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Comment on: "Third-order corrections to the slow-roll expansion: Calculation and constraints with Planck, ACT, SPT, and BICEP/Keck [2025 PDU 47 101813]"

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Abstract

We point out that several terms in the third-order corrections to the slow-roll power spectra presented by Ballardini et al. [1] are incorrect. The authors of that work present their result as differing from the ones originally presented by Auclair and Ringeval [2] due to some different approximation schemes. However, in our original work, all terms at all orders have been derived exactly and any difference between two expansions performed at the same pivot wavenumber signals a problem. As we show in this comment, Ballardini et al. [1] have misevaluated some definite three-dimensional integrals by integrating a truncated Taylor expansion instead of Taylor expanding an integral. Our claim is backed-up with a Monte-Carlo numerical integration of the incriminated three-dimensional integrals, which, unsurprisingly, matches the analytical value derived in Auclair and Ringeval [2].

Keywords: Cosmic inflation, Slow-roll power spectra, N3LO expansion, Integrals and Taylor expansions

1. Introduction

Cosmic inflation, an early era of accelerated expansion of the spacetime, is one of the most compelling paradigms to solve various problems of the Friedmann-Lemaître hot Big-Bang model while providing a physical mechanism at the origin of the cosmic microwave background anisotropies and of the large scale structures of the universe [3–14]. In its simplest incarnation, cosmic inflation can be realized by a single scalar field slowly rolling down its potential, the quantum fluctuations of which acting as the seeds of all the structures observed today. Single-field inflation is a landscape populated by hundreds of different models that have been put to the test with ever-increasingly accurate cosmological data [15–19]. In order to predict the actual shape of the primordial scalar and tensor power spectra associated with any inflationary models, one can rely on exact numerical integrations [20–25]. However, for the single-field slow-roll models, there exists a unified framework, based on a perturbative expansion, that allows us to derive the functional shape of the power spectra [26–33]. These expansions have been shown to be very precise [34–37] and their accuracy can always be increased by pushing them to higher orders. They are constructed over a hierarchy of the so-called "Hubble-flow" functions defined by

$$\epsilon_{i+1}(N) \equiv \frac{\mathrm{d}\ln|\epsilon_i|}{\mathrm{d}N}, \qquad \epsilon_1(N) \equiv -\frac{\mathrm{d}\ln H}{\mathrm{d}N}\,. \tag{1}$$

Here $N = \ln a$ stands for the number of e-folds, $a(\eta)$ being the Friedmann-Lemaître scale factor while $H(N)$ is the Hubble parameter during inflation, which is almost constant for quasi-de Sitter spacetime thereby ensuring the smallness of the ϵ_i functions.

Motivated by the soon-to-be released Euclid satellite data [38–40], we have pushed the Hubble-flow expansion of the scalar and tensor power spectra to third order (N3LO) in Auclair and Ringeval [2]. Our result is based on the so-called Green's functions method introduced by Gong and Stewart [29] and it applies to all single-field slow-roll models having minimal and non-minimal kinetic terms. Later on, our results have been recovered and extended to other effective field theories, including a broader class of modified gravity models, by Bianchi and Gamonal [41].

In simple terms, one perturbatively solves the evolution of cosmological perturbations during inflation, starting from initial conditions set by quantum fluctuations in the Bunch-Davis vacuum. By doing so, one can finally derive the scalar power spectrum, $\mathcal{P}_\zeta(k)$, of the comoving curvature perturbation and the tensor power spectrum, $\mathcal{P}_h(k)$, for the primordial gravitational waves. Their final expressions, at third order, take the form

$$\mathcal{P}_\zeta(k) = \frac{H_\diamond^2}{8\pi^2 M_{\mathrm{P}}^2 \epsilon_{1\diamond} c_{\mathrm{s}\diamond}}\left[b_{0\diamond}^{(\mathrm{S})} + b_{1\diamond}^{(\mathrm{S})}\ln\left(\frac{k}{k_\diamond}\right) + b_{2\diamond}^{(\mathrm{S})}\ln^2\left(\frac{k}{k_\diamond}\right) + b_{3\diamond}^{(\mathrm{S})}\ln^3\left(\frac{k}{k_\diamond}\right)\right], \tag{2}$$

