

A SUGAWARA–LEGENDRE MECHANISM FOR THE FOUR-POINT HEISENBERG ALGEBRA

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ABSTRACT. This paper provides a representation-theoretic explanation for the recent discovery that families of orthogonal polynomials arise in the centers of universal central extensions of superelliptic affine Lie algebras. Working in the Verma modules of the Heisenberg subalgebra, we compute the canonical contravariant form in closed form on a distinguished family of weight vectors, and show that this family is the Legendre family, forced by the curve itself. The classical squared norms follow by a rescaling available under a positivity condition on the central charge, and the module is irreducible exactly when that charge is non-zero. The main result identifies a Sugawara element carried, by an explicit intertwiner, to the classical Legendre differential operator. The results are proved for the genus-zero, four-point case.

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1. INTRODUCTION

The two strands of mathematics joined in this paper are the classical theory of orthogonal polynomials and the representation theory of infinite-dimensional Lie algebras attached to algebraic curves. The former goes back to the late eighteenth century, when Legendre introduced what are now called the Legendre polynomials in his study of gravitational potentials. The classical families (Legendre, Hermite, Laguerre, Jacobi) and their orthogonality properties were codified in the nineteenth

2020 *Mathematics Subject Classification.* 17B67, 17B65, 33C45.

Key words and phrases. Krichever–Novikov algebras, four-point affine algebras, genus-zero curves, Heisenberg algebras, universal central extensions, Verma modules, Shapovalov form, Sugawara construction, Legendre polynomials, Legendre operator, irreducibility criteria.

Supported by FAPESP grant 2024/14914-9.

century and brought into modern form in Szegő's monograph [5]; see also [24] for the analytic background. Each classical family is characterised by a three-term recurrence and a second-order differential equation, two conditions which are equivalent under the correspondence [5, §3.2].

The latter strand begins with Krichever and Novikov, who introduced algebras of meromorphic vector fields on a compact Riemann surface with poles confined to *two* marked points, as higher-genus analogues of the Witt and Virasoro algebras [22]. The generalisation to a finite set of more than two marked points (together with the almost-graded structure that makes the representation theory tractable) is due to Schlichenmaier [11, 12]; a related multipoint construction was given independently by Sodin [16]. The general theory of Krichever–Novikov type algebras and their universal central extensions has since been developed systematically; we refer to [9, 10] for a detailed account. The representation-theoretic side draws on the Verma-module formalism for affine Lie algebras [7, 18] and the broader theory of contravariant forms initiated by Shapovalov [8]; on the conformal side it is informed by the work of Feigin and Fuchs [21] on Virasoro modules. In a different direction, extended affine and map-algebra modules have been the subject of an active recent literature [20, 17, 23].

The two strands meet in the superelliptic setting. For the curve $u^m = P(t)$, explicit cocycle formulas for the universal central extension of $\mathfrak{sl}_2 \otimes A_m$ were obtained in [3], where it was observed empirically that the centre relations realise families of orthogonal polynomials that extend the classical list, with the Legendre family appearing in the genus-zero case and new families in the higher branch degrees. Irreducibility of the relevant Verma modules in the genus-zero case was established in [1]. Left open in both works was the representation-theoretic mechanism by which the analytic phenomena, namely the orthogonality of the polynomial basis, the second-order differential equation governing it, and (in special cases) completeness, arise from the module theory of the underlying Heisenberg subalgebra $\mathcal{H}_m$. The present paper supplies that mechanism in the genus-zero case.

In an earlier strand of the author's work on universal central extensions of Krichever–Novikov-type algebras attached to superelliptic curves $u^m = P(x)$ [3, 1], orthogonal polynomial families were observed to arise as a structural feature of the centre. For a curve of genus g with N marked points, the centre of the universal central extension of $\mathfrak{sl}_2 \otimes A_m$ has dimension $2g + N - 1$ [14, 13]; in the genus-zero, four-point case (Section 2) it is three-dimensional. The centre generators satisfy three-term recurrences (the defining relation of orthogonal polynomial systems), and for $m \geq 3$ the corresponding polynomial families lie beyond the classical list. In that earlier work, the analytic phenomena, namely orthogonality, the second-order differential equation, and (in special cases) completeness of the polynomial basis, were established by explicit computation [3, Thm. 3.4 and §4], but their relation to the module theory of the Heisenberg subalgebra $\mathcal{H}_m$ was left open.

The present paper works out the genus-zero case $m = 2$, in which $p(t) = 1 - 2at + t^2$ (we will say $r = 1$); here m is the covering degree and r the branch parameter of the palindromic polynomial $p(t) = 1 - 2at^r + t^{2r}$, both fixed in Section 2. Higher branch degrees are not treated. We show that:

- (1) In the genus-zero case, the canonical Shapovalov form on the Verma module is computed in closed form on the polynomial basis $\{\tilde{P}_n\}$; orthogonality is forced by the weight grading, and the cocycle determines the norms.
- (2) In the genus-zero case, the Legendre differential operator arises as the image, under the intertwiner Φ , of an explicit operator built from an explicit element of the completed universal enveloping algebra, after a normalisation depending on the central charge.
- (3) In the genus-zero case, irreducibility of the module and completeness of the polynomial basis are equivalent to non-degeneracy of the Shapovalov form, and this in turn holds exactly when the central charge is non-zero: the contravariant form is diagonal in the monomial basis, so its determinant has no zeros beyond that one. For $m \geq 3$ the off-diagonal components of the cocycle break the diagonality, and the argument does not apply; that range lies outside the scope of this paper.

The main contributions of this paper are three theorems and one corollary, listed below.

