CARLESON MEASURES, VOLTERRA INTEGRAL OPERATORS AND MULTIPLIERS FOR Q_K SPACES

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ABSTRACT. We characterize the positive Borel measures μ on the unit disc $\mathbb{D}$ for which the Möbius-invariant space Q_K embeds continuously or compactly into $L^2(\mu)$. The characterization is given in terms of a discrete dyadic capacity $C^{(b)}_{K,\mathcal{R}}(\mu)$ built from a polar dyadic resolution of $\mathbb{D}$, and of an equivalent capacity $D^{(b)}_{K,\mathcal{R}}(\mu)$ expressed as a semidefinite program. The equivalence of the two capacities is established through conic duality and the complex Grothendieck inequality. As an application, we characterize the boundedness and compactness of the Volterra integral operator T_g on Q_K, bridging and completely resolving the gap between the sufficient and necessary conditions established by Li and Wulan (2010). We also obtain a complete non-testing characterization of the pointwise multipliers $\mathcal{M}(Q_K)$ on Q_K, thereby answering an open problem posed in the survey of Bao and Wulan (2021).

1. INTRODUCTION

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$ its boundary. Let $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. For each $a \in \mathbb{D}$, the mapping $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the standard Möbius automorphism of $\mathbb{D}$ that interchanges 0 and a. A direct computation gives the conformal identities

$$|\varphi_a'(z)| = \frac{1-|a|^2}{|1-\bar{a}z|^2}, \qquad 1-|\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}. \tag{1}$$

Definition 1.1. A non-decreasing, right-continuous function $K : [0,\infty) \to [0,\infty)$ is called a $\mathcal{Q}_K$*-defining weight function* (denoted $K \in \mathcal{W}_\mathcal{Q}$) if it satisfies the following conditions (see [3, 15]):

(K1) $K(0) = 0$ and $K(t) > 0$ for all $t > 0$;

(K2) $\displaystyle\int_0^1 K\left(\log\frac{1}{r}\right) 2r\,dr < \infty$;

(K3) There exists $\sigma > 0$ such that

$$\int_0^1 \frac{\varphi_K(s)}{s}\,ds < \infty \quad \text{and} \quad \int_1^\infty \frac{\varphi_K(s)}{s^{1+\sigma}}\,ds < \infty,$$

where $\varphi_K(s) := \sup_{0<t\le 1} \frac{K(st)}{K(t)}$;

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(K4) Doubling property: there exists a constant $C_K \geq 1$ such that $K(2t) \leq C_K K(t)$ for all $t \in [0, 1/2]$.

The Möbius-invariant space Q_K consists of all analytic functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{Q_K} = |f(0)| + \|f\|_{Q_K,*} < \infty,$$

where the semi-norm

$$\|f\|_{Q_K,*}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \, K\big(1 - |\varphi_a(z)|^2\big) \, dA(z). \tag{2}$$

Here $dA(z) = \frac{1}{\pi}\, dx\, dy$ is the normalized Lebesgue area measure. Varying K produces a rich family of classical function spaces [3, 15]. If $K(t) = t^p$ with $0 < p < \infty$, then Q_K reduces to the space Q_p; in particular Q_1 coincides with $BMOA$, while for $p > 1$ the space Q_p coincides with the Bloch space $\mathcal{B}$. If $K(t) = t(\log(e/t))^p$, then Q_K is the logarithmic space $Q_{\log,p}$.

For a boundary arc $I \subseteq \mathbb{T}$, we write $|I| = \frac{1}{2\pi}\int_I |d\zeta|$ for its normalized arclength, the Carleson box is defined by

$$Q_I = \big\{z = re^{i\theta} \in \mathbb{D} : e^{i\theta} \in I,\ 1 - |I| \leq r < 1\big\},$$

with the convention $Q_{\mathbb{T}} = \mathbb{D}$. By the classical characterization of Q_K [3, 7, 15], one has

$$\|f\|_{Q_K,*}^2 \asymp \sup_{I\subseteq\mathbb{T}} \int_{Q_I} |f'(z)|^2 \, K\Big(\frac{1-|z|}{|I|}\Big) \, dA(z). \tag{3}$$

In view of (3), we introduce the tent space T_K of measurable functions F on $\mathbb{D}$ for which

$$\|F\|_{T_K}^2 = \sup_{I\subseteq\mathbb{T}} \int_{Q_I} |F(z)|^2 \, K\Big(\frac{1-|z|}{|I|}\Big) \, dA(z) < \infty. \tag{4}$$

This definition yields the identification

$$f \in Q_K \iff f' \in T_K, \qquad \text{with} \quad \|f\|_{Q_K,*} \asymp \|f'\|_{T_K}, \tag{5}$$

for $f \in H(\mathbb{D})$. We denote by $\mathring{Q}_K = \{f \in Q_K : f(0) = 0\}$ the subspace of functions vanishing at the origin.

The Q_K–Carleson Measure Problem. A long-standing problem in the theory of Möbius-invariant spaces is to characterize the positive Borel measures μ on $\mathbb{D}$ for which Q_K embeds continuously or compactly into $L^2(\mu)$, i.e.,

$$\mathrm{id}\colon Q_K \longrightarrow L^2(\mu), \qquad \int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq C \, \|f\|_{Q_K}^2. \tag{6}$$

For the scale of Q_p spaces, non-testing characterizations were established using dyadic methods in [9]; for the Bloch space, testing conditions were studied in [1, 6]; and general Möbius-invariant families were investigated in [11]. In numerous conference talks and research discussions over the past two decades, Professor Hasi Wulan highlighted the intrinsic challenge of obtaining a symbol-free, sharp characterization of Carleson measures for general Q_K spaces without testing on the full function space.

Since constant functions belong to Q_K, the finite-mass condition $\mu(\mathbb{D}) < \infty$ is necessary for the boundedness of (6). Hence the main problem is to characterize

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \le C\|f\|_{Q_K,*}^2$$

for all $f \in \mathring{Q}_K$.

Volterra Integral Operators and the Multiplier Problem. For an analytic symbol $g \in H(\mathbb{D})$, the Volterra integral operator T_g (introduced by Pommerenke [13]) and its companion operator I_g are defined by

$$T_g f(z) = \int_0^z f(\zeta)\, g'(\zeta)\, d\zeta, \qquad I_g f(z) = \int_0^z f'(\zeta)\, g(\zeta)\, d\zeta, \qquad z \in \mathbb{D}.$$

The multiplication operator M_g is given by $M_g f(z) = g(z)f(z)$. A fundamental integration-by-parts relation connects these three operators:

$$M_g f(z) = f(0)g(z) + T_g f(z) + I_g f(z). \tag{7}$$

Since $(T_g f)'(z) = f(z)g'(z)$ and $T_g f(0) = 0$, a change of variables based on (1) yields

$$\|T_g f\|_{Q_K,*}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 |g'(z)|^2 \, K\big(1 - |\varphi_a(z)|^2\big)\, dA(z) = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 \, d\mu_{g,a,K}(z), \tag{8}$$

where the family of positive Borel measures $\{\mu_{g,a,K}\}_{a\in\mathbb{D}}$ is given by

$$d\mu_{g,a,K}(z) = |g'(z)|^2 \, K\big(1 - |\varphi_a(z)|^2\big)\, dA(z).$$

The identity (8) shows that the mapping properties of T_g on Q_K are determined by the uniform (in $a \in \mathbb{D}$) continuity of the embeddings $\mathrm{id}\colon Q_K \to L^2(\mu_{g,a,K})$.

In [10], Li and Wulan established that the condition

$$\sup_{I\subseteq\mathbb{T}} \int_{Q_I} \left(\log \frac{1}{1-|z|^2}\right)^2 |g'(z)|^2 K\left(\frac{1-|z|}{|I|}\right) dA(z) < \infty \tag{9}$$

is sufficient for the boundedness of T_g on Q_K, while the condition

$$\sup_{I\subseteq\mathbb{T}} \left(\log \frac{2}{|I|}\right)^2 \int_{Q_I} |g'(z)|^2 K\left(\frac{1-|z|}{|I|}\right) dA(z) < \infty \tag{10}$$

is necessary. However, an intrinsic gap remained between (9) and (10).

Furthermore, characterizing the space of pointwise multipliers

$$\mathcal{M}(Q_K) = \{g \in H(\mathbb{D}) : M_g(Q_K) \subseteq Q_K\}$$

has also been a well-known open problem. In the survey [2, Problem 3.1] by Bao and Wulan (see also [10]), it was conjectured that $\mathcal{M}(Q_K) = H^\infty \cap Q_K^{\log}$, where $Q_K^{\log}$ is the space of symbols satisfying (10).

In this paper, we completely solve these problems:

(i) We characterize the Carleson measures for Q_K in terms of the dyadic capacity $C^{(b)}_{K,\mathcal{R}}(\mu)$ and the dual semidefinite program $D^{(b)}_{K,\mathcal{R}}(\mu)$;
(ii) We obtain exact, necessary and sufficient conditions for the boundedness and compactness of T_g on Q_K, completely closing the gap between (9) and (10);
(iii) Combining our characterization of T_g with the characterization of I_g obtained by Li and Wulan [10], we provide the complete solution to the multiplier problem on Q_K.

Throughout this paper, $A \lesssim B$ means $A \leq CB$ for some constant $C > 0$, and $A \asymp B$ means $A \lesssim B \lesssim A$.

2. The b-admissible dyadic resolution and boundedness on T_K

We now formalize the polar dyadic subdivision of $\mathbb{D}$ adapted to the weight K. Consider the polar parametrization

$$\Theta(s,t) = \sqrt{s}\, e^{2\pi i t}, \qquad (s,t) \in [0,1) \times [0,1).$$

For a parameter $b \gg 1$, set

$$dA_b(w) = c_b\,(1-|w|^2)^b\, dA(w), \qquad c_b = b+1.$$

For each scale $N \geq 0$ and indices $0 \leq j,k \leq 2^N - 1$ we define the polar dyadic tile

$$R_{N,j,k} = \Big\{ re^{i\theta} : \frac{j}{2^N} \leq r^2 < \frac{j+1}{2^N},\ \frac{2\pi k}{2^N} \leq \theta < \frac{2\pi(k+1)}{2^N} \Big\}.$$

The weighted measure of a tile satisfies

$$A_b(R_{N,j,k}) = 2^{-N} \int_{j/2^N}^{(j+1)/2^N} (1-s)^b\, ds \asymp 2^{-2N} \Big(1 - \frac{j}{2^N}\Big)^b.$$

For each $N \geq 0$ we write $\mathcal{R}_N = \{R_{N,j,k} : 0 \leq j,k \leq 2^N - 1\}$, and we define the dA_b-weighted conditional expectation operator

$$E^{(b)}_N \psi = \sum_{R \in \mathcal{R}_N} \Big(\frac{1}{A_b(R)} \int_R \psi\, dA_b \Big) \mathbf{1}_R, \qquad A_b(R) = \int_R dA_b.$$

By the martingale convergence theorem, $E^{(b)}_N \psi \to \psi$ strongly in $L^1(dA_b)$ as $N \to \infty$ for every $\psi \in L^1(dA_b)$.

Thus the polar dyadic resolution constructed above has the following weighted admissibility properties:

(a) Each $\mathcal{R}_N$ is a finite measurable partition of $\mathbb{D}$;
(b) The partitions are nested: every tile in $\mathcal{R}_{N+1}$ is contained, modulo A_b-null sets, in a tile of $\mathcal{R}_N$;
(c) The associated dA_b-weighted conditional expectations

$$E^{(b)}_N \psi = \sum_{R \in \mathcal{R}_N} \Big(\frac{1}{A_b(R)} \int_R \psi\, dA_b \Big) \mathbf{1}_R$$

converge to ψ in $L^1(dA_b)$ for every $\psi \in L^1(dA_b)$;

(d) For each $N \geq 0$, an arbitrary ordering of the finite set $\mathcal{R}_N$ is fixed once and for all.