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and

$$\mathcal{P}_h(k) = \frac{2H_\diamond^2}{\pi^2 M_{\rm P}^2}\left[b_{0\diamond}^{(\rm T)} + b_{1\diamond}^{(\rm T)}\ln\left(\frac{k}{k_\diamond}\right) + b_{2\diamond}^{(\rm T)}\ln^2\left(\frac{k}{k_\diamond}\right) + b_{3\diamond}^{(\rm T)}\ln^3\left(\frac{k}{k_\diamond}\right)\right], \tag{3}$$

where the $b_{i\diamond}^{(\rm S)}$ and $b_{i\diamond}^{(\rm T)}$ are determined functionals of the Hubble-flow functions $\epsilon_{i\diamond} \equiv \epsilon_i(\eta_\diamond)$ evaluated at a precise time $\eta_\diamond$. This time corresponds to the instant at which the reference pivot wavenumber $k_\diamond$ (usually set at $0.05\,\mathrm{Mpc}^{-1}$), appearing in Eqs. (2) and (3), crosses the sound radius during inflation, i.e., it is solution of $k_\diamond \eta_\diamond c_{\rm s}(\eta_\diamond) = -1$, $c_{\rm s}(\eta)$ being the sound speed of the cosmological perturbations during (K-) inflation[1]. Most of the action in these expansions consists in determining all the $b_{i\diamond}^{(\rm S)}$ and $b_{i\diamond}^{(\rm T)}$ functionals and their (long) expression can be found in Ref. [2] as Eqs. (54), and (74) to (77).

2. Different functionals

Two years after Ref. [2], Ballardini et al. [1] have presented "an independent approach to the solution of the integrals compared to the one previously presented in the literature", in which they have attempted to reproduce our results in the particular case where $c_{\rm s} = 1$, i.e., for minimal kinetic terms. Although most of the calculations appear to be very similar to our previously published results, including the usage of Green's functions, some of the $b_{i\diamond}^{(\rm S)}$ and $b_{i\diamond}^{(\rm T)}$ functionals end up differing when compared to ours.

Independently of the complexity of the underlying calculations, Eqs. (2) and (3) are just Taylor expansions in $\ln(k/k_\diamond)$. As such, the "coefficients" $b_{i\diamond}^{(\rm S)}$ and $b_{i\diamond}^{(\rm T)}$ are universal and should numerically match. As a matter of fact, and at least up to second order, one can check that other approximate solving methods, such as the uniform approximation, indeed lead to coefficients having numerical values very close to the ones obtained with Green's functions [42, 43].

Ballardini et al. [1] find a different expression than ours for $b_{0\diamond}^{(\rm S)}$ and $b_{0\diamond}^{(\rm T)}$ in various coefficients multiplying terms of order three $\mathcal{O}\left(\epsilon_i^3\right)$. The differences find their root in the evaluation of triple integrals of the form

$$F_{000}(x) = \int_x^\infty \frac{e^{+2iy}}{y} F_{00}(y)\,\mathrm{d}y\,, \tag{4}$$

where

$$F_{00}(x) = \int_x^\infty \frac{e^{-2iy}}{y} F_0(y)\,\mathrm{d}y\,, \tag{5}$$

and

$$F_0(x) \equiv \int_x^\infty \frac{e^{+2iy}}{y}\,\mathrm{d}y\,. \tag{6}$$

[1] The functionals $b_{i\diamond}^{(\rm S)}$ and $b_{i\diamond}^{(\rm T)}$ are also dependent on a hierarchy of sound-flow functions $\delta_i(\eta_\diamond)$ built upon $c_{\rm s}(\eta)$, but this is not important for the present discussion.

In Ref. [2], we have found a generating functional for all integrals of the form F_{0^n} and this has allowed us to evaluate $F_{000}(x)$ exactly (see appendix A of that reference). Indeed, defining

$$h(\nu, x) \equiv \sum_{k=0}^{+\infty} I_k(x)\nu^k, \tag{7}$$

where

$$I_{2n}(x) = \overline{F_{0^{2n}}}(x), \qquad I_{2n+1}(x) = F_{0^{2n+1}}(x), \tag{8}$$

we have proven that

$$h(\nu, x) = -xe^{ix}\left\{\sin\left(\frac{\pi\nu}{2}\right)\left[j_\nu(x) + ij_{\nu-1}(x)\right] + \cos\left(\frac{\pi\nu}{2}\right)\left[y_\nu(x) + iy_{\nu-1}(x)\right]\right\}. \tag{9}$$

In the previous expressions, $j_\nu(x)$ and $y_\nu(x)$ are the spherical Bessel functions of order ν and $\overline{F_{0^{2n}}}$ stands for the complex conjugate of $F_{0^{2n}}$. One therefore sees that all the F_{0^n} integrals can be exactly obtained by successive differentiation of Eq. (9) with respect to ν.

In fact, for the Hubble-flow expansion we are concerned with, one only needs the limit at small x of these integrals, and we find

$$F_{000}(x) = -\frac{7}{3}\zeta(3) - \frac{\pi^2}{4}B - \frac{1}{6}B^3 - \left(\frac{\pi^2}{4} + \frac{B^2}{2}\right)\ln(x) - \frac{B}{2}\ln^2(x) - \frac{1}{6}\ln^3(x) + \mathcal{O}(x), \tag{10}$$

where the constant $B = \gamma_{\rm E} + \ln(2) - \frac{i\pi}{2}$, $\gamma_{\rm E}$ being the Euler-Mascheroni constant.

