(z) := \mathbb{E}[\widehat{P}(z + \widehat{U})], \qquad z \in \mathbb{R}\,.$$

The version clause of Lemma 2.3 shows that $\widehat{P}(z+\widehat{U}) \in L^1$ for every $z \in \mathbb{R}$ and that $\widehat{F}$ is continuous. Because

$$P(x+U) = \mu + \tau \widehat{P}\Big(\frac{x}{\sigma} + \widehat{U}\Big)\,,$$

it follows that $P(x+U) \in L^1$ for every $x \in \mathbb{R}$ and that

$$F(x) := \mathbb{E}[P(x+U)] = \mu + \tau \widehat{F}\Big(\frac{x}{\sigma}\Big)\,, \qquad x \in \mathbb{R}\,,$$

is continuous. Consequently, for every $v, x \in \mathbb{R}$,

$$\mathbb{E}\big[(v - P(x+U))x\big] = x\big(v - F(x)\big)\,.$$

The almost-everywhere profit-maximisation condition now gives

$$\phi(v) \in \arg\max_{x\in\mathbb{R}} x\big(v - F(x)\big) \qquad \text{for Lebesgue-almost every } v \in \mathbb{R}\,.$$

Set

$$f(x,v) := x\big(v - F(x)\big), \qquad W(v) := \sup_{x\in\mathbb{R}} f(x,v)\,.$$

The function W is convex as the pointwise supremum of affine functions of v, and $W \geq 0$ because $x = 0$ is admissible. Let $A \subseteq \mathbb{R}$ be a set of full Lebesgue measure on which $\phi(v)$ attains the supremum. Then

$$W(v) = f(\phi(v), v) < \infty \qquad \text{for every } v \in A\,.$$

Fix $v \in \mathbb{R}$, and choose $a, b \in A$ with $a < v < b$. Convexity gives

$$W(v) \leq \frac{b-v}{b-a} W(a) + \frac{v-a}{b-a} W(b) < \infty\,.$$

Thus, W is finite on $\mathbb{R}$ and hence continuous.

For each $v \in \mathbb{R}$, define

$$M(v) := \arg\max_{x\in\mathbb{R}} f(x,v)\,.$$

We show that $M(v)$ is nonempty and compact. Choose $a < v < b$. For $x \geq 0$,

$$f(x,v) = f(x,b) - x(b-v) \leq W(b) - x(b-v)\,,$$

whereas, for $x \leq 0$,

$$f(x,v) = f(x,a) + x(v-a) \leq W(a) + x(v-a)\,.$$

It follows that $f(x,v) \to -\infty$ as $|x| \to \infty$. Because $x \mapsto f(x,v)$ is continuous, it attains its maximum. The same limit makes the set of maximisers bounded, while continuity makes it closed. Thus, $M(v)$ is nonempty and compact.

Let $v_1 < v_2$ and choose $x_i \in M(v_i)$. Optimality gives

$$f(x_1,v_1) \geq f(x_2,v_1) \qquad \text{and} \qquad f(x_2,v_2) \geq f(x_1,v_2)\,.$$

Adding these inequalities yields

$$(x_2 - x_1)(v_2 - v_1) \geq 0\,,$$

so $x_1 \leq x_2$. Therefore, the function

$$\psi(v) := \min M(v), \qquad v \in \mathbb{R}\,,$$

is nondecreasing and hence Borel measurable.

For every $v \in \mathbb{R}$ and $x \in M(v)$,

$$W(w) \geq f(x,w) = W(v) + x(w-v) \qquad \text{for every } w \in \mathbb{R}\,.$$

Thus, every element of $M(v)$ is a subgradient of W at v. At every point where W is differentiable, its subgradient is the singleton $\{W'(v)\}$; because $M(v)$ is nonempty, this implies

$$M(v) = \{W'(v)\}\,.$$

A finite convex function is differentiable Lebesgue-almost everywhere. Intersecting this full-measure differentiability set with A gives

$$\psi(v) = \phi(v) \qquad \text{for Lebesgue-almost every } v \in \mathbb{R}\,.$$

By construction,

$$\psi(v) \in \arg\max_{x\in\mathbb{R}} x\big(v - F(x)\big) \qquad \text{for every } v \in \mathbb{R}\,.$$

Because the law of V is absolutely continuous and $\psi = \phi$ Lebesgue-almost everywhere,

$$\psi(V) = \phi(V) \qquad \text{and} \qquad Y_\psi = Y_\phi \qquad \text{almost surely}\,.$$

Hence market efficiency gives

$$P(Y_\psi) = \mathbb{E}[V \mid Y_\psi] \qquad \text{almost surely}\,.$$

Moreover, for every $v, x \in \mathbb{R}$, the payoff representation and the pointwise optimality of $\psi(v)$ give

$$\begin{aligned}\mathbb{E}\big[(v - P(\psi(v)+U))\psi(v)\big] &= f(\psi(v), v)\\ &\geq f(x,v)\\ &= \mathbb{E}\big[(v-P(x+U))x\big]\,.\end{aligned}$$

Thus, (ψ, P) is an equilibrium. Theorem 1.1 gives

$$\psi(v) = \frac{\sigma}{\tau}(v-\mu) \qquad \text{for every } v \in \mathbb{R}$$

and

$$P(Y_\psi) = \mu + \frac{\tau}{2\sigma}Y_\psi \qquad \text{almost surely}\,.$$

Because $\psi = \phi$ Lebesgue-almost everywhere and $Y_\psi = Y_\phi$ almost surely, these are exactly the two conclusions of the lemma. The covariance formula for λ follows from the corresponding formula in Theorem 1.1 and the almost-sure equality of Y_ψ and Y_ϕ. Q.E.D.

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