A Method for Preserving Geometric Meaning in the Calculus of Variations

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Abstract

We show that the Calculus of Moving Surfaces offers distinct advantages over the Euler-Lagrange equation for optimization problems originating in Geometry. An extension of Tensor Calculus, it provides tools for analyzing invariant representations rather than specific coordinate representations of relevant quantities, which allows us to avoid the myriad of hardships associated with the use of coordinates from untenable complexity of analytical expressions to virtual impossibility of recovering the geometric interpretation of the final result. As an illustration, we solve the brachistochrone problem and give a geometric characterization for the desired curve in terms of its curvature.

1 Introduction

Every geometric problem in the Calculus of Variations is a problem in the calculus of moving surfaces. It therefore admits a solution more elegant, more beautiful, and more geometrically revealing. We will make this evident by treating the brachistochrone.

In 1744 Leonhard Euler published his celebrated work *A Method for Finding Curves Enjoying the Property of Maximum or Minimum [1]*, in which he laid the foundations of the Calculus of Variations. From its very first pages, where in Paragraph 4 of Chapter I we read *...the precise study of curves requires that they be referred to some fixed axis...*, we see a strong drive towards a coordinate algebraization of mathematics. This trend will prove so effective that it continues to this day.

Joseph Louis Lagrange was inspired by this approach and took it to the next level in his 1762 *Essay on a New Method for Finding the Maxima And Minima of Indefinite Integral Expressions [5]*. In this essay, he reduced the Calculus of variations to a purely analytical method which culminated with the Euler-Lagrange equations. It greatly broadened the range of problems that can be solved and, in particular, extended the analysis from curves to surfaces. In one of the letters between Euler and Lagrange that predated the publication of the essay, Euler wrote: *Your analytical solution of the isoperimetric problem, from what I can see, leaves nothing to be desired in this field of inquiry, and I am delighted beyond measure that it has been your good fortune to carry this subject – which, from its inception, I had cultivated almost entirely on my own – to such a high degree of perfection.*

However, the remarkable effectiveness of the resulting analytical method came at the cost of losing the geometric meaning. As a matter of fact, this statement applies not only to the Calculus of Variations but also to the method of coordinates in general. When an analysis comes untethered from the underlying geometry, one woe follows another. The first woe is the disorienting flatness of the analytical expressions which deprives us of logical thrust – the kind of thrust that is found in all great mathematical analyses. The second woe is the growing computational complexity that often becomes untenable quickly and forces us to retreat in the face of computational difficulties. Finally, if we are able to proceed and are so fortunate as to reach a conclusion, we are unlikely to know the simple meaning of the final answer, thus preventing it from becoming a discovery.

Euler and Lagrange's own works bear witness to this fact. In Chapter 5, Paragraph 47 of the *Method*, Euler – with his singular ability to find the simplest nontrivial example that possesses the crux of a general problem – solved for the shape of revolution of least area. He showed that the profile $y(x)$ satisfies the equation

$$y_{xx}y - y_x^2 - 1 = 0 \tag{1}$$

and promptly noted that the solution is the famous *catenary*, i.e. the shape of a hanging chain. A shape that minimizes surface area is called a *minimal surface*, while the shape of revolution discovered by Euler is known as the *catenoid*.

Tellingly, Euler did not find – nor did he care to find – the geometric interpretation of this equation, which is that the sum of the principal curvatures, also known as the *mean curvature* B_α^α, vanishes, i.e.

$$B_\alpha^\alpha = 0. \tag{2}$$

We must note, of course, that the concept of principal curvatures will be introduced by Euler himself in 1760 [2], and was therefore not available to him in 1744.

The same lack of concern for geometric interpretation is found in Lagrange's work. He showed that a minimal surface represented by a function $z(x,y)$ is characterized by the equation

$$\left(1+z_y^2\right) z_{xx} + 2z_x z_y z_{xy} + \left(1+z_x^2\right) z_{yy} = 0. \tag{3}$$

However, much like Euler, Lagrange was not interested in the geometric interpretation, nor did he have any possibility of success in this endeavor had the question intrigued him.

The proper connection between minimal surfaces and mean curvature was found in 1776 by the young French mathematician Jean Baptiste Meusnier [6].

Future generations of mathematicians have remedied this shortcoming of coordinate systems. One of the avenues of investigation led to the development of Tensor Calculus by Gregorio Ricci and Tullio Levi-Civita at the turn of the twentieth century. The extension of Tensor Calculus to moving surfaces is known as the Calculus of Moving Surfaces to which we now will turn our attention.