Ballardini et al. [1] propose two methods in their appendix B to find the limit at $x \to 0$ of $F_{000}(x)$:

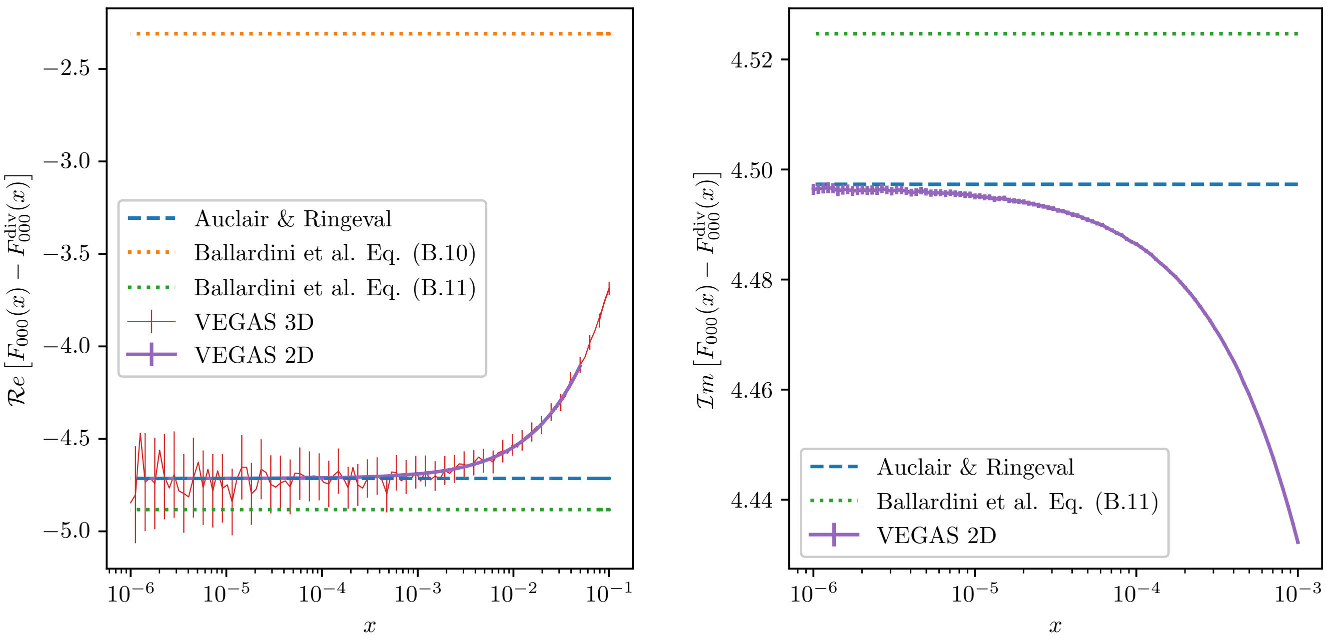
- The first method is described by the authors below their Eq. (B.9):

  > "to solve the triple integral entering eq. (32), we take the limit for $u \to 0$ of the double integral in the integrand, which is eq. (B.6), and then we integrate the leading contributions of order $\mathcal{O}\left(x^0\right)$, multiplied by e^{2iu}/u, between x and ∞."

  In other words, Ballardini et al. [1] integrated a truncated Taylor expansion around 0 up to infinity.

- The second method is described above their Eq. (B.11):

  > "We have divided the integral into two parts: one from x to 1 and one from 1 to $+\infty$. The contribution of the integral from 1 to $+\infty$ is negligible because the integrand $f(u)$ decreases rapidly for $u \gg 1$."



(a) Real part.

(b) Imaginary part.

Figure 1: Finite part of $F_{000}(x)$ computed using the VEGAS algorithm over a two-dimensional and a three-dimensional domain. All the points in the curves were computed using ten iterations of 10^8 samples each. The error bars show five times the estimated standard deviation.

For both methods, they analytically perform *the integral of a truncated Taylor expansion* of $F_{00}(y)$ at small y, either on the full domain – therefore outside the domain of validity of the expansion – or only within $[x, 1]$, the contribution coming from the other domain being neglected. Let us note that the integral from x to 1 diverges logarithmically as $x \to 0$. As a consequence, even if the integral from $]1, \infty[$ might appear negligeable in comparison, it is still relevant to determine the finite part of $F_{000}(x)$. Moreover, integrals may have order unity values even when their integrand decreases rapidly.

Their results end up differing from Eq. (10) by a constant and instead of having the term in $7\zeta(3)/3 \simeq 2.8048$ they obtain either $\mathcal{Z} = \zeta(3)/3 \simeq 0.4007$ or a precise complex number reported to be $\mathcal{Z} \simeq 2.97353 - 0.0273557i$ (see Appendix B of Ref. [1]).

