WEIGHTED SINGULAR VECTORS IN COMMON-BASE SELF-SIMILAR SETS

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ABSTRACT. We prove a lower bound for the Hausdorff dimension of weighted totally irrational singular vectors in affine-spanning common-base integral self-similar sets satisfying the open set condition. For the middle-third Cantor square, the bound improves the previously known explicit lower bound.

1. INTRODUCTION

Dirichlet's theorem (1842), one of the foundational results in Diophantine approximation, asserts that, for every $x \in \mathbb{R}^n$ and every $T > 1$, there are $q \in \mathbb{Z}$ and $\mathbf{p} \in \mathbb{Z}^n$ such that

$$0 < q \leq T, \qquad \|qx - \mathbf{p}\| \leq T^{-1/n}. \tag{1.1}$$

Here $\|y\| := \max_{1\leq i\leq n} |y_i|$ denotes the max norm on $\mathbb{R}^n$. The vector x is *singular* if Dirichlet's inequality can be improved by an arbitrary factor: for every $\varepsilon > 0$, the same conclusion as in (1.1) holds with $\varepsilon T^{-1/n}$ in place of $T^{-1/n}$ for all sufficiently large T. We denote the set of singular vectors in $\mathbb{R}^n$ by $\mathrm{Sing}(n)$. Khintchine introduced singular vectors in [Khi26]. In dimension one, the singular vectors are precisely the rational numbers, so the dimension problem is trivial; it becomes nontrivial in higher dimension. We therefore assume $n \geq 2$ throughout. With $\dim_H$ denoting Hausdorff dimension, Cheung [Che11] proved that $\dim_H \mathrm{Sing}(2) = 4/3$, and Cheung–Chevallier [CC16] proved that $\dim_H \mathrm{Sing}(n) = n^2/(n+1)$ for every $n \geq 2$. The corresponding matrix problems were studied by Kadyrov et al. [KKLM17] and Das et al. [DFSU24]. We restrict attention to vectors.

Weighted singularity allows different approximation rates in the coordinates. For a weight $\mathbf{w} = (w_1, \dots, w_n)$, where $1 > w_1 \geq \cdots \geq w_n > 0$ and $\sum_{i=1}^n w_i = 1$, a vector $x \in \mathbb{R}^n$ is $\mathbf{w}$-*singular* if, for every $\varepsilon > 0$, there is $T_0 > 1$ such that for every $T > T_0$ there are $q \in \mathbb{Z}$ and $\mathbf{p} \in \mathbb{Z}^n$ satisfying

$$0 < q < T, \qquad \max_{1\leq i\leq n} |qx_i - p_i|^{1/w_i} < \frac{\varepsilon}{T}. \tag{1.2}$$

We denote the set of such vectors by $\mathrm{Sing}(\mathbf{w})$. When $w_i = 1/n$ for every i, this definition recovers $\mathrm{Sing}(n)$. In dimension two, Liao–Shi–Solan–Tamam [LSST20] proved $\dim_H \mathrm{Sing}(\mathbf{w}) = 2 - 1/(1 + w_1)$. For general n, the authors [KP24] obtained the lower bound $\dim_H \mathrm{Sing}(\mathbf{w}) \geq n - 1/(1 + w_1)$. A vector

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$x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is called *totally irrational* if $1, x_1, \dots, x_n$ are linearly independent over $\mathbb{Q}$. We set

$$\mathrm{Sing}^*(\mathbf{w}) := \{x \in \mathrm{Sing}(\mathbf{w}) : x \text{ is totally irrational}\}.$$

For equal weights we use the abbreviation $\mathrm{Sing}^*(n)$.

Singular vectors have also been studied on submanifolds and fractals. Kleinbock–Moshchevitin–Weiss [KMW21] constructed totally irrational singular vectors with large uniform exponents on broad classes of analytic manifolds and fractals, while Shah–Yang [SY24] obtained Hausdorff-dimension upper bounds on affine subspaces in terms of their Diophantine exponents. In the weighted setting, Datta–Tamam [DT24] proved inheritance and existence results, and Kleinbock–Moshchevitin–Warren–Weiss [KMWW25] established strong intersection properties and quantitative rates for uniform approximation.

A basic unresolved dimension problem in the fractal setting already appears for the middle-third Cantor square. Let C denote the middle-third Cantor set, and write $C^2 = C \times C$. Bugeaud–Cheung–Chevallier [BCC19, Problem 6] asked for the Hausdorff dimension of

$$\mathrm{Sing}(2) \cap C^2.$$

To the best of our knowledge, its exact value remains unknown. Khalil's upper bound [Kha20], specialized to C^2, gives

$$\dim_H(\mathrm{Sing}(2) \cap C^2) \le \frac{2}{3} \dim_H(C^2) = \frac{4 \log 2}{3 \log 3}.$$

In the weighted setting, Aggarwal–Ghosh [AG26, Theorem 1.3, case (2), and Remark 1.4] give the upper bound

$$\dim_H(\mathrm{Sing}(\mathbf{w}) \cap C^2) \le \frac{2w_1}{1 + w_1} \dim_H(C^2).$$

On the lower-bound side, Schleischitz [Sch22, Theorem 4.2] obtained

$$\dim_H(\mathrm{Sing}^*(2) \cap C^2) > 0.1255.$$

The following theorem gives a weighted lower bound and, in the unweighted case, improves this estimate.