There is an obvious disconnect between the way optimization problems arise in Geometry and the expected starting point for a problem in the Calculus of Variations. When a problem in Geometry calls for some quantity to be minimized or maximized (possibly subject to a constraint or a set of constraints) that quantity (along with some of the constraints) is usually expressed in the form of a *geometric* integral, such as the integral

$$\int_S dS \tag{4}$$

for the total area of a surface. The term *geometric* in this context means that the integral is defined in purely geometric terms. Intuitively speaking, the integral

$$\int_S F dS \tag{5}$$

captures the total amount of the quantity F over the surface. While today's mathematical mind may thirst for a more rigorous definition, we will not concern ourselves with those formalities here.

Meanwhile, the conventional Calculus of Variations approach based on the Euler-Lagrange equation requires the quantity to be minimized or maximized to be expressed as an *arithmetic* integral

$$\int \cdots \int F\left(x^i, z^\alpha, \frac{\partial z^\alpha}{\partial x^i}\right) dx^1 \cdots dx^n, \tag{6}$$

such as

$$\int \int \sqrt{1 + z_x^2 + z_y^2} dx dy. \tag{7}$$

Thus it all but demands that the problem be referred to a specific coordinate system, thereby relinquishing much of the connection to the original geometry along with almost any chance of regaining it in the end.

By contrast, the Calculus of Moving Surfaces provides analytical tools that can be applied directly to the geometric formulation of the problem. Since the Calculus of Moving Surfaces is an extension of Tensor Calculus, it does, at its core, rely on the use of coordinates, but it groups the analytical expressions in a way that allows for the preservation of the geometric meaning.

2 The variational formulation based on moving surfaces

Suppose that given a Euclidean manifold S – be it a curve, a surface, or a volume – we are presented with an optimization problem where the quantity E

to be minimized or maximized is expressed in terms of line, surface, or volume integrals, while the independent variation is the surface S itself, i.e. its shape and location. We will limit ourselves to situations where S is a hypersurface, i.e. a curve embedded in a plane or a surface, or a surface embedded in space. And, naturally, the optimization may be subject to a constraint or several constraints expressed in integral form or some other analytically manageable form.

As a model, let us adopt the classical isoperimetric problem, that is to minimize the total surface area

$$E = \int_S dS \tag{8}$$

of a closed surface S subject to given enclosed volume, i.e.

$$\int_\Omega d\Omega = V_0, \tag{9}$$

where Ω is the enclosed domain.

Let us proceed by imitating the key device in the derivation of the Euler-Lagrange equations, which is to consider a family of surfaces $S(t)$ such that the desired optimal surface occurs at $t = 0$. We can intuitively think of t as time and $S(t)$ as a *moving surface*. Naturally, the quantity E can be evaluated at any value t and therefore can be treated as a function of it. For example,

$$E(t) = \int_{S(t)} dS, \tag{10}$$

and the constraint likewise becomes dependent on time, i.e.

$$\int_{V(t)} dV = V_0. \tag{11}$$

Then for any family $S(t)$, the function $E(t)$ attains a minimum (or a maximum) at $t = 0$ and therefore

$$E'(0) = 0. \tag{12}$$

This is the very condition from which, by taking advantage of the arbitrariness of $S(t)$, we can derive the equilibrium equations.

What is required to turn this approach into an actionable algorithm is the machinery of the Calculus of Moving Surfaces which we will now outline.

3 The essential elements of the Calculus of Moving Surfaces

For the sake of brevity, we will name, but not define or discuss, the essential elements of the Calculus of Moving Surfaces required for our presentation. For a thorough introduction to the subject, we would like to refer the reader to the paper *A Better Calculus of Moving Surfaces* [3] that gives a brief historical overview of the subject and defines the key differential operator in its present form. Meanwhile, a detailed exposition of the subject can be found in the textbook [4].

The motion of the surface $S(t)$ is characterized by the invariant *normal velocity of the interface* C. The scalar field C can be intuitively understood as the rate of deformation of the surface in the normal direction. In particular, the sign of C depends on the chosen direction of the *normal* $\mathbf{N}$. Furthermore, the combination $C\mathbf{N}$, which is independent of the choice of normal, can be thought of as the *vector* rate of deformation in the normal direction.