In various places of their paper, Ballardini et al. [1] suggest that the actual value of $\mathcal{Z}$ is unknown and dependent on some approximation scheme chosen to evaluate $F_{000}(x)$, our result being only one approximation among others. Indeed, one may read on page 2 of Ballardini et al. [1]:

> "While these results have been already presented by Auclair and Ringeval, we obtain them with a different approach to the integrals."

Then, on page 5, they write:

> "$\mathcal{Z}$ is a constant encoding the difference due to the approximation scheme used to calculate the triple integral appearing in Eq. (32), see appendix B".

Let us stress that this is not the case, the *exact value* of $\mathcal{Z}$ is the one we have derived, namely $7\zeta(3)/3$, simply because we have calculated exactly the integral and, then, taken its limit at small x. In other words, we have performed a Taylor expansion of an exact expression for the integral whereas Ballardini et al. [1] have performed an integral of a truncated Taylor expansion.

Eventually, the authors of Ref. [1] write on page 20:

> "Different choices of $\mathcal{Z}$ do not lead to numerically significant differences in the final PPS".

First, we stress that $\mathcal{Z}$ is a known number, it is not a "choice". Second, the error on $\mathcal{Z}$ concerns only third-order terms, scaling as $\mathcal{O}(\epsilon_i^3)$, and, indeed, changing these terms by order unity cannot drastically change the final shape of the power spectra. However, third-order corrections are mathematically well-defined and relevant for future observations. What would be the purpose of deriving third-order corrections in the first place if one does not pay attention to their actual value?

3. Numerical evaluation of the integrals

We present in this section a numerical integration of $F_{000}(x)$, defined by Eqs. (4) to (6). One has to perform a three-dimensional integral of a rapidly oscillating function, from close to zero to infinity, which is a non-trivial technical problem. For this purpose, we have used the VEGAS algorithm, an adaptive multidimensional Monte Carlo integration package now adapted to Python [44, 45].

We have first brute-forcibly computed $F_{000}(x)$ over the full three-dimensional domain. This approach proves to be quite inefficient with relatively large error bars, but its real part has been reported and referred to as VEGAS 3D in Fig. 1.

Another more accurate method is to remark that the integral $F_0(x)$ of Eq. (6) can be analytically performed, thus reducing the integration dimensionality explored by VEGAS to two dimensions (referred to as VEGAS 2D in Fig. 1). Indeed, one has [46]

$$F_0(x) = \mathrm{E}_1(-2ix) = -\,\mathrm{Ei}(2ix) + i\pi \tag{11}$$

where E_1 and Ei are the Exponential integral functions. Note that this expression disagrees with Eq. (B.1) in Ballardini et al. [1]. The reason is that Ei is defined on the real domain, but the continuation to the complex plane can be ambiguous due to singularities at 0 and ∞. We have consistently used the `scipy` implementation of the exponential integral and one can easily check that $\mathrm{Ei}(\pm 2ix) \to \pm i\pi$. A factor $i\pi$ is therefore necessary to recover the expected behavior of $F_0(x)$ at infinity. This problem does not occur if one uses E_1 instead of Ei.

Since the disagreement concerns only the finite part of $F_{000}(x)$, we can subtract from the numerical integration the known diverging part when $x \to 0$. From Eq. (10), we define

$$F_{000}^{\mathrm{div}}(x) \equiv -\left(\frac{\pi^2}{4} + \frac{B^2}{2}\right)\ln(x) - \frac{B}{2}\ln^2(x) - \frac{1}{6}\ln^3(x). \tag{12}$$

Fig. 1 shows the real and imaginary parts of $F_{000}(x) - F_{000}^{\mathrm{div}}(x)$ obtained by the numerical integrations in two and three-dimensions and we find

$$\mathcal{Z}_{\mathrm{vegas}} = (2.8051 \pm 0.0014) + (0.00089 \pm 0.00096)i. \tag{13}$$

Both the real and imaginary parts match, and only match, the value of $\mathcal{Z} = 7\zeta(3)/3$ that we found in Auclair and Ringeval [2]. The two values of $\mathcal{Z}$ proposed in Ballardini et al. [1] are excluded: $\mathcal{Z} = \zeta(3)/3$ fails to be consistent with the real part, and $\mathcal{Z} \simeq 2.97353 - 0.0273557i$ gives an incorrect real *and* imaginary part.

4. Conclusion

In conclusion, the right form for $F_{000}(x)$ is given by Eq. (10) and this implies that the correct expression for the $b_{i\diamond}^{(\mathrm{S})}$ and $b_{i\diamond}^{(\mathrm{T})}$ appearing in the power spectra of Eqs. (2) and (3) should be taken from Auclair and Ringeval [2].

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