Theorem 1.1. *Let C be the middle-third Cantor set and let $\mathbf{w} = (w_1, w_2)$, where $1 > w_1 \ge w_2 > 0$ and $w_1 + w_2 = 1$. Then*

$$\dim_H(\mathrm{Sing}^*(\mathbf{w}) \cap C^2) \ge \left(\frac{1 - w_1}{1 + w_1}\right)^2 \dim_H(C^2) = 2\left(\frac{1 - w_1}{1 + w_1}\right)^2 \frac{\log 2}{\log 3}.$$

In particular,

$$\dim_H(\mathrm{Sing}^*(2) \cap C^2) \ge \frac{2 \log 2}{9 \log 3}.$$

Since $2 \log 2/(9 \log 3) \approx 0.1402$, Theorem 1.1 improves the explicit estimate of Schleischitz. Beyond the Cantor square, these bounds extend in different directions: Khalil's upper-bound theorem applies to self-similar fractals satisfying the open set condition (OSC), the weighted upper bound of Aggarwal–Ghosh applies to Cartesian products of homogeneous one-dimensional self-similar sets satisfying OSC, and Schleischitz's lower bounds apply to Cartesian products of common-base missing-digit sets. The construction behind Theorem 1.1 yields the following lower bound for common-base integral self-similar sets.

For $A \subset \mathbb{R}^n$, write $\mathrm{aff}_{\mathbb{R}}(A)$ for the smallest affine subspace of $\mathbb{R}^n$ containing A. For a common-base integral self-similar set K with digit set $\mathcal{D} \subset \mathbb{Z}^n$ as in Definition 2.1, if $\mathrm{aff}_{\mathbb{R}}(\mathcal{D}) \neq \mathbb{R}^n$, then K is contained in a rational affine hyperplane and hence $K \subset \mathrm{Sing}(\mathbf{w})$. This case is trivial, so we restrict to $\mathrm{aff}_{\mathbb{R}}(\mathcal{D}) = \mathbb{R}^n$.

Theorem 1.2. *Let $K \subset \mathbb{R}^n$ be a common-base integral self-similar set with digit set $\mathcal{D} \subset \mathbb{Z}^n$ as in Definition 2.1, and suppose that its defining system satisfies the open set condition as in Definition 2.2. Assume that $\mathrm{aff}_{\mathbb{R}}(\mathcal{D}) = \mathbb{R}^n$. Then*

$$\dim_H(\mathrm{Sing}^*(\mathbf{w}) \cap K) \geq \left(\frac{1 - w_1}{1 + w_1}\right)^2 \dim_H K.$$

The relation among these vector-case dimension results is summarized in Table 1.

Reference	Setting	Conclusion
[Che11]	Ambient $\mathbb{R}^2$, unweighted	$\dim_H \mathrm{Sing}(2) = 4/3$
[CC16]	Ambient $\mathbb{R}^n$, unweighted	$\dim_H \mathrm{Sing}(n) = n^2/(n+1)$
[LSST20]	Ambient $\mathbb{R}^2$, weighted	$\dim_H \mathrm{Sing}(\mathbf{w}) = 2 - 1/(1+w_1)$
[KP24]	Ambient $\mathbb{R}^n$, weighted	$\dim_H \mathrm{Sing}(\mathbf{w}) \geq n - 1/(1+w_1)$
[Kha20]	Self-similar fractals, unweighted	Hausdorff-dimension upper bound $\dim_H(\mathrm{Sing}(2) \cap C^2) \leq \frac{2}{3} \dim_H(C^2)$
[AG26]	Homogeneous self-similar products, weighted	Hausdorff-dimension upper bound $\dim_H(\mathrm{Sing}(\mathbf{w}) \cap C^2) \leq \frac{2w_1}{1+w_1} \dim_H(C^2)$
[Sch22]	Common-base missing-digit products, unweighted	Hausdorff-dimension lower bound $\dim_H(\mathrm{Sing}^*(2) \cap C^2) > 0.1255$
This paper, Theorem 1.1	Middle-third Cantor square, weighted	$\dim_H(\mathrm{Sing}^*(\mathbf{w}) \cap C^2) \geq ((1-w_1)/(1+w_1))^2 \dim_H(C^2)$
This paper, Theorem 1.2	Affine-spanning common-base integral self-similar sets, weighted	$\dim_H(\mathrm{Sing}^*(\mathbf{w}) \cap K) \geq ((1-w_1)/(1+w_1))^2 \dim_H K$

Table 1. Selected Hausdorff-dimension results in the vector case.

We next briefly describe the main idea of the proof. In the base-b coding of K, we require all digits within each of a sequence of disjoint blocks to equal one fixed digit. Each block gives the required rational approximation for a range of values of the parameter T in (1.2). The blocks are arranged so that these ranges overlap and cover all sufficiently large T, making every point in the resulting set $\mathbf{w}$-singular. The digits outside the prescribed blocks remain free. Under OSC, their lower density yields the Hausdorff-dimension estimate used below; optimizing the block lengths and positions gives the stated lower bound. Finally, the full affine-span assumption ensures that rational affine hyperplanes have zero measure for this construction, so the same bound holds after restricting to totally irrational points.