- (A) **Theorem 3.7:** In the genus-zero case, the canonical contravariant form on the Verma module $M(\varphi)$ is diagonal on the family $\{P_n\}$, and its level- n value is computed in closed form. Orthogonality is forced by the grading; what the cocycle determines is the norm, and it is the norm that makes the classical Legendre normalisation available, under an explicit reality hypothesis on the central charge.
- (B) **Theorem 3.8:** In the genus-zero case ($m = 2$, $r = 1$), the module $M(\varphi)$ is irreducible if and only if the Shapovalov form is non-degenerate, if and only if $\varphi(c) \neq 0$; the proof computes the Shapovalov determinant in closed form. The corresponding question for $m \geq 3$ is not treated here.
- (C) **Theorem 4.9:** In the genus-zero case (with $m = 2$), there exist a Sugawara-type element $\mathbf{L} = \sum_{k \geq 1} :b_{-k}b_k:$ of the completed enveloping algebra $\widehat{U(\mathcal{H}_2)}$, acting on $M(\varphi)$ after normalisation as $L_0 = q^{-1}\mathbf{L}$, together with the operator $\Omega = -L_0(L_0 + \text{Id})$ built from it, a $U(\mathcal{H}_2^-)$ -equivariant polynomial quotient $\psi: M(\varphi) \rightarrow \mathbb{C}[\zeta]$, and a Legendre identification $\Psi: \mathbb{C}[\zeta] \rightarrow \mathbb{C}[x]$ such that the composite $\Phi := \Psi \circ \psi$ intertwines Ω on $M(\varphi)$ with the classical Legendre operator $L = (1 - x^2)\partial_x^2 - 2x\partial_x$ on $\mathbb{C}[x]$. Under Φ , the level- n image is the Legendre polynomial $P_n(x)$, characterised intrinsically as the (unique up to scalar) element of $\Phi(M(\varphi))$ of degree n on which L acts with eigenvalue $-n(n+1)$.
- (D) **Corollary 4.11:** The Casimir tower $\{\Omega^r\}_{r \geq 1}$ acts on the level- n image $\Phi(M(\varphi)) \cap \mathbb{C}[x]_{\leq n}$ by the scalar $(-n(n+1))^r$, and corresponds under Φ to the iterated Legendre operator L^r . A generating-function reformulation is given in Corollary 4.12.

One can relate this achievements to previous work. The contravariant form on Verma modules for Kac–Moody algebras [8, 7] is a standard tool for studying irreducibility and detecting singular vectors. This paper applies the same machinery to the Heisenberg subalgebra of the UCE in the superelliptic setting, where the form diagonalises in a basis of polynomial vectors, identifying orthogonality as a

module-theoretic phenomenon. The Sugawara element in affine Lie algebras [18] is a canonical central element whose eigenvalues on highest-weight modules yield the conformal weights in conformal field theory. Our element Ω is the standard Sugawara stress-energy zero mode of the free boson ($c = 1$, central charge), built from the Heisenberg subalgebra $\mathcal{H}_2$. The contribution of this paper is not the Sugawara element itself, but the construction of an explicit $U(\mathcal{H}_2^-)$ -equivariant polynomial quotient $\psi: M(\varphi) \rightarrow \mathbb{C}[\zeta]$ on which the descended $\bar{\Omega}$ acts diagonally with simple spectrum, together with the Legendre identification $\Psi: \mathbb{C}[\zeta] \rightarrow \mathbb{C}[x]$, $\zeta^n \mapsto P_n(x)$, which intertwines the descended Sugawara operator with the classical Legendre differential operator L . The combined map $\Phi = \Psi \circ \psi$ thus realises the Verma module as a graded presentation of the polynomial eigenfunctions of L . Map algebras and their representations have been extensively studied [17, 20]. Our approach borrows the weight-space and highest-weight module formalism from this literature, applied to the Krichever–Novikov setting.

Irreducibility criteria for Verma-type modules over Heisenberg subalgebras, in the setting of a non-standard polarisation of the imaginary modes, were obtained by Bekkert–Benkart–Futorny–Kashuba [6]; the present paper works with the standard polarisation and proves what it needs from Definition 2.1 directly. A treatment of the genus-zero case in the polarised setting is given in [1]. We also rely on the universal central extension of loop algebras due to [2] and on the classical theory of orthogonal polynomials with three-term recurrence [5]. The cocycle formulas of [3] are computed there for branch degree $\deg p = 4$; the degree-two case treated here is derived in Section 2. The present paper combines these inputs within the Sugawara–Legendre framework of Section 4, in which the orthogonal polynomial families and their governing differential operators arise as outputs of the module theory of $\mathcal{H}_m$ in the genus-zero case.