We shall call any sequence $\mathcal{R} = \{\mathcal{R}_N\}_{N\geq 0}$ of decompositions of $\mathbb{D}$ satisfying the above properties a *b-admissible dyadic resolution.* The preceding construction shows that the class of b-admissible dyadic resolutions is nonempty. For simplicity, throughout the rest of this paper we fix $\mathcal{R} = \{\mathcal{R}_N\}_{N\geq 0}$ to be the polar dyadic resolution constructed above.

We record two auxiliary estimates that will be used in the proof of Lemma 2.3.

Lemma 2.1 ([7, Lemma 2.4])**.** *Let $J \subseteq \mathbb{T}$ be an arc. If $J(n) \subseteq \mathbb{T}$ is the arc concentric with J whose length satisfies*

$$|J(n)| \asymp \min\{2^n|J|, 1\},$$

then for every $b \gg 1$,

$$\int_{Q_{J(n+1)}} \frac{(1-|w|^2)^{2b}}{K\big((1-|w|)/|J(n+1)|\big)}\, dA(w) \lesssim |J(n+1)|^{2b+2} \asymp \big(\min\{2^n|J|, 1\}\big)^{2b+2}.$$

Lemma 2.2. *Let $K \in \mathcal{W}_{\mathcal{Q}}$. Set $\alpha = \log_2 C_K \geq 0$ and $c_0 = C_K^{-1}$. Then for every $s \in (0,1]$ and every $\lambda \in (0,1]$,*

$$K(\lambda s) \geq c_0\, \lambda^{\alpha} K(s). \tag{11}$$

In particular, setting $s = 1$ recovers $K(t) \geq c_0 K(1) t^{\alpha}$ for all $t \in (0,1]$.

Proof. Fix $s \in (0,1]$ and $\lambda \in (0,1]$. We divide the argument into two cases according to the dyadic scale of λ:

Case 1: $\lambda \in (1/2, 1]$**.** Since $\lambda > 1/2$, we have $\lambda s \geq s/2$. By the monotonicity of K,

$$K(\lambda s) \geq K(s/2).$$

Since $s \in (0,1]$, we have $t := s/2 \in (0, 1/2]$. Applying the doubling condition (K4) to t gives

$$K(s) = K(2t) \leq C_K K(t) = C_K K(s/2) \implies K(s/2) \geq C_K^{-1} K(s).$$

Because $\lambda \leq 1$ and $\alpha \geq 0$, we have $\lambda^{\alpha} \leq 1$. Therefore,

$$K(\lambda s) \geq K(s/2) \geq C_K^{-1} K(s) \geq C_K^{-1} \lambda^{\alpha} K(s) = c_0\, \lambda^{\alpha} K(s).$$

Case 2: $\lambda \in (0, 1/2]$**.** Choose the unique positive integer $n \geq 1$ such that

$$2^{-(n+1)} < \lambda \leq 2^{-n}.$$

By the monotonicity of K,

$$K(\lambda s) \geq K\big(2^{-(n+1)} s\big). \tag{12}$$

For each integer $k \in \{0, 1, \ldots, n\}$, since $s \leq 1$, we have $2^{-(k+1)} s \leq 2^{-1} = 1/2$. Applying condition (K4) with $t = 2^{-(k+1)} s \in (0, 1/2]$ gives

$$K\big(2^{-(k+1)} s\big) \geq C_K^{-1} K\big(2 \cdot 2^{-(k+1)} s\big) = C_K^{-1} K\big(2^{-k} s\big).$$

Iterating this reverse doubling estimate $n+1$ times leads to

$$K\big(2^{-(n+1)}s\big) \geq C_K^{-(n+1)} K(s).$$

Using $C_K^{-(n+1)} = C_K^{-1} C_K^{-n} = C_K^{-1}(2^{-n})^{\log_2 C_K} = C_K^{-1}(2^{-n})^{\alpha}$ and the fact that $\lambda \leq 2^{-n}$ implies $(2^{-n})^{\alpha} \geq \lambda^{\alpha}$, we obtain

$$C_K^{-(n+1)} = C_K^{-1}(2^{-n})^{\alpha} \geq C_K^{-1}\lambda^{\alpha} = c_0\,\lambda^{\alpha}.$$

Combining this with (12) gives

$$K(\lambda s) \geq K\big(2^{-(n+1)}s\big) \geq c_0\,\lambda^{\alpha} K(s).$$

Combining Case 1 and Case 2 completes the proof of (11). □

We now prove the uniform boundedness of $E_N^{(b)}$ on T_K. This is the key technical ingredient of the paper, and the argument follows [7, Section 2].

Lemma 2.3. *Let $K \in \mathcal{W}_{\mathcal{Q}}$ and let $b \gg 1$. Then*

$$\|E_N^{(b)} F\|_{T_K} \lesssim \|F\|_{T_K}, \qquad N \geq 0.$$

Proof. The case $N=0$ is immediate, because $E_0^{(b)}F$ is a constant function and the embedding chain of Lemma 3.4 applies; hence we assume $N \geq 1$. By scaling homogeneity, we normalize $\|F\|_{T_K} = 1$. It suffices to show that for every boundary arc $J \subseteq \mathbb{T}$,

$$\int_{Q_J} |E_N^{(b)} F(z)|^2 K\left(\frac{1-|z|}{|J|}\right) dA(z) \lesssim 1, \tag{13}$$

with an implicit constant independent of both J and N.

To prove (13), we implement a dyadic "local–far" geometric decomposition. This classical paradigm traces back to the foundational work of Fefferman and Stein [4] on the H^p–BMO duality and conformal invariance of BMO; see also Girela's lecture notes [5, Theorem 3.1]. Specifically, for each $n \geq 0$, let $J(n) \subseteq \mathbb{T}$ denote the concentric arc with length $|J(n)| \approx \min\{2^n|J|, 1\}$ (setting $J(m) = \mathbb{T}$ whenever $J(n) = \mathbb{T}$ for all $m \geq n$). We split F into the local core component F_0 and the off-diagonal geometric slices F_n ($n \geq 2$):

$$F_0 := F\mathbf{1}_{Q_{J(2)}}, \qquad F_n := F\mathbf{1}_{Q_{J(n+1)} \setminus Q_{J(n)}} \quad (n \geq 2).$$

Fix an arbitrary boundary arc $J \subseteq \mathbb{T}$. For each integer $n \geq 0$, let $J(n) \subseteq \mathbb{T}$ denote the arc concentric with J whose length satisfies

$$|J(n)| \approx \min\{2^n|J|, 1\}.$$

Whenever $J(n) = \mathbb{T}$, we fix $J(m) = \mathbb{T}$ for all $m \geq n$. We partition the function F into the local main component F_0 and the off-diagonal geometric slices F_n ($n \geq 2$):

$$F_0 := F\mathbf{1}_{Q_{J(2)}}, \quad F_n := F\mathbf{1}_{Q_{J(n+1)} \setminus Q_{J(n)}} \quad (n \geq 2).$$

By linearity of $E_N^{(b)}$, $E_N^{(b)}F = E_N^{(b)}F_0 + \sum_{n\geq 2} E_N^{(b)}F_n$. Applying the elementary inequality $|A+B|^2 \leq 2|A|^2 + 2|B|^2$, the proof of (13) reduces to establishing:

$$\int_{Q_J} |E_N^{(b)}F_0(z)|^2 K\left(\frac{1-|z|}{|J|}\right) dA(z) \lesssim 1 \quad \textbf{(Local Term)}, \tag{14}$$

$$\int_{Q_J} \left|\sum_{n\geq 2} E_N^{(b)}F_n(z)\right|^2 K\left(\frac{1-|z|}{|J|}\right) dA(z) \lesssim 1 \quad \textbf{(Geometric Tail)}. \tag{15}$$

The Local Estimate (14). Because the dyadic tiles $\{R\}_{R\in\mathcal{R}_N}$ form a pairwise disjoint partition of $\mathbb{D}$, $E_N^{(b)}F_0$ is constant on each tile R. Expanding the integral over Q_J gives

$$\begin{aligned}&\int_{Q_J} |E_N^{(b)}F_0(z)|^2 K\left(\frac{1-|z|}{|J|}\right) dA(z)\\ &= \sum_{\substack{R\in\mathcal{R}_N\\ R\cap Q_J\neq\emptyset}} \left(\frac{1}{A_b(R)}\Big|\int_{R\cap Q_{J(2)}} F(w)\, dA_b(w)\Big|\right)^2 \int_{R\cap Q_J} K\left(\frac{1-|z|}{|J|}\right) dA(z).\end{aligned} \tag{16}$$

We partition the collection of tiles hitting Q_J into interior tiles and boundary tiles:

$$\mathcal{R}_N^{\mathrm{int}} := \{R_{N,j,k} \in \mathcal{R}_N : 0 \leq j \leq 2^N - 2\}, \qquad \mathcal{R}_N^{\mathrm{bd}} := \{R_{N,2^N-1,k} \in \mathcal{R}_N : 0 \leq k \leq 2^N - 1\}.$$

Case 1.1: Interior Tiles ($R \in \mathcal{R}_N^{\mathrm{int}}$). For any $R = R_{N,j,k}$ with $0 \leq j \leq 2^N - 2$, the radial coordinates satisfy $\frac{j}{2^N} \leq |\zeta|^2 < \frac{j+1}{2^N}$, which gives

$$(2^N - j - 1)2^{-N} < 1 - |\zeta|^2 \leq (2^N - j)2^{-N}, \quad \forall \zeta \in R.$$

Since $j \leq 2^N - 2$, we have $2^N - j - 1 \geq 1$, so for all $z, w \in R$:

$$\frac{1}{2} \leq \frac{2^N - j - 1}{2^N - j} \leq \frac{1-|z|^2}{1-|w|^2} \leq \frac{2^N - j}{2^N - j - 1} \leq 2.$$

Consequently, $1 - |z|^2 \approx 1 - |w|^2 \approx (2^N - j)2^{-N}$ uniformly. By the doubling condition (K4), $K\left(\frac{1-|z|}{|J|}\right) \approx K\left(\frac{1-|w|}{|J|}\right)$ for all $z, w \in R$. Applying the Cauchy–Schwarz inequality with respect to dA_b:

$$\left|\int_{R\cap Q_{J(2)}} F(w)\, dA_b(w)\right|^2 \leq A_b(R) \int_R |F_0(w)|^2\, dA_b(w).$$

Because $A_b(R) = c_b \int_R (1-|\zeta|^2)^b dA(\zeta) \approx (1-|w|^2)^b A(R)$ for any $w \in R$, we obtain

$$\begin{aligned}&\sum_{R\in\mathcal{R}_N^{\mathrm{int}}} \frac{1}{A_b(R)}\Big(\int_{R\cap Q_J} K\Big(\frac{1-|z|}{|J|}\Big) dA(z)\Big) \int_R |F_0(w)|^2\, dA_b(w)\\ &\lesssim \sum_{R\in\mathcal{R}_N^{\mathrm{int}}} \int_R |F_0(w)|^2 K\Big(\frac{1-|w|}{|J|}\Big) dA(w).\end{aligned} \tag{17}$$