The paper is organized as follows. In Section 2, we develop the common-base setting, prove a dimension lower bound for sets obtained by prescribing digits, and establish that the associated measures, when there are infinitely many free positions, give zero mass to affine hyperplanes. Section 3 derives the approximation supplied by a constant-symbol block. The block construction is carried out in Section 4, and the required parameter estimates are proved in Section 5.

Use of artificial intelligence. OpenAI Codex was used as an auxiliary tool in checking selected calculations and arguments and in refining the exposition. The authors independently verified every mathematical argument, determined the final content and wording, and take full responsibility for the manuscript.

2. Common-base fractals and digit-freezing measures

This section introduces the common-base self-similar setting. We then derive a uniform cylinder estimate from OSC and use it to obtain a Hausdorff-dimension lower bound for sets obtained by prescribing digits. Under the full affine-span hypothesis, we also show that the associated measures give zero mass to affine hyperplanes provided that infinitely many positions remain free.

Definition 2.1 (Common-base integral self-similar sets and cylinders)**.** Fix an integer $b \geq 2$ and a finite digit set $\mathcal{D} \subset \mathbb{Z}^n$ with $\#\mathcal{D} \geq 2$. The maps

$$F_{\mathbf{a}}(x) = \frac{x + \mathbf{a}}{b}, \qquad \mathbf{a} \in \mathcal{D},$$

form a *common-base integral self-similar system*. This is a special case of Hutchinson's general self-similar construction in which all similarities have the same linear part $b^{-1}I$ and the digit vectors lie in $\mathbb{Z}^n$ [Hut81]. The unique nonempty compact set K satisfying

$$K = \bigcup_{\mathbf{a} \in \mathcal{D}} F_{\mathbf{a}}(K)$$

is its *common-base integral self-similar set*.

For a word $\xi = (\xi_1, \ldots, \xi_N) \in \mathcal{D}^N$, let

$$F_\xi := F_{\xi_1} \circ \cdots \circ F_{\xi_N}, \qquad K_\xi := F_\xi(K),$$

and call K_ξ the corresponding level-N cylinder.

The associated coding map is the surjection

$$\pi : \mathcal{D}^{\mathbb{N}} \longrightarrow K, \qquad \pi(\eta) := \sum_{r=1}^{\infty} \eta_r b^{-r},$$

so that $K = \pi(\mathcal{D}^{\mathbb{N}})$. For every $\xi = (\xi_1, \ldots, \xi_N) \in \mathcal{D}^N$ and $y \in K$, direct iteration gives

$$F_\xi(y) = \sum_{r=1}^{N} \xi_r b^{-r} + b^{-N} y, \qquad K_\xi = \{\pi(\eta) : \eta|_N = \xi\}, \tag{2.1}$$

where $\eta|_N = (\eta_1, \ldots, \eta_N)$.

Definition 2.2 (Open set condition)**.** The system in Definition 2.1 satisfies the *open set condition* (OSC) if there is a nonempty open set $O \subset \mathbb{R}^n$ such that

$$F_{\mathbf{a}}(O) \subset O \quad \text{for every } \mathbf{a} \in \mathcal{D}, \qquad F_{\mathbf{a}}(O) \cap F_{\mathbf{a}'}(O) = \varnothing \quad \text{if } \mathbf{a} \neq \mathbf{a}'.$$

Remark 2.3. If the common-base system satisfies OSC, then Hutchinson's dimension theorem [Hut81, Theorem 5.3] gives

$$\dim_H K = \frac{\log(\#\mathcal{D})}{\log b}.$$

Equivalently, $\#\mathcal{D} = b^{\dim_H K}$.

The next lemma records a uniform cylinder-counting estimate under OSC. Throughout, $B(x, r)$ denotes the max-norm ball of radius r centered at x.

Lemma 2.4. *Suppose that the common-base system satisfies OSC. Then there is $C_K < \infty$ such that*

$$\sup_{N \geq 1} \sup_{x \in \mathbb{R}^n} \# \left\{ \xi \in \mathcal{D}^N : K_\xi \cap B(x, b^{-N}) \neq \varnothing \right\} \leq C_K.$$

Proof. Fix $N \geq 1$ and $x \in \mathbb{R}^n$. Let O be an open set as in Definition 2.2. Choose $z \in O$ and $r > 0$ such that $B(z, r) \subset O$, and put

$$M := \max_{y \in K} \|z - y\|.$$

Set

$$\mathcal{I}_{N,x} := \{\xi \in \mathcal{D}^N : K_\xi \cap B(x, b^{-N}) \neq \varnothing\}.$$