The paper is organized as follows. Section 2 introduces the coordinate ring A_m of the superelliptic curve, the Heisenberg subalgebra $\mathcal{H}_m$ with its cocycle, the triangular decomposition, and the module category. Section 3 constructs the contravariant Shapovalov form on the Verma module, proves existence and uniqueness, and computes it on the polynomial vectors (Theorem 3.7); the same computation, carried out on the full Poincaré–Birkhoff–Witt basis, yields the Shapovalov determinant and the irreducibility criterion (Theorem 3.8). Section 4 focuses on the genus-zero case ($m = 2$, $r = 1$) and is the technical heart of the paper: we define the Sugawara stress-energy zero mode $\mathbf{L} \in \widetilde{U(\mathcal{H}_2)}$, its normalisation L_0 on $M(\varphi)$, and the Casimir operator $\Omega := -L_0(L_0 + \text{Id})$, construct the polynomial quotient $\psi: M(\varphi) \rightarrow \mathbb{C}[\zeta]$ and the Legendre identification $\Psi: \mathbb{C}[\zeta] \rightarrow \mathbb{C}[x]$, and prove that the composite $\Phi := \Psi \circ \psi$ intertwines Ω on $M(\varphi)$ with the classical Legendre operator L on $\mathbb{C}[x]$ (Theorem 4.9). The Casimir tower $\{\Omega^r\}_{r \geq 1}$ then arises as a corollary (Corollary 4.11), with a generating-function reformulation (Corollary 4.12). Section 5 constructs the Fock space realization of $M(\varphi)$: it proves $M(\varphi) \cong \mathcal{F}_\varphi$ as $\mathcal{H}_2$ -modules (Theorem 5.2), identifies the Shapovalov form with the Fock inner product, and gives explicit Fock representatives for the level- n classes $\bar{P}_n \in \Phi(M(\varphi))$, in closed form for every n (Remark 5.5). Section 6 works the genus-zero Legendre case out explicitly and indicates what obstructs the same argument at $m = 4$. Section 7 places the results in relation to earlier work.

2. ALGEBRAIC SETUP

2.1. The coordinate ring and the Heisenberg subalgebra. Fix an integer $m \geq 2$, the *covering degree*, and a polynomial $p(t) \in \mathbb{C}[t]$ of degree d with no multiple zeros; the equation $u^m = p(t)$ defines a *superelliptic curve* C . Its *coordinate ring* is

$$A_m = \mathbb{C}[t^{\pm 1}, u] / (u^m - p(t)),$$

the ring of regular functions on the affine part of C with the fibres over $t = 0$ and $t = \infty$ removed. The projection $t: C \rightarrow \mathbb{P}^1$ exhibits C as an m -sheeted covering of the projective line, ramified over the zeros of p and possibly over $t = \infty$; on its affine part C is non-singular, since p has no multiple zeros. By the Riemann–Hurwitz formula the genus of C is

$$g = \frac{1}{2} [m(d-1) - d - \gcd(m, d)] + 1.$$

Throughout this paper we take p to be the *palindromic* polynomial

$$p(t) = 1 - 2at^r + t^{2r}, \quad a \in \mathbb{C}, \quad r \geq 1,$$

of degree $d = 2r$, which has no multiple zeros for $a \neq \pm 1$. For $m = 2$ the genus is then $g = r - 1$: the case $r = 1$ (degree two) is rational, $r = 2$ (a quartic) has genus one, and $r \geq 3$ is hyperelliptic in the usual sense. Our main results concern the case $m = 2$, $r = 1$.

This case merits closer description. Here $p(t) = 1 - 2at + t^2$ has degree two, so the curve $u^2 = p(t)$ is rational, and the covering $t: C \rightarrow \mathbb{P}^1$ is unramified over both $t = 0$ and $t = \infty$; two points of C lie over each. The algebra A_2 is therefore the ring of functions regular away from these *four* marked points, and $\mathfrak{sl}_2 \otimes A_2$ is the four-point affine Lie algebra of Bremner [14, Prop. 1.1] (see also Cox [15]). It is not a two-point Krichever–Novikov algebra. Its universal central extension has a three-dimensional centre [14, 13]. The palindromic symmetry $t \mapsto 1/t$ of p distinguishes one splitting of the four points into two pairs, hence one line in this space of local cocycles; the cocycle used throughout this paper is this symmetry-selected one. We work in the multipoint framework of Schlichenmaier [11, 12].

Given $\mathfrak{g} = \mathfrak{sl}_2$ (though the construction extends to any simple Lie algebra), consider the current algebra $\mathfrak{g} \otimes A_m$ and its universal central extension (UCE) $\widetilde{\mathfrak{g} \otimes A_m}$. By Kassel–Loday [2] the defining cocycle

$$\gamma(x \otimes f, y \otimes g) = (x, y) \overline{f dg}$$

takes its values in $\Omega_{A_m}^1 / dA_m$, which is the centre described above and which, in the genus-zero four-point case, is three-dimensional.

Two symbols are needed here and we keep them apart, because they denote different objects. Let $c \in \Omega_{A_m}^1 / dA_m$ span the line singled out by the palindromic symmetry, and let c^* be the corresponding coordinate functional. For the Heisenberg-type generators introduced below we set

$$(1) \quad \psi_{kl}^{(ij)}(a) := c^*(\gamma(b_k^{(i)}, b_l^{(j)})).$$

Thus γ is the cocycle of the universal central extension, with values in a three-dimensional space, whereas $\psi_{kl}^{(ij)}(a)$ is a scalar: the component of γ along one of the three available directions, namely the one the symmetry selects.

Being a component of γ , the scalar $\psi_{kl}^{(ij)}(a)$ is skew-symmetric,

$$(2) \quad \psi_{kl}^{(ij)}(a) = -\psi_{lk}^{(ji)}(a),$$

for all k, l and all i, j . This is inherited from Kassel–Loday and not an extra hypothesis. The Killing form $(\cdot, \cdot)$ is symmetric, while $\overline{f dg} = -\overline{g df}$ in $\Omega_{A_m}^1/dA_m$ because $d(fg) = f dg + g df$ is exact. Closed expressions for ψ in the superelliptic setting are computed in [3]; for the case treated in this paper they are given in (3) below.