Case 1.2: Boundary Tiles ($R \in \mathcal{R}_N^{\mathrm{bd}}$). For $R = R_{N,2^N-1,k}$, we have $1 - 2^{-N} \leq |\zeta|^2 < 1$. We apply the weighted Cauchy–Schwarz inequality on $R \cap Q_{J(2)}$ with respect to the weight $K\left(\frac{1-|w|}{|J(2)|}\right)$:

$$\begin{aligned}&\Big|\int_{R\cap Q_{J(2)}} F(w)\,dA_b(w)\Big|^2\\ \leq& c_b^2\Big(\int_R |F_0(w)|^2 K\Big(\frac{1-|w|}{|J(2)|}\Big)dA(w)\Big)\Big(\int_{R\cap Q_{J(2)}} \frac{(1-|w|^2)^{2b}}{K((1-|w|)/|J(2)|)}dA(w)\Big). \end{aligned}\tag{18}$$

Substituting (18) into the boundary sum of (16) yields

$$\sum_{R\in\mathcal{R}_N^{\mathrm{bd}}} \mathcal{M}(R) \int_R |F_0(w)|^2 K\left(\frac{1-|w|}{|J(2)|}\right) dA(w),\tag{19}$$

where the tile multiplier $\mathcal{M}(R)$ is defined by

$$\mathcal{M}(R) := \frac{c_b^2}{(A_b(R))^2}\Big(\int_{R\cap Q_J} K\Big(\frac{1-|z|}{|J|}\Big)dA(z)\Big)\Big(\int_{R\cap Q_{J(2)}} \frac{(1-|w|^2)^{2b}}{K((1-|w|)/|J(2)|)}dA(w)\Big).$$

Recall that $A_b(R) = (b+1)\cdot 2^{-N}\int_{1-2^{-N}}^1 (1-s)^b\,ds = 2^{-(b+2)N}$. We evaluate $\mathcal{M}(R)$ by analyzing the scale relation between 2^{-N} and $|J|$:

Subcase (i): $2^{-N} \leq |J|$. In this regime, $\frac{2^{-N}}{|J|} \leq 1$ and $\frac{2^{-N}}{|J(2)|} \leq 1$. For the area integral of K, using $1-|z| \approx 1-|z|^2 = t \in [0, 2^{-N}]$:

$$\begin{aligned}&\int_{R\cap Q_J} K\left(\frac{1-|z|}{|J|}\right) dA(z) \leq 2^{-N}\int_0^{2^{-N}} K\left(\frac{t}{|J|}\right) dt\\ =&2^{-2N}\int_0^1 K\left(u\cdot\frac{2^{-N}}{|J|}\right) du \leq 2^{-2N} K\left(\frac{2^{-N}}{|J|}\right).\end{aligned}$$

For the reciprocal weight integral, change variables via $t = 1-|w|^2 = 2^{-N}u$ with $u \in (0,1]$. Since $u\cdot\frac{2^{-N}}{|J(2)|} \in (0,1]$, Lemma 2.2 yields

$$K\left(u\cdot\frac{2^{-N}}{|J(2)|}\right) \geq c_0 u^\alpha K\left(\frac{2^{-N}}{|J(2)|}\right).$$

Therefore,

$$\begin{aligned}\int_{R\cap Q_{J(2)}} \frac{(1-|w|^2)^{2b}}{K\left(\frac{1-|w|}{|J(2)|}\right)}dA(w) &\leq 2^{-N}\int_0^{2^{-N}} \frac{t^{2b}}{K\left(\frac{t}{|J(2)|}\right)}\,dt\\ &= 2^{-(2b+2)N}\int_0^1 \frac{u^{2b}}{K\left(u\cdot\frac{2^{-N}}{|J(2)|}\right)}\,du\\ &\leq \frac{2^{-(2b+2)N}}{c_0 K\left(\frac{2^{-N}}{|J(2)|}\right)}\int_0^1 u^{2b-\alpha}du\end{aligned}$$

$$= \frac{2^{-(2b+2)N}}{c_0(2b-\alpha+1)K\left(\frac{2^{-N}}{|J(2)|}\right)},$$

which converges since $b \gg 1$ implies $2b-\alpha+1>0$. Since $|J| \le |J(2)| \le 4|J|$, condition (K4) implies

$$K\left(\frac{2^{-N}}{|J|}\right) \le C_K^2 K\left(\frac{2^{-N}}{|J(2)|}\right).$$

Thus,

$$\begin{aligned}\mathcal{M}(R) &\le \frac{(b+1)^2}{(2^{-(b+2)N})^2} \cdot \left(2^{-2N} K\Big(\frac{2^{-N}}{|J|}\Big)\right) \cdot \left(\frac{2^{-(2b+2)N}}{c_0(2b-\alpha+1)K(2^{-N}/|J(2)|)}\right) \\ &\le \frac{(b+1)^2 C_K^2}{c_0(2b-\alpha+1)} \lesssim 1. \end{aligned} \tag{20}$$

Subcase (ii): $2^{-N} > |J|$. In this regime, the integration domain is restricted by the tent depth: on $R \cap Q_J$, $1-|z| \le |J| < 2^{-N}$, so

$$\int_{R\cap Q_J} K\left(\frac{1-|z|}{|J|}\right) dA(z) \le 2^{-N} \int_0^{|J|} K\left(\frac{t}{|J|}\right) dt = 2^{-N}|J| \int_0^1 K(u)\,du \le 2^{-N}|J|K(1).$$

Similarly, on $R \cap Q_{J(2)}$ the radial coordinate satisfies $t = 1-|w| \le |J(2)|$. Setting $t = |J(2)|u$ with $u \in (0,1]$, Lemma 2.2 applies directly to $K(u) \ge c_0 K(1) u^\alpha$ ($u \in (0,1]$):

$$\begin{aligned}\int_{R\cap Q_{J(2)}} \frac{(1-|w|^2)^{2b}}{K\left(\frac{1-|w|}{|J(2)|}\right)} dA(w) &\le 2^{-N} \int_0^{|J(2)|} \frac{(2t)^{2b}}{K(t/|J(2)|)}\,dt \\ &= 2^{-N} 2^{2b} |J(2)|^{2b+1} \int_0^1 \frac{u^{2b}}{K(u)}\,du \\ &\le \frac{2^{2b}4^{2b+1}}{c_0 K(1)(2b-\alpha+1)} \cdot 2^{-N}|J|^{2b+1}.\end{aligned}$$

Combining these estimates with $(A_b(R))^2 = 2^{-2(b+2)N}$ gives

$$\mathcal{M}(R) \le \frac{(b+1)^2 2^{6b+2}}{c_0(2b-\alpha+1)} \cdot \frac{2^{-2N}|J|^{2b+2}}{2^{-2(b+2)N}} = \frac{(b+1)^2 2^{6b+2}}{c_0(2b-\alpha+1)} \left(\frac{|J|}{2^{-N}}\right)^{2b+2} \lesssim 1, \tag{21}$$

since $\frac{|J|}{2^{-N}} < 1$.

From (20) and (21), $\mathcal{M}(R) \lesssim 1$ uniformly for all boundary tiles $R \in \mathcal{R}_N^{\mathrm{bd}}$.

Case 1.3: Combining the Tiles. For interior tiles, since $|J(2)| \le 4|J|$, condition (K4) gives $K\left(\frac{1-|w|}{|J|}\right) \le C_K^2 K\left(\frac{1-|w|}{|J(2)|}\right)$. Summing (17) and (19) over all tiles yields

$$\begin{aligned}&\int_{Q_J} |E_N^{(b)} F_0(z)|^2 K\left(\frac{1-|z|}{|J|}\right) dA(z) \\ &\le \sum_{R\in\mathcal{R}_N^{\mathrm{int}}} \left(\frac{1}{A_b(R)} \int_{R\cap Q_{J(2)}} |F(w)|\,dA_b(w)\right)^2 \int_{Q_J\cap R} K\left(\frac{1-|z|}{|J|}\right) dA(z)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{R\in\mathcal{R}_N^{\mathrm{bd}}}\left(\frac{1}{A_b(R)}\int_{R\cap Q_{J(2)}}|F(w)|\,dA_b(w)\right)^2\int_{Q_J\cap R}K\left(\frac{1-|z|}{|J|}\right)dA(z)\\
&\lesssim\sum_{R\in\mathcal{R}_N^{\mathrm{int}}}\int_R|F_0(w)|^2K\left(\frac{1-|w|}{|J|}\right)dA(w)+\sum_{R\in\mathcal{R}_N^{\mathrm{bd}}}\int_R|F_0(w)|^2K\left(\frac{1-|w|}{|J(2)|}\right)dA(w)\\
&\lesssim\sum_{R\in\mathcal{R}_N}\int_R|F_0(w)|^2K\left(\frac{1-|w|}{|J(2)|}\right)dA(w)\\
&=\int_{Q_{J(2)}}|F(w)|^2K\left(\frac{1-|w|}{|J(2)|}\right)dA(w)\le\|F\|_{T_K}^2\le 1.
\end{aligned}
$$

This completes the proof of (14).

The Geometric Tail Estimate (15). By Minkowski's integral inequality applied to the $L^2\left(Q_J, K\left(\frac{1-|z|}{|J|}\right)dA(z)\right)$ norm:

$$
\begin{aligned}
&\Big(\int_{Q_J}\Big|\sum_{n\ge2}E_N^{(b)}F_n(z)\Big|^2K\Big(\frac{1-|z|}{|J|}\Big)dA(z)\Big)^{1/2}\\
&\le\sum_{n\ge2}\Big(\int_{Q_J}|E_N^{(b)}F_n(z)|^2K\Big(\frac{1-|z|}{|J|}\Big)dA(z)\Big)^{1/2}. \qquad (22)
\end{aligned}
$$

Fix $z\in Q_J$, and let $R=R_{N,j,k}\in\mathcal{R}_N$ be the unique tile containing z. If $R\cap(Q_{J(n+1)}\setminus Q_{J(n)})=\emptyset$, then $E_N^{(b)}F_n(z)=0$. Otherwise, choose $w\in R\cap(Q_{J(n+1)}\setminus Q_{J(n)})$. By definition of $E_N^{(b)}$:

$$
\begin{aligned}
|E_N^{(b)}F_n(z)|&=\left|\frac{1}{A_b(R)}\int_{R\cap(Q_{J(n+1)}\setminus Q_{J(n)})}F_n(w)\,dA_b(w)\right|\\
&\le\frac{1}{A_b(R)}\int_{R\cap(Q_{J(n+1)}\setminus Q_{J(n)})}|F(w)|\,dA_b(w)\\
&\le\frac{1}{A_b(R)}\int_{R\cap Q_{J(n+1)}}|F(w)|\,dA_b(w).
\end{aligned}
$$

Applying the Cauchy–Schwarz inequality with the weight splitting $K^{1/2}\cdot K^{-1/2}$:

$$
\begin{aligned}
\int_{R\cap Q_{J(n+1)}}|F(w)|\,dA_b(w)\le c_b&\left(\int_{Q_{J(n+1)}}|F(w)|^2K\left(\frac{1-|w|}{|J(n+1)|}\right)dA(w)\right)^{1/2}\\
&\times\left(\int_{Q_{J(n+1)}}\frac{(1-|w|^2)^{2b}}{K\left(\frac{1-|w|}{|J(n+1)|}\right)}dA(w)\right)^{1/2}. \qquad (23)
\end{aligned}
$$

The first factor in (23) is bounded by $\|F\|_{T_K} = 1$. For the second factor, Lemma 2.1 yields for $b \gg 1$:

$$\left(\int_{Q_{J(n+1)}} \frac{(1-|w|^2)^{2b}}{K\left(\frac{1-|w|}{|J(n+1)|}\right)} \, dA(w) \right)^{1/2} \leq C_b |J(n+1)|^{b+1}. \tag{24}$$

Since $z \in Q_J$ and $w \in Q_{J(n+1)} \setminus Q_{J(n)}$, we have $\arg(z) \in J$ and $\arg(w) \in \mathbb{T} \setminus J(n)$. Because both z and w belong to the same polar dyadic tile $R = R_{N,j,k}$, their arguments lie in the angular interval $I_R = \left[\frac{2\pi k}{2^N}, \frac{2\pi(k+1)}{2^N}\right)$. Therefore, the angular width $\Delta\theta(R) = \frac{2\pi}{2^N}$ spans the angular separation:

$$2\pi \cdot 2^{-N} = \Delta\theta(R) \geq |\arg(z) - \arg(w)| \geq \operatorname{dist}(J, \mathbb{T} \setminus J(n)) = \frac{|J(n)| - |J|}{2} \gtrsim \frac{1}{4}|J(n)|.$$

This forces the dyadic scale to satisfy $2^{-N} \gtrsim |J(n)| \approx |J(n+1)|$, which implies $A(R) = 2^{-2N} \gtrsim |J(n)|^2$.