The symbol $\dot{\nabla}$ represents the *invariant time derivative with respect to the moving surface.* When applied to an invariant, such as C, $\mathbf{N}$, or the mean curvature B^{α}_{α} (or any non-invariant object of order zero), it is interpreted as the rate of change of that quantity along the normal direction. The analytical definition of $\dot{\nabla}$ is quite elaborate and can be found in [3] and [4]. The one property of $\dot{\nabla}$ that is relevant to our narrative is the *chain rule.* For a time-independent field F that is a surface restriction of an ambient field F, such as the gravitational potential, the derivative $\dot{\nabla}F$ is given by the formula

$$\dot{\nabla}F = C\frac{\partial F}{\partial n}, \tag{13}$$

where $\partial F/\partial n$ is the normal derivative of F. Note that, given our Tensor Calculus context, $N^i\nabla_i F$ would have been a more appropriate way to express this quantity, but we will continue with $\partial F/\partial n$ because it is more familiar for most readers and because it will help us avoid the discussion of the covariant derivative. For the same reason, it would have perhaps been wiser to use the more common symbol κ for B^{α}_{α}, but in that case we opted for the more tensorial symbol to remind the reader of the robust analytical framework operating behind the scenes.

With these essential elements in place, we are ready to turn our attention to the problems at hand. Note that all elements presented here are valid for any hypersurface, be it a surface in space or a curve in the plane. Furthermore, if we were to omit any mention of the vector normal $\mathbf{N}$, all elements could be extended to higher dimensions, as well.

4 Time evolution of integrals

First consider the volume integral

$$\int_{\Omega} F d\Omega \tag{14}$$

of an invariant field F defined over the time-dependent domain Ω with boundary S. Note that we have dropped the explicit dependence of Ω and S on t since all domains and surfaces are now understood to be time-dependent. Then the rate of change of this integral is given by the formula

$$\frac{d}{dt}\int_{\Omega} F d\Omega = \int_{\Omega}\frac{\partial F}{\partial t}d\Omega + \int_{S} CF dS, \tag{15}$$

where C is referred to the *exterior* normal. This formula is easily seen to be a direct generalization of the Fundamental Theorem of Calculus written in the form

$$\frac{d}{dt}\int_a^{b(t)} F\left(t,x\right)dx = \int_a^{b(t)} \frac{\partial F\left(t,x\right)}{\partial t}dx + b'\left(t\right)F\left(t,b\right) \tag{16}$$

Next, let us turn our attention to the surface integral

$$\int_S F dS, \tag{17}$$

of an invariant field F defined on the surface S. The evolution of this integral is governed by the formula

$$\frac{d}{dt}\int_S F dS = \int_S \dot{\nabla} F ds - \int_S C F B_\alpha^\alpha dS. \tag{18}$$

Note that, unlike the volume integral, the choice of normal here is immaterial since the sign of the mean curvature B_α^α also depends on the choice of normal and therefore the combination CB_α^α is independent of it.

5 The isoperimetric problem

Let us begin with the isoperimetric problem. To accommodate the constraint of given volume, form the Lagrangian

$$L\left(t\right) = \int_S F dS + \lambda\left(\int_\Omega d\Omega - V_0\right), \tag{19}$$

where we once again remind the reader that the time-dependence is due to the virtual evolution $\Omega\left(t\right)$ and $S\left(t\right)$ of the domain and its boundary that passes through the optimal configuration at $t=0$.

According to the laws contained in equations (15) and (18), we find:

$$L'\left(t\right) = \int_S C\left(-B_\alpha^\alpha + \lambda\right)dS. \tag{20}$$

Equating $L'\left(0\right)$ to 0, we find that the desired minimal surface satisfies the equation

$$\int_S C\left(-B_\alpha^\alpha + \lambda\right)dS = 0 \tag{21}$$

for any possible interface velocity C. Since C may be treated as an independent variation, the quantity it multiplies in the integrand must vanish, i.e.

$$B_\alpha^\alpha = \lambda. \tag{22}$$

In other words, the equilibrium equation states that the solution is a surface of constant *mean curvature*. This implies a circle in two dimensions and a sphere in three dimensions.

If instead we consider a surface patch S with a fixed boundary, but no volume constraint, the analysis simplifies even further and takes us straight from the integral

$$E(t) = \int_S dS \tag{23}$$

to the equilibrium condition

$$B_\alpha^\alpha = 0. \tag{2}$$

In other words, the *mean curvature vanishes.* When interpreted in specific coordinate systems, this criterion will deliver both Euler's equation (1) as well as Lagrange's equation (3), but it does so in a way that preserves the geometric interpretation throughout.