We write $\mathcal{H}_m \subset \widetilde{\mathfrak{g} \otimes A_m}$ for the *Heisenberg subalgebra*, the subalgebra spanned by the Heisenberg-type generators $\{b_n^{(j)} : n \in \mathbb{Z}, 1 \leq j \leq \lfloor m/2 \rfloor\}$ arising from the odd powers of u in the Laurent expansion. (For $\mathfrak{g} = \mathfrak{sl}_2$, the relevant Chevalley generators pair to form such degrees-of-freedom.) Restricting γ to this subalgebra gives it a central extension of its own, which we now record.

Definition 2.1. The *Heisenberg subalgebra* $\mathcal{H}_m \subset \widetilde{\mathfrak{g} \otimes A_m}$ is the two-step nilpotent Lie algebra with basis

$$\{b_k^{(j)} : k \in \mathbb{Z}, 1 \leq j \leq \lfloor m/2 \rfloor\} \cup \{c\},$$

with $b_k^{(j)}$ of degree k and c of degree 0, and brackets

$$[b_k^{(i)}, b_l^{(j)}] = \psi_{kl}^{(ij)} c, \quad [c, \mathcal{H}_m] = 0,$$

where $\psi_{kl}^{(ij)}$ is the scalar of (1). These scalars depend on the coefficients of p ; in the palindromic case $p(t) = 1 - 2at^r + t^{2r}$ treated here the dependence is on the single parameter a , and we write $\psi_{kl}^{(ij)}(a)$.

We fix once and for all that the mode index is k and that the letter m is reserved for the covering degree. Skew-symmetry of the brackets is not an assumption here but a consequence of (2), which is why no antisymmetry hypothesis appears in the definition. Only one case is used below, and we record it now.

For $m = 2, r = 1$ there is a single family of generators, and we drop the superscript $j = 1$. Here $\psi_{kl}(a) = k \delta_{k+l,0} \omega_1$ with $\omega_1 > 0$ a constant depending only on p , so that

$$(3) \quad [b_k, b_l] = k \delta_{k+l,0} \omega_1 c, \quad \text{that is} \quad [b_k, b_{-k}] = k \omega_1 c \quad (k \neq 0),$$

and all other brackets among the b 's vanish. In particular b_0 is central in $\mathcal{H}_m$, every bracket $[b_k, b_l]$ with $k + l \neq 0$ vanishes, and $\mathcal{H}_m$ is the classical infinite-dimensional Heisenberg algebra, the constant ω_1 amounting to a rescaling of the central element.

2.2. Triangular decomposition. The *triangular decomposition* of $\mathcal{H}_m$ is defined by the natural grading on generators with respect to the Laurent degree n :

$$\mathcal{H}_m = \mathcal{H}_{m+} \oplus \mathcal{H}_{m0} \oplus \mathcal{H}_{m-},$$

where

- $\mathcal{H}_{m+} = \bigoplus_{n>0} \mathbb{C} b_n^{(j)}$ (positive-degree generators),
- $\mathcal{H}_{m0} = \bigoplus_j \mathbb{C} b_0^{(j)} \oplus \mathbb{C} c$ (degree-zero Cartan),
- $\mathcal{H}_{m-} = \bigoplus_{n<0} \mathbb{C} b_n^{(j)}$ (negative-degree generators).

This is the standard algebraic triangular decomposition, grading by the Laurent index n . Geometrically, this coincides with the almost-graded structure of Schlichenmaier [9] restricted to the Heisenberg subalgebra of the universal central extension. (The alternative geometric decomposition, the in/out decomposition from the branch points, is distinct; for this paper, we use the algebraic grading.)

The triangular decomposition makes $\mathcal{H}_m$ suitable for the standard construction of Verma modules and highest-weight representations.

2.3. The module category. We work with the standard category of highest-weight $\mathcal{H}_m$ -modules. An $\mathcal{H}_m$ -module M is called a *weight module* if it decomposes as

$$M = \bigoplus_{\mu \in \mathcal{H}_{m_0}^*} M_\mu,$$

where $M_\mu = \{v \in M : b_0^{(j)}v = \mu(b_0^{(j)})v \text{ and } cv = \mu(c)v\}$.

A highest-weight module for $\mathcal{H}_m$ is specified by:

- (1) A linear functional $\varphi : \mathcal{H}_{m_0} \rightarrow \mathbb{C}$ (the *weight*). No condition is imposed on $\varphi(c)$ here. The value $\varphi(c) = 0$ is permitted, and Theorem 3.8 identifies it as exactly the degenerate case.
- (2) A highest-weight vector v_φ satisfying:
 - $b_n^{(j)}v_\varphi = 0$ for all $n > 0$ and all j ,
 - $b_0^{(j)}v_\varphi = \varphi(b_0^{(j)})v_\varphi$,
 - $cv_\varphi = \varphi(c)v_\varphi$.
- (3) The action of $\mathcal{H}_{m-}$ is locally nilpotent, i.e., for each $v \in M$ and each $n < 0$, the operators $b_n^{(j)}$ act nilpotently on the finite-dimensional weight spaces.

Such modules are called φ -modules in the language of [17, 20], where analogous constructions for map algebras are developed.

The main object of study in this paper is the *Verma module*, the universal highest-weight module generated by a highest-weight vector of weight φ .

3. THE VERMA MODULE AND THE CANONICAL FORM

3.1. The Verma module.

Definition 3.1. For a linear functional $\varphi : \mathcal{H}_{m_0} \rightarrow \mathbb{C}$, the *Verma module* of weight φ is the induced module

$$M(\varphi) := U(\mathcal{H}_m) \otimes_{U(\mathcal{H}_{m+} \oplus \mathcal{H}_{m_0})} \mathbb{C}_\varphi,$$

where $\mathbb{C}_\varphi$ is the one-dimensional $\mathcal{H}_{m+} \oplus \mathcal{H}_{m_0}$ -module determined by

- $b_k^{(j)} \cdot 1 = 0$ for all $k > 0$ and all j ,
- $b_0^{(j)} \cdot 1 = \varphi(b_0^{(j)})$,
- $c \cdot 1 = \varphi(c)$.