For each $\xi \in \mathcal{I}_{N,x}$, since $K_\xi = F_\xi(K)$, choose $y_\xi \in K$ such that $F_\xi(y_\xi) \in B(x, b^{-N})$. Since F_ξ has contraction ratio b^{-N},

$$\|F_\xi(z) - x\| \leq b^{-N} \|z - y_\xi\| + b^{-N} \leq (M + 1) b^{-N}.$$

Iterating OSC, the sets $F_\xi(O)$, $\xi \in \mathcal{D}^N$, are pairwise disjoint. Since $B(z,r) \subset O$ and F_ξ is a similarity, the balls

$$A_\xi := F_\xi(B(z,r)) = B(F_\xi(z), rb^{-N}), \qquad \xi \in \mathcal{I}_{N,x},$$

are pairwise disjoint, and the preceding bound gives $A_\xi \subset B(x, (M+1+r)b^{-N})$. Comparing volumes therefore yields

$$\#\mathcal{I}_{N,x}\big(rb^{-N}\big)^n \le \big((M+1+r)b^{-N}\big)^n.$$

Thus the counting estimate holds with $C_K = ((M+1+r)/r)^n$. Since N and x were arbitrary, the bound is uniform. $\square$

Definition 2.5 (Digit-freezing measures)**.** Fix $\mathbf{a}_* \in \mathcal{D}$. For $P \subset \mathbb{N}$, define

$$U_P(N) := \#(\{1, \dots, N\} \setminus P), \qquad N \ge 1, \tag{2.2}$$

and

$$E(P) := \{\pi(\xi) : \xi_r = \mathbf{a}_* \text{ for every } r \in P\}. \tag{2.3}$$

The positions in P are *prescribed*, while those in $\mathbb{N} \setminus P$ are *free*; thus $U_P(N)$ counts the free positions up to N.

To each $P \subset \mathbb{N}$, associate the following measures. Let

$$m_{\mathcal{D}} := \frac{1}{\#\mathcal{D}} \sum_{\mathbf{a} \in \mathcal{D}} \delta_{\mathbf{a}},$$

where $\delta_{\mathbf{a}}$ is the Dirac mass at $\mathbf{a}$, and for $r \ge 1$ set

$$\nu_{P,r} := \begin{cases} \delta_{\mathbf{a}_*}, & r \in P, \\ m_{\mathcal{D}}, & r \notin P. \end{cases}$$

Define

$$\nu_P := \bigotimes_{r=1}^{\infty} \nu_{P,r} \quad \text{on } \mathcal{D}^{\mathbb{N}}, \qquad \mu_P := \pi_* \nu_P \quad \text{on } K.$$

We call μ_P the *digit-freezing measure* associated with P. It is supported on $E(P)$, and in particular $\mu_P(E(P)) = 1$.

Lemma 2.6. *Suppose that the common-base system satisfies OSC. Then*

$$\dim_H E(P) \ge \left(\liminf_{N \to \infty} \frac{U_P(N)}{N} \right) \dim_H K.$$

Proof. Put

$$h := \liminf_{N \to \infty} \frac{U_P(N)}{N}.$$

For the measures in Definition 2.5, every $\xi \in \mathcal{D}^N$ compatible with the prescribed positions satisfies $\nu_P\{\eta : \eta|_N = \xi\} = (\#\mathcal{D})^{-U_P(N)}$.

If $h = 0$, the claim is immediate. Assume $h > 0$, and fix $0 < \tau < h$. For all sufficiently large N, we have $U_P(N) \ge \tau N$. If $b^{-(N+1)} < R \le b^{-N}$, then

$B(x,R) \subset B(x,b^{-N})$. By Lemma 2.4, at most C_K level-N cylinders meet $B(x,R)$. Moreover,

$$\pi^{-1}(B(x,R)) \cap \operatorname{supp} \nu_P \subset \bigcup_{\substack{\xi \in \mathcal{D}^N \\ K_\xi \cap B(x,R) \neq \varnothing}} \{\eta \in \operatorname{supp} \nu_P : \eta|_N = \xi\}.$$

The union on the right has at most C_K nonempty sets, each of ν_P-mass $(\#\mathcal{D})^{-U_P(N)}$. Since $\#\mathcal{D} = b^{\dim_H K}$ by Remark 2.3, we have

$$\mu_P(B(x,R)) \leq C_K (\#\mathcal{D})^{-U_P(N)} \leq C_K\, b^{\tau \dim_H K} R^{\tau \dim_H K}.$$

Here the last inequality uses $U_P(N) \geq \tau N$ and $R > b^{-(N+1)}$. The mass distribution principle [Fal03, Principle 4.2] gives $\dim_H E(P) \geq \tau \dim_H K$. Letting $\tau \uparrow h$ proves the claim. □

Remark 2.7. Under the hypothesis of Lemma 2.6, although only the lower bound is needed below, one can in fact show that equality holds.

To pass to totally irrational points, we use the following property.