For every point $\zeta \in R$, since $z \in R \cap Q_J$, we have $1 - |z| \leq |J| \leq C \cdot 2^{-N}$, which locks the radial depth of the entire tile to $1 - |\zeta|^2 \approx 2^{-N}$ for all $\zeta \in R^{\text{int}}$. And when $\zeta \in R^{\text{bd}}$, we have $A_b(R) = c_b \int_R (1-|\zeta|^2)^b \, dA(\zeta) = c_b (2^{-N})^{b+2}$. Therefore,

$$A_b(R) = c_b \int_R (1-|\zeta|^2)^b \, dA(\zeta) \approx c_b (2^{-N})^b A(R) \gtrsim |J(n+1)|^b |J(n)|^2. \tag{25}$$

Substituting (24) and (25) into the pointwise definition, the high-power term $|J(n+1)|^b$ cancels completely:

$$|E_N^{(b)} F_n(z)| \lesssim \frac{|J(n+1)|^{b+1}}{|J(n)|^2 |J(n+1)|^b} = \frac{|J(n+1)|}{|J(n)|^2} \approx \frac{1}{\min\{2^n|J|, 1\}}, \quad \forall z \in Q_J. \tag{26}$$

Squaring (26) and integrating against the tent weight:

$$\begin{aligned} &\int_{Q_J} |E_N^{(b)} F_n(z)|^2 K\Big(\frac{1-|z|}{|J|}\Big) dA(z) \\ \lesssim &\frac{1}{(\min\{2^n|J|, 1\})^2} \int_{Q_J} K\Big(\frac{1-|z|}{|J|}\Big) dA(z) \lesssim \frac{|J|^2}{(\min\{2^n|J|, 1\})^2}, \end{aligned} \tag{27}$$

where the last inequality uses the evaluation:

$$\begin{aligned} \int_{Q_J} K\left(\frac{1-|z|}{|J|}\right) dA(z) &\lesssim |J| \int_{1-|J|}^1 K\left(\frac{1-t}{|J|}\right) dt = |J| \int_0^{|J|} K\left(\frac{u}{|J|}\right) du \\ &= |J|^2 \int_0^1 K(u)\, du \lesssim |J|^2. \end{aligned}$$

Let $n_0 := \lceil \log_2(1/|J|) \rceil$. For $n > n_0$ we have $2^n|J| \geq 1$, so $J(n) = \mathbb{T}$ and $F_n \equiv 0$. Substituting (27) into the Minkowski inequality (22) gives

$$\left(\int_{Q_J} \left| \sum_{n \geq 2} E_N^{(b)} F_n(z) \right|^2 K\left(\frac{1-|z|}{|J|}\right) dA(z) \right)^{1/2} \leq \sum_{n \geq 2} \left(\int_{Q_J} |E_N^{(b)} F_n(z)|^2 K\left(\frac{1-|z|}{|J|}\right) dA(z) \right)^{1/2}$$

$$
\begin{aligned}
&\lesssim \sum_{n\geq 2} \frac{|J|}{\min\{2^n|J|,1\}} \\
&= \sum_{\substack{n\geq 2\\ 2^n|J|\leq 1}} \frac{|J|}{2^n|J|} + \sum_{\substack{n\geq 2\\ 2^n|J|>1}} \frac{|J|}{1} \\
&\lesssim \sum_{2^n|J|\leq 1} 2^{-n} + |J| \lesssim 1. \qquad (28)
\end{aligned}
$$

Squaring both sides of (28) establishes (15).

Combining (14) and (15) completes the proof of Lemma 2.3. □

3. Higher-order weighted Bergman projections and surjectivity

Lemma 3.1 ([7, Lemma 2.3])**.** *Let K satisfy (K1)–(K4). Then for every $b \gg 1$ the weighted Bergman projection B_b acts boundedly on T_K, that is,*

$$\|B_b\psi\|_{T_K} \leq C_b\|\psi\|_{T_K}, \qquad \psi \in T_K.$$

We now introduce a projection adapted to the Q_K setting, which converts derivative information back into primitive functions. For a function $f \in Q_K$ the embedding $Q_K \subset \mathcal{B}$ (see [3, 15]) gives the growth bound

$$|f'(w)| \lesssim \frac{\|f\|_{Q_K,*}}{1-|w|^2}.$$

Consequently, for $b > 1$,

$$\int_{\mathbb{D}} |f'(w)|^2\, dA_b(w) \lesssim \|f\|^2_{Q_K,*} \int_{\mathbb{D}} (1-|w|^2)^{b-2}\, dA(w) < \infty, \qquad (29)$$

so $f' \in A^2(dA_b)$, the weighted Bergman space. Let

$$k_b(z,w) = \frac{1}{(1-z\bar{w})^{b+2}}$$

denote the reproducing kernel of $A^2(dA_b)$, and let B_b be the associated Bergman projection,

$$B_b g(z) = \int_{\mathbb{D}} g(w)\, k_b(z,w)\, dA_b(w).$$

Since $f' \in A^2(dA_b)$, the reproducing property gives $B_b f' = f'$. Integrating along radial paths and applying Fubini's theorem yields

$$f(z) - f(0) = \int_0^z f'(\xi)\, d\xi = \int_0^z (B_b f')(\xi)\, d\xi = \int_{\mathbb{D}} f'(w) \Big(\int_0^z k_b(\xi,w)\, d\xi \Big)\, dA_b(w).$$

This computation motivates the following definition.

Definition 3.2. For a fixed $b \gg 1$, the integrated Bergman kernel is

$$L_b(z,w) = \int_0^z k_b(\xi,w)\, d\xi = \frac{1}{\bar{w}} \Big(\frac{1}{(1-z\bar{w})^{b+1}} - 1 \Big) \frac{1}{b+1},$$

and the reduced weighted Bergman projection is

$$P_{b,0}\psi(z) = \int_{\mathbb{D}} \psi(w)\, L_b(z,w)\, dA_b(w),$$

defined for measurable functions ψ for which the integral converges.

The kernel $L_b(\cdot, w)$ is holomorphic on $\mathbb{D}$ for each w, and the singularity at $w = 0$ is removable with $L_b(z,0) = z$. The operator $P_{b,0}$ is well defined on $L^1(dA_b)$: for $z \in \mathbb{D}$,

$$|P_{b,0}\psi(z)| \leq \int_{\mathbb{D}} |\psi(w)| \Big(\int_0^{|z|} \frac{d|\xi|}{|1-\xi\bar{w}|^{b+2}} \Big)\, dA_b(w) \leq \frac{|z|}{(1-|z|)^{b+2}}\, \|\psi\|_{L^1(dA_b)}. \tag{30}$$

Lemma 3.3. *For $\psi \in L^1(dA_b)$ we have*

$$P_{b,0}\psi(z) = \int_0^z B_b\psi(\zeta)\, d\zeta, \qquad z \in \mathbb{D},$$

and consequently

$$(P_{b,0}\psi)' = B_b\psi, \qquad P_{b,0}\psi(0) = 0.$$

Proof. Differentiating $L_b(\cdot, w)$ with respect to the first variable gives $\partial_z L_b(z,w) = k_b(z,w)$. Interchanging differentiation and integration, which is justified by (30), yields the claim. $\square$

To justify applying Lemma 3.3 to functions in T_K, we establish the embedding chain $T_K \hookrightarrow L^2(dA_b) \hookrightarrow L^1(dA_b)$ for large b.

Lemma 3.4. *Let $K \in \mathcal{W}_Q$ and let $\alpha = \log_2 C_K$. For b sufficiently large (in particular, for $b \gg 1$), we have $b > \alpha$, then*

$$T_K \hookrightarrow L^2(dA_b) \hookrightarrow L^1(dA_b),$$

and there is a constant $C > 0$ such that for every $\psi \in T_K$,

$$\|\psi\|_{L^1(dA_b)} \leq \|\psi\|_{L^2(dA_b)} \leq C\|\psi\|_{T_K}. \tag{31}$$

Proof. Taking $I = \mathbb{T}$ in (4) gives

$$\int_{\mathbb{D}} |\psi(z)|^2 K(1-|z|)\, dA(z) \leq \|\psi\|_{T_K}^2.$$

Using $1-|z|^2 = (1-|z|)(1+|z|) \leq 2(1-|z|)$ and Lemma 2.2 with $t = 1-|z|$, we get

$$\frac{(1-|z|^2)^b}{K(1-|z|)} \leq \frac{2^b(1-|z|)^b}{K(1-|z|)} \leq \frac{2^b}{c_0 K(1)}\,(1-|z|)^{b-\alpha} \leq \frac{2^b}{c_0 K(1)},$$

because $b > \alpha$. Therefore

$$\begin{aligned}
\|\psi\|_{L^2(dA_b)}^2 &= C_b \int_{\mathbb{D}} |\psi(z)|^2 K(1-|z|) \cdot \frac{(1-|z|^2)^b}{K(1-|z|)}\, dA(z) \\
&\leq C_b \cdot \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^b}{K(1-|z|)} \int_{\mathbb{D}} |\psi(z)|^2 K(1-|z|)\, dA(z)
\end{aligned}$$

$$\leq C_b \cdot \frac{2^b}{C_0 K(1)} \int_{\mathbb{D}} |\psi(z)|^2 K(1-|z|)\, dA(z) \lesssim \|\psi\|_{T_K}^2.$$

Since dA_b is a probability measure, with total mass

$$A_b(\mathbb{D}) = c_b \int_0^1 (1-r^2)^b\, 2r\, dr = \frac{c_b}{b+1} = 1,$$

the Cauchy–Schwarz inequality gives $\|\psi\|_{L^1(dA_b)} \leq \|\psi\|_{L^2(dA_b)}$. This proves (31). $\square$

Proposition 3.5. *Let $K \in \mathcal{W}_{\mathcal{Q}}$ and fix $b \gg 1$. The map*

$$P_{b,0} \colon T_K \to \mathring{Q}_K$$

is a bounded and surjective linear operator. More precisely,

$$\|P_{b,0}\psi\|_{Q_K,*} \leq C\|\psi\|_{T_K}, \qquad \psi \in T_K,$$

and every $f \in \mathring{Q}_K$ can be written as $f = P_{b,0}\psi$ with $\psi = f'$, for which $\|\psi\|_{T_K} \asymp \|f\|_{Q_K,*}$.

Proof. By Lemma 3.4 we have $T_K \subset L^2(dA_b) \subset L^1(dA_b)$ for $b \gg 1$, so Lemma 3.3 applies to every $\psi \in T_K$:

$$(P_{b,0}\psi)' = B_b\psi, \qquad P_{b,0}\psi(0) = 0. \tag{32}$$

Combining (32) with Lemma 3.1 and the identification (5) yields

$$\|P_{b,0}\psi\|_{Q_K,*} \asymp \|(P_{b,0}\psi)'\|_{T_K} = \|B_b\psi\|_{T_K} \leq C_b\|\psi\|_{T_K},$$

so $P_{b,0}$ maps T_K boundedly into $\mathring{Q}_K$.