We write v_φ for the image of 1, the highest-weight vector.

Following Definition 2.1 the bracket $[b_k^{(i)}, b_l^{(j)}]$ vanishes whenever $k, l \geq 0$, since $\psi_{kl}^{(ij)}$ is supported on $k + l = 0$ and the structure constant carries the factor k , so $\mathcal{H}_{m+} \oplus \mathcal{H}_{m_0}$ is abelian and every linear functional on it is a character. In particular $M(\varphi)$ is defined for every φ , including $\varphi(c) = 0$.

A word on terminology, since the letter φ is used differently in part of the literature. Bekkert, Benkart, Futorny and Kashuba [6] attach a module to a *polarisation*, a sign function $\phi: \mathbb{N} \rightarrow \{\pm\}$ that decides which imaginary modes are placed in the positive part of the triangular decomposition, and call the result a ϕ -Verma module. Here φ is not a polarisation but a weight, a linear functional on $\mathcal{H}_{m0}$, and the polarisation is the standard one, all positive modes annihilating v_φ . In the notation of [6] this is the case $\phi \equiv +$, for which their module is, in their words, the usual Verma module for the Heisenberg algebra [6, §3.1].

By the PBW theorem, $M(\varphi)$ has a $\mathbb{C}$ -basis consisting of the monomials $b_{-\lambda_1}^{(j_1)} \cdots b_{-\lambda_s}^{(j_s)} v_\varphi$ with $\lambda_1 \geq \cdots \geq \lambda_s > 0$. In the genus-zero case there is a single family of generators, the superscripts are absent, and such a monomial is determined by the partition $\lambda = (\lambda_1, \dots, \lambda_s)$ alone; we then write

$$(4) \quad b_{-\lambda} v_\varphi := b_{-\lambda_1} \cdots b_{-\lambda_s} v_\varphi.$$

The grading is inherited from the degree: the weight- $(-n)$ subspace $M(\varphi)[-n]$ has basis $\{b_{-\lambda} v_\varphi : \lambda \vdash n\}$, of dimension the number $p(n)$ of partitions of n .

3.2. The contravariant form.

Definition 3.2. Define the linear map $\omega: \mathcal{H}_m \rightarrow \mathcal{H}_m$ by

$$\omega(b_n^{(j)}) = b_{-n}^{(j)}, \quad \omega(c) = c.$$

This is an anti-involution: $\omega(\omega(x)) = x$ and $\omega([x, y]) = [\omega(y), \omega(x)]$. Extend ω to $U(\mathcal{H}_m)$ as an anti-automorphism of the universal enveloping algebra, so that

$$\omega(xy) = \omega(y)\omega(x).$$

This is the standard contravariance map for the Heisenberg algebra with respect to the triangular decomposition $\mathcal{H}_m = \mathcal{H}_{m+} \oplus \mathcal{H}_{m0} \oplus \mathcal{H}_{m-}$.

Lemma 3.3. *There exists a unique (up to scalar) symmetric bilinear form $\mathcal{S}_\varphi: M(\varphi) \times M(\varphi) \rightarrow \mathbb{C}$ satisfying:*

- (i) *Normalization:* $\mathcal{S}_\varphi(v_\varphi, v_\varphi) = 1$.
- (ii) *Contravariance:* $\mathcal{S}_\varphi(x \cdot v, w) = \mathcal{S}_\varphi(v, \omega(x) \cdot w)$ for all $x \in U(\mathcal{H}_m)$ and $v, w \in M(\varphi)$.
- (iii) *Weight preservation:* $\mathcal{S}_\varphi(v, w) = 0$ whenever v and w lie in distinct weight spaces of $M(\varphi)$.

Proof. Uniqueness: Suppose $\mathcal{S}_\varphi$ satisfies (i)–(iii). By contravariance,

$$\mathcal{S}_\varphi(uv_\varphi, vv_\varphi) = \mathcal{S}_\varphi(v_\varphi, \omega(u)vv_\varphi).$$

Since v_φ is the unique highest-weight vector (up to scaling) and $\omega(u)v$ may be expressed in the PBW basis, the form is determined by its values on the basis monomials. Normalization fixes the overall scale.

Existence: Define $\mathcal{S}_\varphi$ on the highest-weight vector by $\mathcal{S}_\varphi(v_\varphi, v_\varphi) := 1$. For general $u, v \in U(\mathcal{H}_m)$, set

$$\mathcal{S}_\varphi(uv_\varphi, vv_\varphi) := \varphi(\omega(u)v),$$

where the right side is interpreted as: compute the product $\omega(u)v$ in $U(\mathcal{H}_m)$, then apply φ to the component that is in $\mathcal{H}_{m0}$ (the central element c contributes $\varphi(c)$ times its coefficient). By the PBW theorem and the structure of $\mathcal{H}_m$, this is well-defined and

extends to a contravariant form on all of $M(\varphi)$. The symmetry $\mathcal{S}_\varphi(v, w) = \mathcal{S}_\varphi(w, v)$ follows from the antisymmetry of the cocycle ψ .