Lemma 2.8. *Let $P \subset \mathbb{N}$ have infinitely many free positions, and let ν_P, μ_P be as in Definition 2.5. If $\operatorname{aff}_{\mathbb{R}}(\mathcal{D}) = \mathbb{R}^n$, then every affine hyperplane has μ_P-measure zero; in particular, μ_P-almost every point is totally irrational.*

Proof. Fix an affine hyperplane

$$H = \{x \in \mathbb{R}^n : u \cdot x = t\}, \qquad u \in \mathbb{R}^n \setminus \{0\}, \quad t \in \mathbb{R}.$$

Since $\operatorname{aff}_{\mathbb{R}}(\mathcal{D}) = \mathbb{R}^n$, the set $u \cdot \mathcal{D}$ has at least two elements. Put

$$\delta := \min_{\substack{c,c' \in u\cdot\mathcal{D} \\ c \neq c'}} |c - c'| > 0.$$

Choose $L \geq 1$ so that

$$\frac{\operatorname{diam}(u \cdot \mathcal{D})}{b^L - 1} < \delta,$$

and choose free positions

$$r_1 < r_2 < \cdots, \qquad r_{j+1} - r_j \geq L.$$

Put

$$S := \{r_j : j \geq 1\}, \qquad \nu_S := \bigotimes_{r \in S} m_{\mathcal{D}}, \qquad \nu_{S^c} := \bigotimes_{r \notin S} \nu_{P,r}.$$

Since every position in S is free, the coordinate identification $\mathcal{D}^{\mathbb{N}} \simeq \mathcal{D}^S \times \mathcal{D}^{S^c}$ identifies ν_P with $\nu_S \otimes \nu_{S^c}$. For $\eta \in \mathcal{D}^{S^c}$, define

$$B_\eta := \{\zeta \in \mathcal{D}^S : \pi(\zeta, \eta) \in H\}.$$

Fubini's theorem gives

$$\mu_P(H) = \nu_P(\pi^{-1}(H)) = \int_{\mathcal{D}^{S^c}} \nu_S(B_\eta)\, d\nu_{S^c}(\eta). \tag{2.4}$$

Thus it remains to prove the following.

Claim. For every $\eta \in \mathcal{D}^{S^c}$,

$$\nu_S(B_\eta) = 0.$$

Proof of the claim. By the choice of L and the gaps $r_{j+1} - r_j$, for every $j \geq 1$,

$$\begin{aligned}\operatorname{diam}(u \cdot \mathcal{D}) \sum_{\ell > j} b^{-r_\ell} &\leq \operatorname{diam}(u \cdot \mathcal{D}) b^{-r_j} \sum_{q \geq 1} b^{-qL} \\ &= \frac{\operatorname{diam}(u \cdot \mathcal{D})}{b^L - 1} b^{-r_j} < \delta b^{-r_j}.\end{aligned} \tag{2.5}$$

Fix $\eta \in \mathcal{D}^{S^c}$. The equation defining B_η is

$$\sum_{j \geq 1} (u \cdot \zeta_{r_j}) b^{-r_j} = t - \sum_{r \notin S} (u \cdot \eta_r) b^{-r}. \tag{2.6}$$

For distinct sequences $(c_j)_{j \geq 1}, (c'_j)_{j \geq 1} \in (u \cdot \mathcal{D})^{\mathbb{N}}$, put $k := \min\{j : c_j \neq c'_j\}$. By (2.5),

$$\begin{aligned}\left| \sum_{j \geq 1} (c_j - c'_j) b^{-r_j} \right| &\geq \delta b^{-r_k} - \operatorname{diam}(u \cdot \mathcal{D}) \sum_{\ell > k} b^{-r_\ell} \\ &> 0.\end{aligned}$$

Thus distinct projected sequences give distinct values of the left-hand side of (2.6). Since its right-hand side is fixed by η, (2.6) admits at most one projected sequence. Put

$$m := \max_{c \in u \cdot \mathcal{D}} \#\{\mathbf{a} \in \mathcal{D} : u \cdot \mathbf{a} = c\} < \#\mathcal{D}.$$

The strict inequality holds because $u \cdot \mathcal{D}$ has at least two elements. If $B_\eta = \varnothing$, the claim is immediate. Otherwise, let $(c_j)_{j \geq 1}$ be its projected sequence. For every $N \geq 1$,

$$\begin{aligned}0 \leq \nu_S(B_\eta) &\leq \prod_{j=1}^{N} \frac{\#\{\mathbf{a} \in \mathcal{D} : u \cdot \mathbf{a} = c_j\}}{\#\mathcal{D}} \\ &\leq \left(\frac{m}{\#\mathcal{D}} \right)^N \xrightarrow[N \to \infty]{} 0.\end{aligned}$$

$\square$

The claim and (2.4) give $\mu_P(H) = 0$. Finally, the set of points that are not totally irrational is the countable union

$$\bigcup_{\mathbf{p} \in \mathbb{Z}^n \setminus \{0\}} \bigcup_{p_0 \in \mathbb{Z}} \{x \in \mathbb{R}^n : \mathbf{p} \cdot x = -p_0\}$$

of affine hyperplanes, so it has μ_P-measure zero. $\square$

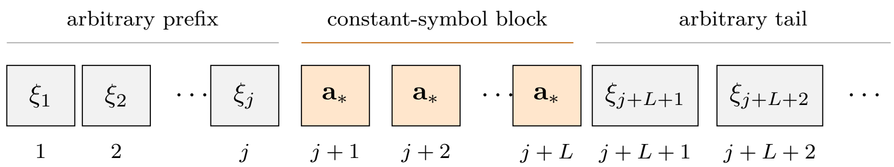

FIGURE 1. A constant-symbol block as in Lemma 3.1. The orange-shaded digits are fixed, while the prefix and tail are arbitrary.