These examples alone demonstrate that the Calculus of Moving Surfaces is a method that achieves an even *higher degree of perfection* than Lagrange's original method. To further convince the reader of this, we will now treat the brachistochrone.

6 The brachistochrone

Consider a material particle of unit mass, initially at rest, constrained to slide frictionlessly along a planar curve segment S under the influence of a general gravitational potential P. Assuming, with no loss of generality, that $P = 0$ at the initial location, the conservation of energy reads

$$P + \frac{1}{2}v^2 = 0. \tag{24}$$

Thus, the velocity v of the material particle is $\sqrt{-2P}$, and therefore the total time T it takes to travel along the curve S is given by the integral

$$T = \int_S \frac{dS}{\sqrt{-2P}}. \tag{25}$$

Since the actual concept of time is present in this problem, let us use τ as the parameter for the virtual motion of the surface S. According to equation (18), the first variation reads

$$T'(\tau) = \int_S \left(\dot{\nabla} \frac{1}{\sqrt{-2P}} - \frac{CB_\alpha^\alpha}{\sqrt{-2P}} \right) dS \tag{26}$$

Subsequently, by the chain rule (13),

$$\dot{\nabla} (-2P)^{-1/2} = C \frac{\partial P / \partial n}{(-2P)^{3/2}} \tag{27}$$

and therefore we find that

$$T'(\tau) = \int_S C \left(\frac{\partial P/\partial n}{(-2P)^{3/2}} - \frac{B_\alpha^\alpha}{(-2P)^{1/2}} \right) dS \tag{28}$$

Once again, since C can be treated as an independent variation, the brachistochrone equation reads

$$\frac{\partial P}{\partial n} = -2PB^{\alpha}_{\alpha}. \tag{29}$$

For consistency with the isoperimetric problem, let us write this equation in the form that isolates the mean curvature, i.e.

$$B^{\alpha}_{\alpha} = -\frac{1}{2P}\frac{\partial P}{\partial n}. \tag{30}$$

As before, we have arrived at the equilibrium criterion expressed entirely in terms of invariant quantities, further solidifying the claim that the Calculus of Moving Surfaces represents an advancement of Lagrange's method.

Finally, this gives us occasion to demonstrate how the invariant criterion (30) can be reduced to equations with ordinary derivatives that would have resulted from a conventional application of the Euler-Lagrange equation. This step is needed when one wishes to find a specific solution to the problem by analytical or numerical methods.

Suppose that the desired curve is given in Cartesian coordinates x, y by the equations

$$x = x(\gamma) \tag{31}$$

$$y = y(\gamma), \tag{32}$$

where the symbol γ represents the curve coordinate S^1. Then the gravitational potential P is given by

$$P(x, y) = mgy \tag{33}$$

where m is the mass of the material particle and g is the acceleration of gravity. Since the components of the gradient of P are $(0, mg)$, and the components of the normal are $(y_\gamma, -x_\gamma)/\sqrt{x_\gamma^2 + y_\gamma^2}$, we have

$$\partial P/\partial n = -\frac{mgx_\gamma}{\sqrt{x_\gamma^2 + y_\gamma^2}}. \tag{34}$$

The expression for the mean curvature corresponding to the above choice of normal is given by

$$B^{\alpha}_{\alpha} = \frac{x_\gamma y_{\gamma\gamma} - y_\gamma x_{\gamma\gamma}}{\left(x_\gamma^2 + y_\gamma^2\right)^{3/2}}. \tag{35}$$

Therefore, the equilibrium equation (30) reduces to

$$\frac{x_\gamma y_{\gamma\gamma} - y_\gamma x_{\gamma\gamma}}{\left(x_\gamma^2 + y_\gamma^2\right)^{3/2}} = -\frac{x_\gamma}{2y\sqrt{x_\gamma^2 + y_\gamma^2}} \tag{36}$$

or, equivalently,

$$2y\left(x_\gamma y_{\gamma\gamma} - y_\gamma x_{\gamma\gamma}\right) + x_\gamma\left(x_\gamma^2 + y_\gamma^2\right) = 0 \tag{37}$$

Finally, when we parameterize the curve by x, i.e. $\gamma = x$, we arrive at the well-known brachistochrone equation

$$2yy_{xx} + y_x^2 + 1 = 0. \tag{38}$$

References

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