For further details and the standard treatment, see [8] and [7, Ch. 2]. $\square$

Denote by G_n the Gram matrix of $\mathcal{S}_\varphi$ on $M(\varphi)[-n]$ in the basis (4), that is, the $p(n) \times p(n)$ matrix with entries $\mathcal{S}_\varphi(b_{-\lambda}v_\varphi, b_{-\mu}v_\varphi)$ for $\lambda, \mu \vdash n$. The Kac determinant analogue $\det(G_n)$ is evaluated in closed form in Theorem 3.8, equation (10). It also gives the condition on the weight under which the constructions below behave well.

Definition 3.4. A linear functional $\varphi: \mathcal{H}_{m0} \rightarrow \mathbb{C}$ is *Shapovalov-regular* if

$$\det(G_n) \neq 0 \quad \text{for every } n \geq 1.$$

This is a condition on the determinant of an explicit matrix, so it can be checked by computation. For $m = 2, r = 1$ it collapses to the single requirement $\varphi(c) \neq 0$ (Theorem 3.8).

3.3. Orthogonality of $\{P_n\}$. The bridge to orthogonal polynomials needs nothing beyond the defining polynomial of the curve and the classical uniqueness of orthogonal polynomial sequences [5]. The following lemma derives what is required for Theorem 3.7.

Lemma 3.5. *Let $m = 2, r = 1$, so that $p(t) = 1 - 2at + t^2$. Then*

(D1) *The formal expansion of $p(t)^{-1/2}$ in t ,*

$$(1 - 2at + t^2)^{-1/2} = \sum_{n \geq 0} c_n(a) t^n,$$

has coefficients $c_n(a)$ determined by $c_0 = 1, c_1 = a$ and

$$(5) \quad (n+1)c_{n+1}(a) = (2n+1)ac_n(a) - nc_{n-1}(a), \quad n \geq 1.$$

(D2) *Consequently $c_n(a) = P_n(a)$, the n -th Legendre polynomial, and $\deg P_n = n$.*

Proof. (D1) Write $F(a, t) = (1 - 2at + t^2)^{-1/2}$. Differentiating, $\partial_t F = (a - t)(1 - 2at + t^2)^{-3/2}$, so

$$(1 - 2at + t^2) \partial_t F + (t - a) F = 0.$$

Substituting $F = \sum_n c_n t^n$ and collecting the coefficient of t^n gives (5); the values $c_0 = 1$ and $c_1 = a$ are read off from the expansion.

(D2) Relation (5) together with $c_0 = 1, c_1 = a$ determines the whole sequence, and it is the classical three-term recurrence of the Legendre polynomials [5, eq. (4.7.17)] with the same initial values. Hence $c_n = P_n$ for all n , and $\deg P_n = n$ follows inductively from (5). $\square$

For $\deg p = 4$ the analogous computation yields associated ultraspherical and Gegenbauer families instead, and those are the subject of [3].

We can now define the polynomial vectors precisely.

Definition 3.6. Let $m = 2, r = 1$ and let φ satisfy $\varphi(c) \neq 0$. For $n \geq 0$ let $\mu_n: M(\varphi)[-n] \rightarrow \mathbb{C}$ be the linear functional taking the value 1 on every basis monomial $b_{-\lambda}v_\varphi, \lambda \vdash n$. The *polynomial vector* $\tilde{P}_n \in M(\varphi)[-n]$ is the representative of μ_n with respect to the Shapovalov form, that is, the unique vector with

$$(6) \quad \mathcal{S}_\varphi(\tilde{P}_n, w) = \kappa_n \mu_n(w) \quad \text{for all } w \in M(\varphi)[-n],$$

the scalar κ_n being fixed by the normalisation of Theorem 3.7 (iv).

Equivalently, $\tilde{P}_n$ is the unique direction in $M(\varphi)[-n]$ whose Shapovalov pairing with a monomial does not depend on which monomial is taken, hence the unique direction orthogonal to all differences $b_{-\lambda}v_\varphi - b_{-\mu}v_\varphi$ with $\lambda, \mu \vdash n$.

Theorem 3.7. *Let $m = 2$, $r = 1$ and let φ be Shapovalov-regular. Write $q := \omega_1 \varphi(c)$ and, for a partition λ with $m_k(\lambda)$ parts equal to k , let $z_\lambda := \prod_{k \geq 1} k^{m_k(\lambda)} m_k(\lambda)!$ and $\ell(\lambda) := \sum_k m_k(\lambda)$ for the number of parts.*

(i) *The vector of Definition 3.6 exists, is unique up to the scalar κ_n , and has the closed form*

$$(7) \quad \tilde{P}_n = \kappa'_n \sum_{\lambda \vdash n} \frac{1}{z_\lambda q^{\ell(\lambda)}} b_{-\lambda} v_\varphi,$$

valid for every $n \geq 0$.

(ii) $\mathcal{S}_\varphi(\tilde{P}_n, \tilde{P}_{n'}) = 0$ for $n \neq n'$.

(iii) With $\kappa'_n = 1$ in (7) one has

$$(8) \quad \mathcal{S}_\varphi(\tilde{P}_n, \tilde{P}_n) = \sum_{\lambda \vdash n} \frac{1}{z_\lambda q^{\ell(\lambda)}} = \binom{q^{-1} + n - 1}{n} = \frac{1}{n!} \prod_{j=1}^{n-1} (1 + jq) \cdot q^{-n}.$$

(iv) *Assume in addition $q \in \mathbb{R}_{>0}$. Then, for $n \geq 1$, $\kappa'_n = \left(\frac{2}{2n+1}\right)^{1/2} \binom{q^{-1} + n - 1}{n}^{-1/2}$ is the unique positive scalar for which*

$$\mathcal{S}_\varphi(\tilde{P}_n, \tilde{P}_{n'}) = \frac{2}{2n+1} \delta_{nn'}, \quad n, n' \geq 1,$$

the right-hand side being the squared norms of the Legendre polynomials of Lemma 3.5.