3. Approximation from constant-symbol blocks

Fix the weight $\mathbf{w} = (w_1, \ldots, w_n)$ from the introduction. The next lemma converts a constant-symbol block into the weighted approximation in (1.2).

Lemma 3.1. *There is a constant $C_0 \geq 1$, depending only on $\mathcal{D}$ and $\mathbf{w}$, with the following property. Suppose $x = \pi(\xi) \in K$ and there are integers $j, L \geq 1$ such that*

$$\xi_{j+1} = \cdots = \xi_{j+L} = \mathbf{a}_*;$$

see Figure 1. Set $q := (b-1)b^j$. Then there is $\mathbf{p} \in \mathbb{Z}^n$ such that

$$\max_{1 \leq i \leq n} |qx_i - p_i|^{1/w_i} \leq C_0 b^{-L/w_1}.$$

Moreover, if $\varepsilon > 0$ and

$$q < T < \frac{\varepsilon}{C_0} b^{L/w_1},$$

then the same q and $\mathbf{p}$ satisfy the inequalities in (1.2).

Proof. Set

$$\Delta_{\mathcal{D}} := \max_{\mathbf{a}, \mathbf{a}' \in \mathcal{D}} \|\mathbf{a} - \mathbf{a}'\|, \qquad C_0 := \Delta_{\mathcal{D}}^{1/w_n}.$$

Since $\mathcal{D} \subset \mathbb{Z}^n$ has at least two elements, $\Delta_{\mathcal{D}} \geq 1$, so $C_0 \geq 1$.

Extend the prefix $(\xi_1, \ldots, \xi_j)$ by repeating $\mathbf{a}_*$ forever. The resulting point lies in K and is the rational vector

$$\frac{\mathbf{p}}{q} := \sum_{r=1}^{j} \xi_r b^{-r} + \sum_{r=j+1}^{\infty} \mathbf{a}_* b^{-r},$$

where $q = (b-1)b^j$ and

$$\mathbf{p} = (b-1) \sum_{r=1}^{j} \xi_r b^{j-r} + \mathbf{a}_* \in \mathbb{Z}^n.$$

Let $y := \pi(\xi_{j+L+1}, \xi_{j+L+2}, \ldots) \in K$ be the tail after the constant-symbol block. Since x and $\mathbf{p}/q$ have the same first $j+L$ digits,

$$x - \frac{\mathbf{p}}{q} = \sum_{r=j+L+1}^{\infty} (\xi_r - \mathbf{a}_*) b^{-r} = b^{-(j+L)} \left(y - \frac{\mathbf{a}_*}{b-1} \right).$$

Multiplying by $q=(b-1)b^j$ gives

$$qx-\mathbf{p}=(b-1)b^{-L}\left(y-\frac{\mathbf{a}_*}{b-1}\right).$$

Moreover,

$$\left\|y-\frac{\mathbf{a}_*}{b-1}\right\|\le\sum_{r=1}^{\infty}\|\xi_{j+L+r}-\mathbf{a}_*\|b^{-r}\le\Delta_{\mathcal{D}}\sum_{r=1}^{\infty}b^{-r}=\frac{\Delta_{\mathcal{D}}}{b-1}.$$

It follows that

$$|qx_i-p_i|\le\Delta_{\mathcal{D}}b^{-L}\qquad(1\le i\le n).$$

For each i, the inequalities $w_n\le w_i\le w_1$ and $b>1$ give

$$|qx_i-p_i|^{1/w_i}\le\Delta_{\mathcal{D}}^{1/w_i}b^{-L/w_i}\le\Delta_{\mathcal{D}}^{1/w_n}b^{-L/w_1}=C_0b^{-L/w_1}.$$

Taking the maximum over i proves the first assertion. Finally, if $q<T<(\varepsilon/C_0)b^{L/w_1}$, then $0<q<T$ and

$$C_0b^{-L/w_1}<\frac{\varepsilon}{T},$$

which proves the second assertion. $\square$

4. Block construction

Fix C_0 as in Lemma 3.1. The following proposition provides the block locations and lengths used in the construction. Its proof is given in Section 5.

Proposition 4.1. *For any fixed $\alpha>w_1/(1-w_1)$, there exist sequences of positive integers $j_k=j_k(\alpha)$ and $L_k=L_k(\alpha)$ such that the sets and denominators*

$$B_k:=\{j_k+1,\dots,j_k+L_k\},\qquad q_k:=(b-1)b^{j_k},\qquad P_\alpha:=\bigcup_{k\ge1}B_k$$

satisfy

$$j_k+L_k<j_{k+1}\qquad(k\ge1).$$

Moreover, for every $\varepsilon>0$, there exists k_0 such that

$$(q_{k_0},\infty)\subset\bigcup_{k\ge k_0}\left(q_k,\frac{\varepsilon}{C_0}b^{L_k/w_1}\right).\tag{4.1}$$

Finally, with the notation of (2.2), set

$$U_\alpha(N):=U_{P_\alpha}(N).$$

Then

$$\liminf_{N\to\infty}\frac{U_\alpha(N)}{N}=h_{\mathbf{w}}(\alpha):=\frac{\alpha(1-w_1)-w_1}{(\alpha-w_1)(1+\alpha)}.\tag{4.2}$$

With the notation of Definition 2.5, define

$$E_\alpha := E(P_\alpha) = \left\{ \pi(\xi) : \begin{array}{l} \xi \in \mathcal{D}^{\mathbb{N}}, \\ \xi_{j_k+r} = \mathbf{a}_* \quad (k \geq 1,\ 1 \leq r \leq L_k) \end{array} \right\}.$$

Thus E_α is obtained by fixing the digits on the blocks B_k, while all remaining digits are free.