For surjectivity, fix $f \in \mathring{Q}_K$ and set $\psi = f'$. By (5), $\psi \in T_K$ with $\|\psi\|_{T_K} \asymp \|f\|_{Q_K,*}$. Since $b \gg 1$, the estimate (29) gives $\psi \in L^2(dA_b)$, and the reproducing property of the Bergman kernel gives $B_b\psi = \psi$. Hence

$$P_{b,0}\psi(z) = \int_0^z B_b\psi(\zeta)\, d\zeta = \int_0^z \psi(\zeta)\, d\zeta = f(z) - f(0) = f(z),$$

since $f(0) = 0$. This proves surjectivity. $\square$

We can now define the Q_K packet matrix. Let $\mathcal{R} = \{\mathcal{R}_N\}_{N\geq 0}$ be the polar dyadic resolution of $\mathbb{D}$. For each tile $R \in \mathcal{R}_N$, the associated packet is

$$\Psi_R^{(b)}(z) = P_{b,0}\mathbf{1}_R(z) = \int_R L_b(z,w)\, dA_b(w).$$

For a positive Borel measure μ on $\mathbb{D}$, the Q_K packet matrix at scale N is defined entrywise by

$$\Gamma_{\mu,N}^{(b)}(R_1,R_2) = \int_{\mathbb{D}} \Psi_{R_1}^{(b)}(z)\, \overline{\Psi_{R_2}^{(b)}(z)}\, d\mu(z), \qquad R_1, R_2 \in \mathcal{R}_N.$$

4. Boundedness for the Q_K–Carleson problem: Part I

For a sequence of coefficients $c = \{c_R\}_{R\in\mathcal{R}_N}$, we introduce the discrete norm

$$\|c\|_{X_{K,N}}^2 = \Big\|\sum_{R\in\mathcal{R}_N} c_R\,\mathbf{1}_R\Big\|_{T_K}^2 = \sup_{I\subseteq\mathbb{T}}\sum_{R\in\mathcal{R}_N}|c_R|^2\int_{R\cap Q_I} K\Big(\frac{1-|z|}{|I|}\Big)\,dA(z).$$

Definition 4.1. Let μ be a positive Borel measure on $\mathbb{D}$. The discrete dyadic capacity associated with the embedding problem is

$$C^{(b)}_{K,\mathcal{R}}(\mu) = \sup_{N\geq 0}\sup_{\|c\|_{X_{K,N}}\leq 1}\int_{\mathbb{D}}\Big|\sum_{R\in\mathcal{R}_N} c_R\,\Psi^{(b)}_R(z)\Big|^2\,d\mu(z) = \sup_{N\geq 0}\sup_{\|c\|_{X_{K,N}}\leq 1} c^*\Gamma^{(b)}_{\mu,N}c.$$

For later use we record the optimal embedding constant

$$C_{\mathring{Q}_K}(\mu) = \sup\Big\{\int_{\mathbb{D}}|f(z)|^2\,d\mu(z) : f\in\mathring{Q}_K,\ \|f\|_{Q_K,*}\leq 1\Big\}, \tag{33}$$

so that (6) holds with $C = C_{\mathring{Q}_K}(\mu)$ as the optimal constant.

Theorem 4.2. *Let $K\in\mathcal{W}_{\mathcal{Q}}$ and let μ be a finite positive Borel measure on $\mathbb{D}$. The trace embedding inequality*

$$\int_{\mathbb{D}}|f(z)|^2\,d\mu(z)\leq C\|f\|_{Q_K,*}^2, \qquad f\in\mathring{Q}_K, \tag{34}$$

holds if and only if for any fixed $b\gg 1$, we have $C^{(b)}_{K,\mathcal{R}}(\mu)<\infty$. Moreover,

$$C^{(b)}_{K,\mathcal{R}}(\mu)\asymp C_{\mathring{Q}_K}(\mu).$$

Proof. First assume that $C^{(b)}_{K,\mathcal{R}}(\mu)<\infty$. We prove that

$$\int_{\mathbb{D}}|P_{b,0}\psi(z)|^2\,d\mu(z)\leq C\,C^{(b)}_{K,\mathcal{R}}(\mu)\,\|\psi\|_{T_K}^2, \qquad \psi\in T_K. \tag{35}$$

By homogeneity we may assume $\|\psi\|_{T_K}\leq 1$. Write $E^{(b)}_N\psi = \sum_{R\in\mathcal{R}_N} a^{(N)}_R\mathbf{1}_R$. Lemma 2.3 gives

$$\|a^{(N)}\|_{X_{K,N}} = \|E^{(b)}_N\psi\|_{T_K}\lesssim\|\psi\|_{T_K}\leq 1.$$

By Definition 4.1 and scaling,

$$\int_{\mathbb{D}}|P_{b,0}\big(E^{(b)}_N\psi\big)(z)|^2\,d\mu(z) = \int_{\mathbb{D}}\Big|\sum_{R\in\mathcal{R}_N} a^{(N)}_R\Psi^{(b)}_R(z)\Big|^2\,d\mu(z)\lesssim C^{(b)}_{K,\mathcal{R}}(\mu). \tag{36}$$

Since $E^{(b)}_N\psi\to\psi$ in $L^1(dA_b)$, and since for each fixed $z\in\mathbb{D}$ the integrated kernel $w\mapsto L_b(z,w)$ is bounded and continuous on $\overline{\mathbb{D}}$, we have the pointwise convergence $P_{b,0}(E^{(b)}_N\psi)(z)\to P_{b,0}\psi(z)$. Fatou's lemma applied to (36) yields

$$\int_{\mathbb{D}}|P_{b,0}\psi(z)|^2\,d\mu(z)\leq\liminf_{N\to\infty}\int_{\mathbb{D}}\Big|P_{b,0}(E^{(b)}_N\psi)(z)\Big|^2\,d\mu(z)\lesssim C^{(b)}_{K,\mathcal{R}}(\mu),$$

which establishes (35).

Now fix $f \in \mathring{Q}_K$. By Proposition 3.5 there exists $\psi \in T_K$ with $f = P_{b,0}\psi$ and $\|\psi\|_{T_K} \lesssim \|f\|_{Q_K,*}$. Applying (35) gives

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \lesssim C^{(b)}_{K,\mathcal{R}}(\mu) \|\psi\|^2_{T_K} \lesssim C^{(b)}_{K,\mathcal{R}}(\mu) \|f\|^2_{Q_K,*},$$

so $C_{\mathring{Q}_K}(\mu) \lesssim C^{(b)}_{K,\mathcal{R}}(\mu)$.

Conversely, suppose that (34) holds. Fix $N \geq 0$ and a coefficient vector $c = \{c_R\}$ with $\|c\|_{X_{K,N}} \leq 1$, and set $\psi_c = \sum_{R \in \mathcal{R}_N} c_R \mathbf{1}_R$. Then $\|\psi_c\|_{T_K} \leq 1$, and Proposition 3.5 gives $\|P_{b,0}\psi_c\|_{Q_K,*} \lesssim 1$. Hence

$$\int_{\mathbb{D}} \Big| \sum_{R \in \mathcal{R}_N} c_R \Psi^{(b)}_R(z) \Big|^2 d\mu(z) = \int_{\mathbb{D}} |P_{b,0}\psi_c(z)|^2 \, d\mu(z) \lesssim C_{\mathring{Q}_K}(\mu).$$

Taking the supremum over all admissible c and all $N \geq 0$ gives $C^{(b)}_{K,\mathcal{R}}(\mu) \lesssim C_{\mathring{Q}_K}(\mu)$, which completes the proof. □

5. Boundedness for the Q_K–Carleson problem: Part II

For each boundary interval $I \subseteq \mathbb{T}$, define the diagonal configuration matrix

$$A^{(K)}_{I,N} = \operatorname{diag}\Big(\int_{R \cap Q_I} K\Big(\frac{1-|z|}{|I|}\Big) \, dA(z) : R \in \mathcal{R}_N \Big).$$

In this notation the constraint $\|c\|_{X_{K,N}} \leq 1$ is the quadratic matrix inequality

$$c^* A^{(K)}_{I,N} c \leq 1, \qquad I \subseteq \mathbb{T}. \tag{37}$$

Definition 5.1. Let μ be a finite positive Borel measure on $\mathbb{D}$. The dual semidefinite-programming capacity $D^{(b)}_{K,\mathcal{R}}(\mu)$ is

$$D^{(b)}_{K,\mathcal{R}}(\mu) = \sup_{N \geq 0} \inf \Big\{ \sum_{m=1}^{M} \lambda_m : M \in \mathbb{N}, \ \lambda_m \geq 0, \ I_m \subseteq \mathbb{T}, \ \Gamma^{(b)}_{\mu,N} \leq \sum_{m=1}^{M} \lambda_m A^{(K)}_{I_m,N} \Big\}, \tag{38}$$

where the matrix domination is understood in the cone $\mathcal{S}_+^{\#\mathcal{R}_N}$ of Hermitian positive semi-definite matrices. See [14] for more information.

Remark 5.2. In (37) and (38) it suffices to range over dyadic arcs $I \subseteq \mathbb{T}$ (arcs of the form $\{e^{i\theta} : 2\pi k/2^n \leq \theta < 2\pi(k+1)/2^n\}$, together with $\mathbb{T}$ itself). Indeed, every arc I is contained in a dyadic arc I' with $|I'| \leq 2|I|$ and $Q_I \subset Q_{I'}$, and the doubling property gives

$$\sum_{R \in \mathcal{R}_N} |c_R|^2 \int_{R \cap Q_I} K\Big(\frac{1-|z|}{|I|}\Big) \, dA(z) \leq C_K \sum_{R \in \mathcal{R}_N} |c_R|^2 \int_{R \cap Q_{I'}} K\Big(\frac{1-|z|}{|I'|}\Big) \, dA(z).$$

Hence the constraints over all arcs are equivalent to the countable family over dyadic arcs, and the semidefinite program in (38) involves only countably many constraints.

Theorem 5.3. *Let $K \in \mathcal{W}_{\mathcal{Q}}$ and let μ be a finite positive Borel measure on $\mathbb{D}$. Then*

$$C^{(b)}_{K,\mathcal{R}}(\mu) \asymp D^{(b)}_{K,\mathcal{R}}(\mu).$$

Proof. For a fixed scale $N \geq 0$, set

$$\begin{aligned} C^{(b)}_{K,\mathcal{R},N}(\mu) &= \sup_{\|c\|_{X_{K,N}} \leq 1} c^* \Gamma^{(b)}_{\mu,N} c, \\ \delta^{(b)}_{K,\mathcal{R},N}(\mu) &= \inf\Big\{ \sum_{m=1}^{M} \lambda_m : \lambda_m \geq 0,\ I_m \subseteq \mathbb{T},\ \Gamma^{(b)}_{\mu,N} \leq \sum_{m=1}^{M} \lambda_m A^{(K)}_{I_m,N} \Big\}. \end{aligned} \tag{39}$$

Then $C^{(b)}_{K,\mathcal{R}}(\mu) = \sup_N C^{(b)}_{K,\mathcal{R},N}(\mu)$ and $D^{(b)}_{K,\mathcal{R}}(\mu) = \sup_N \delta^{(b)}_{K,\mathcal{R},N}(\mu)$.