Proof. (i) By Lemma 3.3 (iii) the form is block-diagonal, and within $M(\varphi)[-n]$ it is diagonal in the monomial basis: moving the positive modes of $\omega(b_{-\lambda})$ across $b_{-\mu}v_\varphi$ by contravariance, a monomial survives only if every part of λ is matched by an equal part of μ , so $\mathcal{S}_\varphi(b_{-\lambda}v_\varphi, b_{-\mu}v_\varphi) = 0$ unless $\lambda = \mu$. For $\lambda = \mu$ the $m_k(\lambda)!$ matchings of equal parts each contribute $\prod_k (kq)^{m_k(\lambda)}$, giving

$$(9) \quad \mathcal{S}_\varphi(b_{-\lambda}v_\varphi, b_{-\lambda}v_\varphi) = \prod_{k \geq 1} (kq)^{m_k(\lambda)} m_k(\lambda)! = z_\lambda q^{\ell(\lambda)}.$$

The exponent is the number of parts $\ell(\lambda)$, not n ; the two agree only for $\lambda = (1^n)$. Since $q \neq 0$, the Gram matrix is invertible and μ_n has a unique representative. Writing $\tilde{P}_n = \sum_\lambda x_\lambda b_{-\lambda}v_\varphi$, condition (6) reads $x_\lambda z_\lambda q^{\ell(\lambda)} = \kappa_n$ for every λ , whence $x_\lambda \propto 1/(z_\lambda q^{\ell(\lambda)})$, which is (7).

(ii) Distinct n give distinct weights, and $\mathcal{S}_\varphi$ vanishes on distinct weight spaces (Lemma 3.3 (iii)).

(iii) By (9) and (7),

$$\mathcal{S}_\varphi(\tilde{P}_n, \tilde{P}_n) = \sum_{\lambda \vdash n} \frac{z_\lambda q^{\ell(\lambda)}}{(z_\lambda q^{\ell(\lambda)})^2} = \sum_{\lambda \vdash n} \frac{1}{z_\lambda q^{\ell(\lambda)}}.$$

Setting $t := q^{-1}$, the exponential formula gives $\sum_{n \geq 0} \sum_{\lambda \vdash n} z_\lambda^{-1} t^{\ell(\lambda)} u^n = \exp(t \sum_{k \geq 1} u^k / k) = (1 - u)^{-t}$, whose u^n coefficient is $\binom{t+n-1}{n}$ [4, Ch. I, §2]. This is (8).

(iv) For $q > 0$ every factor $1 + jq$ in (8) is positive, so the norm is a positive real and κ'_n is a well-defined positive real; then $\mathcal{S}_\varphi(\kappa'_n \tilde{P}_n, \kappa'_n \tilde{P}_n) = 2/(2n+1)$ by construction, and a second positive scalar with the same property would have the same square. Orthogonality for $n \neq n'$ is (ii). $\square$

Formula (7) is the image of the complete homogeneous symmetric function h_n under $p_k \mapsto q^{-1} b_{-k}$, in the standard expansion $h_n = \sum_{\lambda \vdash n} p_\lambda / z_\lambda$ [4, Ch. I, (2.14)]; the weight $q^{-\ell(\lambda)}$ is what the substitution contributes. It is worth separating the parts of the theorem. Part (ii) uses only the grading. Part (iv) fixes a normalisation and needs the reality hypothesis $q \in \mathbb{R}_{>0}$, which is not automatic: φ takes values in $\mathbb{C}$, and for $q \notin \mathbb{R}_{>0}$ the scalars κ'_n are not real and the form is not positive definite on the real span of the monomials. What is specific to $\mathcal{H}_m$ is the value $z_\lambda q^{\ell(\lambda)}$ in (9), produced by the cocycle of Definition 2.1. For $m \geq 3$ the cocycle acquires off-diagonal components, the Gram matrix is no longer diagonal, and neither (7) nor the evaluation in (iii) survives.

3.4. Irreducibility criterion.

Theorem 3.8. *Let $m = 2$, $r = 1$ and let $\varphi : (\mathcal{H}_m)_0 \rightarrow \mathbb{C}$. The following are equivalent:*

- (i) *The Verma module $M(\varphi)$ is irreducible.*
- (ii) *The Shapovalov form $\mathcal{S}_\varphi$ is non-degenerate on $M(\varphi)$.*
- (iii) *φ is Shapovalov-regular in the sense of Definition 3.4.*
- (iv) *$\varphi(c) \neq 0$.*

Moreover, for every φ the Gram matrix of $\mathcal{S}_\varphi$ on $M(\varphi)[-n]$ is diagonal in the Poincaré–Birkhoff–Witt basis, with

$$(10) \quad \det(G_n) = \prod_{\lambda \vdash n} \prod_{k \geq 1} (k \omega_1 \varphi(c))^{m_k(\lambda)} m_k(\lambda)!,$$

where $m_k(\lambda)$ is the multiplicity of k in the partition λ .

Proof. (i) $\Leftrightarrow$ (ii): For a highest-weight module carrying a contravariant form, irreducibility is equivalent to non-degeneracy of the form [7, Ch. 2]: if the form is degenerate its kernel is a non-trivial invariant subspace, and if the module is reducible the maximal proper submodule lies in the kernel.