Corollary 4.2. *One has*

$$E_\alpha \subset \mathrm{Sing}(\mathbf{w}) \cap K.$$

Proof. Fix $\varepsilon > 0$, and choose k_0 as in (4.1). If $T > q_{k_0}$, then there is $k \geq k_0$ such that

$$q_k < T < \frac{\varepsilon}{C_0} b^{L_k/w_1}.$$

For every $x \in E_\alpha$, the k-th prescribed block and Lemma 3.1 give $\mathbf{p}_k \in \mathbb{Z}^n$ such that

$$0 < q_k < T, \qquad \max_{1 \leq i \leq n} |q_k x_i - p_{i,k}|^{1/w_i} < \frac{\varepsilon}{T}.$$

Hence $x \in \mathrm{Sing}(\mathbf{w})$. □

Corollary 4.3. *Suppose that the common-base system satisfies OSC. Then*

$$\dim_H E_\alpha \geq h_{\mathbf{w}}(\alpha) \dim_H K.$$

Proof. This follows from (4.2) and Lemma 2.6, since $U_\alpha = U_{P_\alpha}$. □

Proof of Theorem 1.2. By Corollaries 4.2 and 4.3, the set E_α is contained in $\mathrm{Sing}(\mathbf{w}) \cap K$ and has dimension at least $h_{\mathbf{w}}(\alpha) \dim_H K$ for every $\alpha > w_1/(1-w_1)$. Differentiation gives

$$h'_{\mathbf{w}}(\alpha) = -\frac{\alpha((1-w_1)\alpha - 2w_1)}{(\alpha - w_1)^2(1+\alpha)^2}.$$

The unique critical point in the admissible interval is $\alpha_* := 2w_1/(1-w_1)$. Since the function tends to zero at both ends of the interval, this point gives its maximum, and $h_{\mathbf{w}}(\alpha_*) = ((1-w_1)/(1+w_1))^2$. Put $E = E_{\alpha_*}$ and $\mu := \mu_{P_{\alpha_*}}$, as in Definition 2.5. Since $h_{\mathbf{w}}(\alpha_*) > 0$, there are infinitely many free positions, and Lemma 2.8 shows that μ-almost every point is totally irrational. Since $E \subset \mathrm{Sing}(\mathbf{w})$, the set $E \cap \mathrm{Sing}^*(\mathbf{w})$ has full μ-measure. For every $s < h_{\mathbf{w}}(\alpha_*) \dim_H K$, the ball estimate in the proof of Lemma 2.6 remains valid after restricting μ to this set. Consequently,

$$\dim_H(E \cap \mathrm{Sing}^*(\mathbf{w})) \geq h_{\mathbf{w}}(\alpha_*) \dim_H K.$$

Since $E \subset K$, this proves the theorem. □

Proof of Theorem 1.1. The Cantor square C^2 is the attractor of the base-3 system with digit set $\{0,2\}^2$, which affinely spans $\mathbb{R}^2$. It satisfies OSC with $O = (0,1)^2$, and $\dim_H(C^2) = 2\log 2/\log 3$. The weighted assertion is therefore the special case $b = 3$, $\mathcal{D} = \{0,2\}^2$ of Theorem 1.2. The unweighted statement follows from $w_1 = w_2 = 1/2$. □

5. Block parameter estimates

Proof of Proposition 4.1. Fix $\alpha > w_1/(1-w_1)$.

Step 1: Choice, growth, and separation. Put

$$\rho := \frac{\alpha}{w_1}, \qquad \rho_0 := \frac{1+\rho}{2}.$$

Since both $\rho-\rho_0$ and $\alpha(1-w_1)/w_1 - 1$ are positive, choose j_1 so large that, for every $j \geq j_1$,

$$\min\left\{\rho-\rho_0, \frac{\alpha(1-w_1)}{w_1} - 1\right\} j \geq \sqrt{j} + 3.$$

Define recursively

$$L_k := \lceil \alpha j_k \rceil, \qquad j_{k+1} := \left\lfloor \frac{L_k}{w_1} \right\rfloor - \lceil \sqrt{j_k} \rceil. \tag{5.1}$$

The ceiling and floor bounds in (5.1) give

$$\begin{aligned} \alpha j_k \leq L_k &< \alpha j_k + 1, \\ \sqrt{j_k} \leq \frac{L_k}{w_1} - j_{k+1} &< \sqrt{j_k} + 2, \\ \rho j_k - \sqrt{j_k} - 2 < j_{k+1} &< \rho j_k + \frac{1}{w_1}. \end{aligned} \tag{5.2}$$