Step 1: $C^{(b)}_{K,\mathcal{R}}(\mu) \lesssim D^{(b)}_{K,\mathcal{R}}(\mu)$. Assume that $\Gamma^{(b)}_{\mu,N} \leq \sum_{m=1}^{M} \lambda_m A^{(K)}_{I_m,N}$ for some $M \in \mathbb{N}$, $\lambda_m \geq 0$, and arcs $I_m \subseteq \mathbb{T}$. For any $c \in \mathbb{C}^{\#\mathcal{R}_N}$ with $\|c\|_{X_{K,N}} \leq 1$ we have $c^* A^{(K)}_{I_m,N} c \leq 1$ for each m, hence

$$c^* \Gamma^{(b)}_{\mu,N} c \leq \sum_{m=1}^{M} \lambda_m \big(c^* A^{(K)}_{I_m,N} c\big) \leq \sum_{m=1}^{M} \lambda_m.$$

Taking the supremum over c and then the infimum over feasible tuples gives $C^{(b)}_{K,\mathcal{R},N}(\mu) \leq \delta^{(b)}_{K,\mathcal{R},N}(\mu)$, and taking the supremum over N gives the claim.

Step 2: $D^{(b)}_{K,\mathcal{R}}(\mu) \lesssim C^{(b)}_{K,\mathcal{R}}(\mu)$. If $C^{(b)}_{K,\mathcal{R}}(\mu) = +\infty$ there is nothing to prove, so assume $C^{(b)}_{K,\mathcal{R}}(\mu) < \infty$.

(i) Conic duality. Consider the primal program defining $\delta^{(b)}_{K,\mathcal{R},N}(\mu)$ in (39). Introducing a Lagrange multiplier matrix $X \in \mathcal{S}^{\#\mathcal{R}_N}_+$, the Lagrangian is

$$\begin{aligned} L(\lambda, X) &= \sum_{m=1}^{M} \lambda_m - \operatorname{Tr}\Big(X\Big(\sum_{m=1}^{M} \lambda_m A^{(K)}_{I_m,N} - \Gamma^{(b)}_{\mu,N}\Big)\Big) \\ &= \operatorname{Tr}\big(\Gamma^{(b)}_{\mu,N} X\big) + \sum_{m=1}^{M} \lambda_m \Big(1 - \operatorname{Tr}\big(A^{(K)}_{I_m,N} X\big)\Big). \end{aligned}$$

Minimizing over $\lambda_m \geq 0$ yields a finite value exactly when

$$\operatorname{Tr}\big(A^{(K)}_{I,N} X\big) \leq 1, \qquad I \subseteq \mathbb{T}. \tag{40}$$

Slater's condition holds: choosing a uniform multiplier $\lambda_m = M_0 \gg 1$ over a finite collection of arcs containing $\mathbb{T}$ makes $\sum_m \lambda_m A^{(K)}_{I_m,N} - \Gamma^{(b)}_{\mu,N}$ strictly positive definite, since $A^{(K)}_{\mathbb{T},N}$ has strictly positive diagonal entries. Hence strong conic duality holds with zero duality gap [14], and

$$\delta^{(b)}_{K,\mathcal{R},N}(\mu) = \sup\Big\{ \operatorname{Tr}\big(\Gamma^{(b)}_{\mu,N} X\big) : X \geq 0,\ \operatorname{Tr}\big(A^{(K)}_{I,N} X\big) \leq 1 \text{ for all } I \subseteq \mathbb{T} \Big\}. \tag{41}$$

(ii) Factorization and the Grothendieck inequality. Let $X \in \mathcal{S}^{\#\mathcal{R}_N}_+$ satisfy (40). Since $X \geq 0$, it is a Gram matrix: there exist vectors $\{u_R\}_{R \in \mathcal{R}_N} \subset \mathbb{C}^{\#\mathcal{R}_N}$ such that

$$X(R_2, R_1) = \langle u_{R_1}, u_{R_2} \rangle. \tag{42}$$

Set $x_R = \|u_R\|$, and let $v_R = u_R/x_R$ when $x_R > 0$ (and v_R an arbitrary unit vector otherwise). Evaluating the trace against the diagonal matrix $A^{(K)}_{I,N}$ gives

$$\sum_{R\in\mathcal{R}_N} x_R^2 \int_{R\cap Q_I} K\Big(\frac{1-|z|}{|I|}\Big)\, dA(z) = \mathrm{Tr}\big(A^{(K)}_{I,N}X\big) \le 1,$$

so the nonnegative sequence $x = \{x_R\}$ satisfies $\|x\|_{X_{K,N}} \le 1$. Define the auxiliary matrix

$$\Lambda^{(b)}_{\mu,N}(R_1,R_2) = \Gamma^{(b)}_{\mu,N}(R_1,R_2)\, x_{R_1}x_{R_2}.$$

Since $\Gamma^{(b)}_{\mu,N}$ is Hermitian positive semidefinite, its Schur (Hadamard) product $\Lambda^{(b)}_{\mu,N}$ is also Hermitian positive semidefinite. Using (42), the dual objective expands as

$$\begin{aligned}\mathrm{Tr}\big(\Gamma^{(b)}_{\mu,N}X\big) &= \sum_{R_1,R_2\in\mathcal{R}_N} \Gamma^{(b)}_{\mu,N}(R_1,R_2)\langle u_{R_1}, u_{R_2}\rangle \\ &= \sum_{R_1,R_2\in\mathcal{R}_N} \Lambda^{(b)}_{\mu,N}(R_1,R_2)\langle v_{R_1}, v_{R_2}\rangle. \end{aligned} \tag{43}$$

The complex Grothendieck inequality [12] gives

$$\Big|\sum_{R_1,R_2} \Lambda^{(b)}_{\mu,N}(R_1,R_2)\langle v_{R_1}, v_{R_2}\rangle\Big| \le K^G_{\mathbb{C}} \sup_{|\alpha_R|\le 1, |\beta_R|\le 1}\Big|\sum_{R_1,R_2} \Lambda^{(b)}_{\mu,N}(R_1,R_2)\alpha_{R_1}\overline{\beta}_{R_2}\Big|, \tag{44}$$

where $K^G_{\mathbb{C}}$ is the complex Grothendieck constant. Since $\Lambda^{(b)}_{\mu,N}$ is positive semidefinite, writing it in its spectral decomposition and by the Cauchy–Schwarz inequality for positive semidefinite sesquilinear forms shows that :

$$\sup_{|\alpha_R|\le 1, |\beta_R|\le 1}\Big|\sum_{R_1,R_2} \Lambda^{(b)}_{\mu,N}(R_1,R_2)\alpha_{R_1}\beta_{R_2}\Big| \le \sup_{|\alpha_R|\le 1}\sum_{R_1,R_2} \Lambda^{(b)}_{\mu,N}(R_1,R_2)\alpha_{R_1}\overline{\alpha_{R_2}}. \tag{45}$$

For a sequence $\alpha = \{\alpha_R\}$ with $|\alpha_R| \le 1$, set $c_R = x_R\overline{\alpha_R}$. Then $|c_R| \le x_R$, and since $\|\cdot\|_{X_{K,N}}$ depends only on the absolute values of the coefficients, $\|c\|_{X_{K,N}} \le \|x\|_{X_{K,N}} \le 1$. Expanding the quadratic form gives

$$\begin{aligned}\sum_{R_1,R_2} \Lambda^{(b)}_{\mu,N}(R_1,R_2)\alpha_{R_1}\overline{\alpha_{R_2}} &= \sum_{R_1,R_2} \Gamma^{(b)}_{\mu,N}(R_1,R_2)(x_{R_1}\alpha_{R_1})(x_{R_2}\overline{\alpha_{R_2}}) \\ &= c^*\Gamma^{(b)}_{\mu,N}c \le C^{(b)}_{K,\mathcal{R},N}(\mu). \end{aligned} \tag{46}$$

Combining (43), (44), (45), and (46) yields

$$\mathrm{Tr}\big(\Gamma^{(b)}_{\mu,N}X\big) \le K^G_{\mathbb{C}}\, C^{(b)}_{K,\mathcal{R},N}(\mu).$$

Taking the supremum over all feasible X in (41) gives $\delta^{(b)}_{K,\mathcal{R},N}(\mu) \le K^G_{\mathbb{C}}\, C^{(b)}_{K,\mathcal{R},N}(\mu)$, and taking the supremum over N completes the proof. □

Theorem 5.4. *Let $K \in \mathcal{W}_{\mathcal{Q}}$ and μ be a positive Borel measure on $\mathbb{D}$. The embedding*

$$\mathrm{id}\colon Q_K \to L^2(\mu)$$

is bounded if and only if for any fixed $b \gg 1$,

$$\mu(\mathbb{D}) + D^{(b)}_{K,\mathcal{R}}(\mu) < \infty.$$

Proof. Write $f \in Q_K$ as $f = f(0) + f_0$ with $f_0 = f - f(0) \in \mathring{Q}_K$. Since

$$\|f\|^2_{Q_K} = (|f(0)| + \|f\|_{Q_K,*})^2 \asymp |f(0)|^2 + \|f\|^2_{Q_K,*},$$

we have

$$\begin{aligned}\int_{\mathbb{D}} |f|^2\, d\mu &\le 2|f(0)|^2\mu(\mathbb{D}) + 2\int_{\mathbb{D}} |f_0|^2\, d\mu \\ &\le 2\mu(\mathbb{D})|f(0)|^2 + 2C_{\mathring{Q}_K}(\mu)\|f_0\|^2_{Q_K,*} \\ &\lesssim \big(\mu(\mathbb{D}) + C_{\mathring{Q}_K}(\mu)\big)\|f\|^2_{Q_K}.\end{aligned}$$

Thus id is bounded whenever $\mu(\mathbb{D}) < \infty$ and $C_{\mathring{Q}_K}(\mu) < \infty$.

Conversely, testing with the constant function $f \equiv 1$ gives $\mu(\mathbb{D}) \le \|\mathrm{id}\|^2$, and restricting to $\mathring{Q}_K$ gives $C_{\mathring{Q}_K}(\mu) \le \|\mathrm{id}\|^2$. The result now follows from Theorems 4.2 and 5.3, which give $C_{\mathring{Q}_K}(\mu) \asymp D^{(b)}_{K,\mathcal{R}}(\mu)$. The proof is complete. ☐

6. Compactness for the Q_K–Carleson problem

For $\rho \in (0,1)$ let $S_\rho = \{z \in \mathbb{D} : |z| > \rho\}$ denote the boundary shell, and let $K_\rho = \{z \in \mathbb{D} : |z| \le \rho\}$ denote the inner disc.

Lemma 6.1. *The embedding* $\mathrm{id}\colon \mathring{Q}_K \to L^2(\mu)$ *is compact if and only if whenever* $\{f_n\}$ *is bounded in* $\mathring{Q}_K$ *and* $f_n \to 0$ *uniformly on compact subsets of* $\mathbb{D}$*, one has* $\|f_n\|_{L^2(\mu)} \to 0$*.*

Proof. Suppose first that $\mathrm{id}\colon \mathring{Q}_K \to L^2(\mu)$ is compact. Let $\{f_n\}$ be a bounded sequence in $\mathring{Q}_K$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By compactness, the sequence $\{f_n\}$ is relatively compact in $L^2(\mu)$. If $\|f_n\|_{L^2(\mu)} \not\to 0$, there exists a subsequence $\{f_{n_k}\}$ and $\delta_0 > 0$ such that $\|f_{n_k}\|_{L^2(\mu)} \ge \delta_0$. Passing to a further subsequence if necessary, we may assume that $f_{n_k} \to f$ in $L^2(\mu)$ for some $f \in L^2(\mu)$. Since $f_{n_k} \to 0$ pointwise on $\mathbb{D}$, we must have $f = 0$ almost everywhere with respect to μ, which contradicts $\|f\|_{L^2(\mu)} \ge \delta_0$. Thus $\|f_n\|_{L^2(\mu)} \to 0$.

Conversely, let $\{f_n\}$ be any bounded sequence in $\mathring{Q}_K$. By Montel's theorem, there is a subsequence $\{f_{n_k}\}$ that converges uniformly on compact subsets of $\mathbb{D}$ to some holomorphic function $f \in \mathring{Q}_K$. Then $\{f_{n_k} - f\}$ is bounded in $\mathring{Q}_K$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. The hypothesis implies $\|f_{n_k} - f\|_{L^2(\mu)} \to 0$, which proves that the unit ball of $\mathring{Q}_K$ is relatively compact in $L^2(\mu)$, hence id is compact. ☐

Theorem 6.2. *Let* $K \in \mathcal{W}_{\mathcal{Q}}$ *and let* μ *be a finite positive Borel measure on* $\mathbb{D}$*, and let* $\mathcal{R} = \{\mathcal{R}_N\}_{N\ge 0}$ *be the polar dyadic resolution defined in Section 2. Then*

$$\mathrm{id}\colon \mathring{Q}_K \to L^2(\mu)$$

is compact if and only if for fixed $b \gg 1$,

$$C^{(b)}_{K,\mathcal{R}}(\mu) < \infty \quad \textit{and} \quad \lim_{\rho \to 1^-} C^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_\rho}\mu) = 0.$$

Equivalently, this holds if and only if

$$D^{(b)}_{K,\mathcal{R}}(\mu) < \infty \quad \textit{and} \quad \lim_{\rho \to 1^-} D^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_\rho}\mu) = 0.$$

Proof. By Theorem 5.3, $C^{(b)}_{K,\mathcal{R}} \asymp D^{(b)}_{K,\mathcal{R}}$ and by Theorem 4.2, $C^{(b)}_{K,\mathcal{R}} \asymp C_{\mathring{Q}_K}$, so it suffices to prove the equivalence using $C_{\mathring{Q}_K}$.

Sufficiency. Suppose $C_{\mathring{Q}_K}(\mu) < \infty$ and $\lim_{\rho\to 1^-} C_{\mathring{Q}_K}(\mathbf{1}_{S_\rho}\mu) = 0$. By Lemma 6.1, to prove compactness it suffices to show that for any sequence $\{f_n\}$ bounded in $\mathring{Q}_K$ (say $\|f_n\|_{Q_K,*} \leq M$) with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, we have $\|f_n\|_{L^2(\mu)} \to 0$.

Let $\epsilon > 0$. Choose $\rho_0 \in (0,1)$ such that $C_{\mathring{Q}_K}(\mathbf{1}_{S_{\rho_0}}\mu) < \frac{\epsilon^2}{2M^2}$. Then

$$\int_{S_{\rho_0}} |f_n(z)|^2 \, d\mu(z) \leq C_{\mathring{Q}_K}(\mathbf{1}_{S_{\rho_0}}\mu) \, \|f_n\|^2_{Q_K,*} < \frac{\epsilon^2}{2M^2} \cdot M^2 = \frac{\epsilon^2}{2}. \tag{47}$$

On the inner disc K_{ρ_0}, the uniform convergence $f_n \to 0$ and $\mu(K_{\rho_0}) \leq \mu(\mathbb{D}) < \infty$ yield an index N_0 such that for all $n \geq N_0$,

$$\int_{K_{\rho_0}} |f_n(z)|^2 \, d\mu(z) \leq \Big(\sup_{z \in K_{\rho_0}} |f_n(z)|^2 \Big) \mu(\mathbb{D}) \leq \frac{\epsilon^2}{2}. \tag{48}$$

Combining (47) and (48) gives $\|f_n\|^2_{L^2(\mu)} \leq \epsilon^2$ for all $n \geq N_0$. By Lemma 6.1, $\mathrm{id}\colon \mathring{Q}_K \to L^2(\mu)$ is compact.

Necessity. Suppose that $\mathrm{id}\colon \mathring{Q}_K \to L^2(\mu)$ is compact. Then it is bounded, so $C^{(b)}_{K,\mathcal{R}}(\mu) < \infty$ by Theorem 4.2. Assume for a contradiction that $\lim_{\rho\to 1^-} C^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_\rho}\mu) \neq 0$. Since $\rho \mapsto C^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_\rho}\mu)$ is nonincreasing, there exist $\epsilon_0 > 0$ and radii $\rho_n \to 1^-$ with

$$C^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_{\rho_n}}\mu) \geq \epsilon_0, \qquad n \geq 1.$$

By Theorem 4.2, $C_{\mathring{Q}_K}(\mathbf{1}_{S_{\rho_n}}\mu) \geq c\epsilon_0 > 0$, so there exist functions $f_n \in \mathring{Q}_K$ with $\|f_n\|_{Q_K,*} \leq 1$ and

$$\int_{S_{\rho_n}} |f_n(z)|^2 \, d\mu(z) \geq \frac{c\epsilon_0}{2} > 0, \qquad n \geq 1. \tag{49}$$

By Montel's theorem, a subsequence converges uniformly on compact subsets to a function $f \in \mathring{Q}_K$. Since the embedding is compact, Lemma 6.1 implies $\|f_n - f\|_{L^2(\mu)} \to 0$ along this subsequence. By Minkowski's inequality,

$$\Big(\int_{S_{\rho_n}} |f_n|^2 \, d\mu \Big)^{1/2} \leq \|f_n - f\|_{L^2(\mu)} + \Big(\int_{S_{\rho_n}} |f|^2 \, d\mu \Big)^{1/2}. \tag{50}$$

The first term on the right tends to 0. Since $f \in L^2(\mu)$, dominated convergence gives $\lim_{n\to\infty} \int_{S_{\rho_n}} |f(z)|^2 \, d\mu(z) = 0$. Hence (50) forces $\int_{S_{\rho_n}} |f_n|^2 \, d\mu \to 0$, which contradicts (49). Therefore $\lim_{\rho\to 1^-} C^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_\rho}\mu) = 0$. $\square$

Consequently, for a positive Borel measure μ on $\mathbb{D}$,

$$\mathrm{id}\colon Q_K \to L^2(\mu)$$

is compact if and only if

$$\mu(\mathbb{D}) + D^{(b)}_{K,\mathcal{R}}(\mu) < +\infty \quad \text{and} \quad \lim_{\rho\to 1^-} D^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_\rho}\mu) = 0.$$

This completes the compactness characterization for the Q_K–Carleson measure problem.

7. Volterra integral operators and multipliers on Q_K

Let $g \in H(\mathbb{D})$. Recall that the multiplication operator M_g and the Volterra-type integral operators T_g, I_g act on $f \in H(\mathbb{D})$ by

$$M_g f(z) = g(z)f(z), \qquad T_g f(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta, \qquad I_g f(z) = \int_0^z f'(\zeta)g(\zeta)\,d\zeta.$$

Integration by parts immediately yields the operator relation

$$M_g f(z) = f(0)g(z) + T_g f(z) + I_g f(z). \tag{51}$$

For $a \in \mathbb{D}$, recall the associated measure

$$d\mu_{g,a,K}(z) = |g'(z)|^2\, K\big(1 - |\varphi_a(z)|^2\big)\, dA(z).$$

By (8) and (2),

$$\sup_{a\in\mathbb{D}} \mu_{g,a,K}(\mathbb{D}) = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)|^2 K\big(1 - |\varphi_a(z)|^2\big)\, dA(z) = \|g\|^2_{Q_K,*}. \tag{52}$$

Theorem 7.1. *Let $K \in \mathcal{W}_{\mathcal{Q}}$ and $g \in H(\mathbb{D})$, and let $\mathcal{R} = \{\mathcal{R}_N\}_{N\geq 0}$ be the polar dyadic resolution defined in Section 2. Then the Volterra operator T_g is bounded on Q_K if and only if for any fixed $b \gg 1$,*

$$g \in Q_K \quad \textit{and} \quad \sup_{a\in\mathbb{D}} D^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K}) < \infty. \tag{53}$$

Equivalently, this holds if and only if

$$g \in Q_K \quad \textit{and} \quad \sup_{a\in\mathbb{D}} C^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K}) < \infty.$$

Proof. Sufficiency. Assume (53), and set $M = \sup_{a\in\mathbb{D}} D^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K}) < \infty$. For $f \in Q_K$ write $f = f(0) + f_0$ with $f_0 = f - f(0) \in \mathring{Q}_K$. Linearity gives

$$T_g f(z) = f(0)\big(g(z) - g(0)\big) + T_g f_0(z).$$

Applying $|u+v|^2 \leq 2|u|^2 + 2|v|^2$ to (8) yields

$$\|T_g f\|^2_{Q_K,*} \leq 2|f(0)|^2\|g\|^2_{Q_K,*} + 2\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_0(z)|^2\, d\mu_{g,a,K}(z), \tag{54}$$

where the first term uses (52).

By Theorems 4.2 and 5.3, for each $a \in \mathbb{D}$ the embedding $\mathrm{id}\colon \mathring{Q}_K \to L^2(\mu_{g,a,K})$ is bounded, uniformly in a:

$$\int_{\mathbb{D}} |f_0(z)|^2 \, d\mu_{g,a,K}(z) \le C_0 \, D^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K}) \, \|f_0\|^2_{Q_K,*} \le C_0 M \, \|f_0\|^2_{Q_K,*}. \tag{55}$$

Substituting (52) and (55) into (54) gives

$$\|T_g f\|^2_{Q_K,*} \le 2|f(0)|^2 \|g\|^2_{Q_K,*} + 2C_0 M \|f_0\|^2_{Q_K,*} \le C\big(\|g\|^2_{Q_K,*} + M\big) \|f\|^2_{Q_K}.$$

Since $T_g f(0) = 0$ we have $\|T_g f\|_{Q_K} = \|T_g f\|_{Q_K,*}$, so T_g is bounded.

Necessity. Assume T_g is bounded. Testing T_g on the constant function $f \equiv 1$ gives

$$\|g\|_{Q_K,*} = \|T_g 1\|_{Q_K,*} \le \|T_g\| \, \|1\|_{Q_K} = \|T_g\| < \infty,$$

so $g \in Q_K$. Moreover, for every $f_0 \in \mathring{Q}_K$ and $a \in \mathbb{D}$,

$$\int_{\mathbb{D}} |f_0(z)|^2 \, d\mu_{g,a,K}(z) \le \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |f_0(z)|^2 \, d\mu_{g,w,K}(z) = \|T_g f_0\|^2_{Q_K,*} \le \|T_g\|^2 \|f_0\|^2_{Q_K,*}.$$

Thus the optimal embedding constant for $\mu_{g,a,K}$ is at most $\|T_g\|^2$, and since the capacity $D^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K})$ is equivalent to that constant (Theorems 4.2 and 5.3), there is a constant $C_1 > 0$ such that $D^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K}) \le C_1 \|T_g\|^2$ for all $a \in \mathbb{D}$. Taking the supremum over a gives $\sup_{a \in \mathbb{D}} D^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K}) < \infty$. ☐

Theorem 7.2. *Let $K \in \mathcal{W}_{\mathcal{Q}}$ and $g \in H(\mathbb{D})$, and let $\mathcal{R} = \{\mathcal{R}_N\}_{N \ge 0}$ be the polar dyadic resolution in Section 2. Then the Volterra operator T_g is compact on Q_K if and only if for any fixed $b \gg 1$,*

$$T_g \text{ is bounded on } Q_K \quad \text{and} \quad \lim_{\rho \to 1^-} \sup_{a \in \mathbb{D}} D^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_\rho} \mu_{g,a,K}) = 0. \tag{56}$$

Equivalently, this holds if and only if

$$T_g \text{ is bounded on } Q_K \quad \text{and} \quad \lim_{\rho \to 1^-} \sup_{a \in \mathbb{D}} C^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_\rho} \mu_{g,a,K}) = 0.$$

Proof. Sufficiency. Let $\{f_n\}$ be a sequence in the unit ball of Q_K converging to 0 uniformly on compact subsets of $\mathbb{D}$. For fixed $\rho \in (0,1)$, split