Suppose that $j_k \geq j_1$. Then the first condition in the choice of j_1, together with (5.2), gives

$$\begin{aligned} j_{k+1} &> \rho j_k - \sqrt{j_k} - 2 \\ &= \rho_0 j_k + \big((\rho-\rho_0) j_k - \sqrt{j_k} - 2\big) \geq \rho_0 j_k + 1. \end{aligned} \tag{5.3}$$

Since $\rho_0 > 1$, induction shows that (5.3) holds for every k and that $j_k \geq \rho_0^{k-1} j_1$. In particular, $j_k \to \infty$. Moreover,

$$\begin{aligned} j_{k+1} - j_k - L_k &> L_k \left(\frac{1}{w_1} - 1\right) - j_k - \sqrt{j_k} - 2 \\ &\geq \left(\frac{\alpha(1-w_1)}{w_1} - 1\right) j_k - \sqrt{j_k} - 2 \\ &\geq 1. \end{aligned}$$

Hence $j_k + L_k < j_{k+1}$.

Step 2: Covering all large T. Set

$$\varepsilon_k := 2(b-1) C_0 b^{-(L_k/w_1 - j_{k+1})}.$$

By the middle estimate in (5.2),

$$0 < \varepsilon_k \leq 2(b-1) C_0 b^{-\sqrt{j_k}} \longrightarrow 0.$$

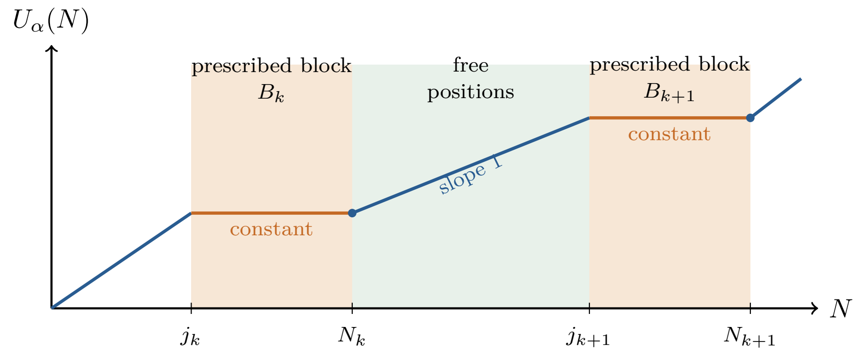

FIGURE 2. The count is constant on prescribed blocks and has slope one elsewhere, so $U_\alpha(N)/N$ is minimized at N_k.

Furthermore, a direct cancellation gives

$$\begin{aligned}\frac{\varepsilon_k}{C_0} b^{L_k/w_1} &= 2(b-1)b^{-(L_k/w_1 - j_{k+1})} b^{L_k/w_1}\\ &= 2(b-1)b^{j_{k+1}} = 2q_{k+1} > q_{k+1}.\end{aligned}$$

Given $\varepsilon > 0$, choose k_0 so that $\varepsilon_k < \varepsilon$ for $k \ge k_0$. If $T > q_{k_0}$, choose $k \ge k_0$ such that $q_k < T \le q_{k+1}$, which is possible since $q_k \to \infty$. Then

$$q_k < T \le q_{k+1} < \frac{\varepsilon_k}{C_0} b^{L_k/w_1} < \frac{\varepsilon}{C_0} b^{L_k/w_1}.$$

Thus T belongs to the k-th interval in (4.1).

Step 3: Density of free positions. Let $N_k := j_k + L_k$. Since the prescribed blocks are disjoint, U_α is constant on $[j_k, N_k]$ and increases with slope one on $[N_k, j_{k+1}]$. Consequently, $U_\alpha(N)/N$ decreases on the first interval and increases on the second. Its successive local minima therefore occur at N_k, as illustrated in Figure 2, and

$$\liminf_{N\to\infty} \frac{U_\alpha(N)}{N} = \liminf_{k\to\infty} \frac{j_k - \sum_{h<k} L_h}{j_k + L_k}. \tag{5.4}$$

By (5.2) and the fact that $j_k \to \infty$,

$$\frac{L_k}{j_k} \longrightarrow \alpha, \qquad \frac{j_{k+1}}{j_k} \longrightarrow \rho.$$

Hence

$$\frac{L_k}{j_{k+1} - j_k} = \frac{L_k/j_k}{j_{k+1}/j_k - 1} \longrightarrow \frac{\alpha}{\rho - 1}.$$

Since j_k is strictly increasing and tends to infinity, the Stolz–Cesàro theorem gives

$$\lim_{k\to\infty} \frac{\sum_{h<k} L_h}{j_k} = \lim_{k\to\infty} \frac{L_k}{j_{k+1} - j_k} = \frac{\alpha}{\rho - 1}.$$

Consequently,

$$\frac{U_\alpha(N_k)}{N_k} = \frac{j_k - \sum_{h<k} L_h}{j_k + L_k} \longrightarrow \frac{1-\alpha/(\rho-1)}{1+\alpha} = \frac{\alpha(1-w_1)-w_1}{(\alpha-w_1)(1+\alpha)}.$$

Together with (5.4), this proves (4.2). ☐

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