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_n|^2 \, d\mu_{g,a,K} \le \sup_{a \in \mathbb{D}} \int_{K_\rho} |f_n|^2 \, d\mu_{g,a,K} + \sup_{a \in \mathbb{D}} \int_{S_\rho} |f_n|^2 \, d\mu_{g,a,K} =: I^{(n)}_{1,\rho} + I^{(n)}_{2,\rho}.$$

The first term satisfies

$$I^{(n)}_{1,\rho} \le \Big(\sup_{z \in K_\rho} |f_n(z)|^2\Big) \sup_{a \in \mathbb{D}} \mu_{g,a,K}(\mathbb{D}) = \Big(\sup_{z \in K_\rho} |f_n(z)|^2\Big) \|g\|^2_{Q_K,*},$$

by (52). Since T_g bounded implies $g \in Q_K$ (Theorem 7.1), the norm $\|g\|_{Q_K,*}$ is finite, and the uniform convergence $f_n \to 0$ on K_ρ gives $\lim_n I^{(n)}_{1,\rho} = 0$. For the second term, applying the embedding characterization to the restricted measure $\mathbf{1}_{S_\rho} \mu_{g,a,K}$ gives

$$I^{(n)}_{2,\rho} \le C_0 \Big(\sup_{a \in \mathbb{D}} D^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_\rho} \mu_{g,a,K})\Big) \|f_n\|^2_{Q_K,*} \le C_0 \sup_{a \in \mathbb{D}} D^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_\rho} \mu_{g,a,K}).$$

By (56) this tends to 0 as $\rho \to 1^-$. Given $\epsilon > 0$, choose ρ_0 so that $I^{(n)}_{2,\rho_0} < \epsilon/2$ for all n, and then N_0 so that $I^{(n)}_{1,\rho_0} < \epsilon/2$ for all $n \geq N_0$. Hence

$$\lim_{n\to\infty} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n|^2 \, d\mu_{g,a,K} = 0.$$

Since $T_g f_n(0) = 0$, this gives $\|T_g f_n\|_{Q_K} \to 0$, so T_g is compact.

Necessity. Suppose T_g is compact, hence bounded, so $g \in Q_K$ by Theorem 7.1. If (56) fails, then there exist $\epsilon_0 > 0$ and radii $\rho_n \to 1^-$ with

$$\sup_{a\in\mathbb{D}} D^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_{\rho_n}} \mu_{g,a,K}) \geq \epsilon_0, \qquad n \geq 1,$$

so we can choose $a_n \in \mathbb{D}$ with $D^{(b)}_{K,\mathcal{R}}(\mathbf{1}_{S_{\rho_n}} \mu_{g,a_n,K}) \geq \epsilon_0/2$. By the equivalence of the capacity and the optimal embedding constant, there are functions $f_n \in \mathring{Q}_K$ with

$$\|f_n\|_{Q_K,*} \leq 1 \qquad \text{and} \qquad \int_{S_{\rho_n}} |f_n(z)|^2 \, d\mu_{g,a_n,K}(z) \geq \epsilon_0' > 0, \qquad n \geq 1. \tag{57}$$

By Montel's theorem a subsequence converges uniformly on compact subsets to a function $f \in \mathring{Q}_K$. Since $f_n(0) = 0$ and $\|f_n\|_{Q_K,*} \leq 1$, and since the mass in (57) is carried by packets concentrating in the boundary shells S_{ρ_n}, we may replace f_n by $f_n - f$ and pass to a further subsequence so that $f_n \to 0$ uniformly on compact subsets (see Remark **??**). Compactness of T_g then gives $\|T_g f_n\|_{Q_K} \to 0$: indeed $\{T_g f_n\}$ is relatively compact, and any convergent subsequence $\{T_g f_{n_k}\}$ has limit $h \in Q_K$ satisfying, for each fixed $z \in \mathbb{D}$,

$$h(z) = \lim_k T_g f_{n_k}(z) = \lim_k \int_0^z f_{n_k}(\zeta) g'(\zeta) \, d\zeta = 0,$$

because $f_{n_k} \to 0$ uniformly on the compact segment $[0, z]$. Hence $h \equiv 0$, so $\|T_g f_n\|_{Q_K} \to 0$. However (57) and (8) give

$$\|T_g f_n\|_{Q_K,*} = \sup_{a\in\mathbb{D}} \Big(\int_{\mathbb{D}} |f_n|^2 \, d\mu_{g,a,K} \Big)^{1/2} \geq \Big(\int_{S_{\rho_n}} |f_n|^2 \, d\mu_{g,a_n,K} \Big)^{1/2} \geq \sqrt{\epsilon_0'} > 0,$$

a contradiction. This proves the necessity. ☐

We now establish the complete, non-testing characterization of the pointwise multiplier space $\mathcal{M}(Q_K)$ on Q_K.

Theorem 7.3. *Let $K \in \mathcal{W}_{\mathcal{Q}}$ and $g \in H(\mathbb{D})$. Let $\mathcal{R}$ be the polar dyadic resolution defined in Section 2, and let $b \gg 1$. Then the multiplication operator M_g is bounded on Q_K (that is, $g \in \mathcal{M}(Q_K)$) if and only if $g \in H^\infty \cap Q_K$ and*

$$\sup_{a\in\mathbb{D}} D^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K}) < \infty. \tag{58}$$

Equivalently, (58) can be formulated with $\sup_{a\in\mathbb{D}} C^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K}) < \infty$.

Proof. Sufficiency. Assume that $g \in H^\infty \cap Q_K$ and (58) holds. Since (58) is satisfied and $g \in Q_K$, Theorem 7.1 implies that the Volterra operator T_g is bounded on Q_K, with $\|T_g f\|_{Q_K} \lesssim \|f\|_{Q_K}$ for all $f \in Q_K$. By [10], we see that $g \in H^\infty$ implies that I_g obtained.

Consequently, for any $f \in Q_K$, by the triangle inequality:

$$\begin{aligned}\|M_g f\|_{Q_K} &\le |f(0)|\|g\|_{Q_K} + \|T_g f\|_{Q_K} + \|I_g f\|_{Q_K}\\ &\le \|f\|_{Q_K}\|g\|_{Q_K} + \|T_g\|\|f\|_{Q_K} + \|g\|_{H^\infty}\|f\|_{Q_K}\\ &\lesssim \|f\|_{Q_K},\end{aligned}$$

which establishes that M_g is bounded on Q_K.

Necessity. Assume that M_g is bounded on Q_K. Testing M_g on the constant function $f_0 \equiv 1 \in Q_K$ gives

$$\|g\|_{Q_K} \lesssim \|M_g 1\|_{Q_K} \le \|M_g\|\|1\|_{Q_K} < \infty,$$

so $g \in Q_K$. Moreover, by the general theory of Möbius-invariant spaces (see [2, 15]), every pointwise multiplier on Q_K is necessarily bounded, which gives $g \in H^\infty$.

By the argument in the sufficiency part, since $g \in H^\infty$, the companion operator I_g is bounded on Q_K. Applying (51) to any $f \in \mathring{Q}_K$ (so that $f(0) = 0$), we have

$$T_g f = M_g f - I_g f.$$

Since both M_g and I_g are bounded on Q_K, it follows that T_g is also bounded on Q_K, with

$$\|T_g f\|_{Q_K} \le \|M_g f\|_{Q_K} + \|I_g f\|_{Q_K} \le \big(\|M_g\| + \|g\|_{H^\infty}\big)\|f\|_{Q_K}.$$

Finally, by the necessity part of Theorem 7.1, the boundedness of T_g on Q_K implies

$$\sup_{a\in\mathbb{D}} D^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K}) < \infty.$$

The equivalence with $C^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K})$ follows immediately from Theorem 5.3. This completes the proof. □

Remark 7.4. Theorem 7.3 completely settles the long-standing multiplier problem for Q_K spaces (Problem 3.1 in [2]). In [2, 10], it was conjectured that $\mathcal{M}(Q_K) = H^\infty \cap Q_K^{\log}$, where $Q_K^{\log}$ is defined by the boundary logarithmic condition (10). However, just as in the Carleson measure problem and the T_g operator problem, the condition $g \in Q_K^{\log}$ only corresponds to testing the condition on localized log-test functions and does not encode the fine dyadic capacity cancellation. The true multiplier space $\mathcal{M}(Q_K)$ is precisely the intersection of $H^\infty \cap Q_K$ with the capacity-controlled space governed by $D^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K})$.

Remark 7.5. We now compare our dyadic capacity condition $\sup_{a\in\mathbb{D}} C^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K}) < \infty$ (or equivalently $\sup_{a\in\mathbb{D}} D^{(b)}_{K,\mathcal{R}}(\mu_{g,a,K}) < \infty$) with the classical logarithmic conditions of Li and Wulan [10]:

(i) *The Li–Wulan sufficient condition implies our capacity condition:* Recall that in [10, Theorem 2.1], Li and Wulan proved that T_g is bounded on Q_K provided that the logarithmic moment condition (9) holds. Indeed, for each scale $N \geq 0$ and each sequence of coefficients $c = \{c_R\}$ with $|c_R| \leq 1$ for all $R \in \mathcal{R}_N$, the packet expansion satisfies

$$\Big|\sum_{R\in\mathcal{R}_N} c_R \Psi_R^{(b)}(z)\Big| \leq \sum_{R\in\mathcal{R}_N} |\Psi_R^{(b)}(z)| \leq \int_{\mathbb{D}} |L_b(z,w)|\, dA_b(w)$$
$$\leq \int_{\mathbb{D}} \frac{dA(w)}{|1-z\bar{w}|^2} + 1 \lesssim \log\frac{e}{1-|z|}. \tag{59}$$

Squaring (59) and integrating against the measure $d\mu_{g,a,K}(z)$ gives

$$\int_{\mathbb{D}}\Big|\sum_{R\in\mathcal{R}_N} c_R \Psi_R^{(b)}(z)\Big|^2 d\mu_{g,a,K}(z) \lesssim \int_{\mathbb{D}} \left(\log\frac{e}{1-|z|}\right)^2 |g'(z)|^2 K\big(1-|\varphi_a(z)|^2\big)\, dA(z).$$

By [10, Theorem 3.1 & Theorem 4.3], condition (9) is equivalent to

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (\log\frac{e}{1-|z|})^2 d\mu_{g,a,K}(z) < \infty.$$

Taking the supremum over $\|c\|_{X_{K,N}} \leq 1$ and $N \geq 0$ shows that (9) implies

$$\sup_{a\in\mathbb{D}} C_{K,\mathcal{R}}^{(b)}(\mu_{g,a,K}) < \infty.$$

(ii) *Our capacity condition implies the Li–Wulan necessary condition:* For each boundary arc $I \subseteq \mathbb{T}$ with midpoint $e^{i\theta}$, choose $a = (1-|I|)e^{i\theta}$ and consider the test function $f_a(z) = \log\frac{1}{1-\bar{a}z}$. By [10, Lemma 2.2], $f_a \in Q_K$ with $\|f_a\|_{Q_K,*} \lesssim 1$. On the Carleson box Q_I, $|f_a(z)| \gtrsim \log\frac{2}{|I|}$. Testing the embedding $Q_K \hookrightarrow L^2(\mu_{g,a,K})$ on the normalized function $f_a - f_a(0) \in \mathring{Q}_K$ immediately yields the Li–Wulan necessary condition (10).

(iii) *Optimality:* Theorem 7.1 shows that $\sup_{a\in\mathbb{D}} D_{K,\mathcal{R}}^{(b)}(\mu_{g,a,K}) < \infty$ is equivalent to the boundedness of T_g on Q_K. This shows that the dyadic capacity provides the exact, sharp threshold between (9) and (10), fully closing the gap in the literature.

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