(ii) $\Leftrightarrow$ (iii): $\mathcal{S}_\varphi$ is block-diagonal with respect to the weight grading (Lemma 3.3(iii)), so non-degeneracy of $\mathcal{S}_\varphi$ on $M(\varphi)$ is non-degeneracy of G_n for every $n \geq 1$, which is Definition 3.4.

(iii) $\Leftrightarrow$ (iv), and (10): By Definition 2.1 with $m = 2$, $r = 1$, the zero mode b_0 is central and the only non-vanishing brackets are $[b_k, b_{-k}] = k \omega_1 c$. Let $\lambda, \mu \vdash n$ and let $b_{-\lambda} v_\varphi, b_{-\mu} v_\varphi$ be the corresponding PBW monomials. Moving the positive modes of $\omega(b_{-\lambda}) = b_\lambda$ to the right past $b_{-\mu}$ produces, at each step, either a scalar $k \omega_1 \varphi(c)$ from a matched pair (b_k, b_{-k}) or an operator b_k with $k > 0$, which annihilates v_φ . A monomial $b_{-\mu} v_\varphi$ therefore contributes only when every part of λ is matched by an equal part of μ , that is, when $\lambda = \mu$; hence $\mathcal{S}_\varphi(b_{-\lambda} v_\varphi, b_{-\mu} v_\varphi) = 0$ for $\lambda \neq \mu$ and G_n is diagonal. For $\lambda = \mu$ the $m_k(\lambda)!$ ways of matching the equal parts each contribute

$(k\omega_1\varphi(c))^{m_k(\lambda)}$, which gives the diagonal entry and hence (10). Since $\omega_1 > 0$, every factor of (10) is a non-zero multiple of a power of $\varphi(c)$; therefore $\det(G_n) \neq 0$ for every $n \geq 1$ if and only if $\varphi(c) \neq 0$. $\square$

Theorem 3.8 says that the genus-zero, four-point Heisenberg algebra exhibits no new behaviour, and identifies the source. By Definition 2.1 the cocycle is supported on $k+l=0$, so the contravariant form is diagonal and its determinant cannot acquire zeros away from $\varphi(c)=0$. The interest of the Shapovalov-regularity condition of Definition 3.4 therefore lies entirely in the range $m \geq 3$, where off-diagonal cocycle components destroy the diagonality used above.

4. SUGAWARA ELEMENT, POLYNOMIAL QUOTIENT, AND LEGENDRE IDENTIFICATION

Throughout this section we work in the genus-zero case: $m=2$, $r=1$, so $\mathcal{H}_m = \mathcal{H}_2$ with cocycle $\psi_{kl}(a) = k\delta_{k+l,0}\omega_1$, see (3).

The goal of this section is to construct an explicit chain of $\mathcal{H}_2$ -equivariant maps

$$M(\varphi) \xrightarrow{\psi} \mathbb{C}[\zeta] \xrightarrow{\Psi} \mathbb{C}[x]$$

and a quadratic operator Ω on $M(\varphi)$ whose pushforward under $\Phi := \Psi \circ \psi$ is the Legendre differential operator $L = (1-x^2)\partial_x^2 - 2x\partial_x$. The element Ω is built from the Sugawara stress-energy zero mode L_0 of the free boson; the map ψ collapses each PBW level to a single power ζ^n ; and Ψ identifies ζ^n with the Legendre polynomial $P_n(x)$.

4.1. The Sugawara stress-energy zero mode. For two Heisenberg generators b_k, b_l ($k, l \in \mathbb{Z}$), define the *normal-ordered product*

$$:b_k b_l: = \begin{cases} b_k b_l & \text{if } k \leq 0, \\ b_l b_k & \text{if } k > 0, \end{cases}$$

i.e., negative-mode (and zero-mode) operators are placed to the left of positive-mode operators. Because $[b_k, b_l] = \psi_{kl}c$ is a scalar on $M(\varphi)$, the normal-ordered product differs from the bare product by a scalar correction whenever $k+l=0$, and equals the bare product otherwise. In particular, the sum $\sum_{k \geq 1} :b_{-k} b_k: = \sum_{k \geq 1} b_{-k} b_k$ is already normal-ordered as written (each $-k$ is negative, so each factor stays in place).

Definition 4.1. Set

$$(11) \quad \mathbb{L} := \sum_{k \geq 1} :b_{-k} b_k: = \sum_{k \geq 1} b_{-k} b_k.$$

This expression involves no weight, it is an element of the completion $\widetilde{U(\mathcal{H}_2)}$ of $U(\mathcal{H}_2)$ along the grading, and on a weight vector of weight $-n$ only the finitely many terms with $k \leq n$ act non-trivially, so no analytic condition is needed to make it act.

Now let φ be Shapovalov-regular, so that $q := \omega_1\varphi(c) \neq 0$ by Theorem 3.8, and define the operator

$$(12) \quad L_0 := \frac{1}{q} \mathbb{L} \Big|_{M(\varphi)}.$$

The scalar q^{-1} belongs to the action, not to the element: $\mathbf{L}$ is fixed once and for all, while L_0 is its normalisation on the particular module $M(\varphi)$.

The element $\mathbf{L}$ is the zero mode of the standard Sugawara stress-energy tensor of the free boson (central charge $c_{\text{Vir}} = 1$), the element that generates the energy grading on the free-boson Fock space; see [18, 19] for the general construction. Its eigenvalue on a level- n monomial is qn , so it records the level only up to the factor q , which depends on the cocycle normalisation and on the central charge. Dividing by q is what makes the eigenvalue the integer n itself, and thereby makes the intertwining with the classical Legendre operator exact, with no parameter-dependent constants. The price is that L_0 , and with it Ω , are attached to $M(\varphi)$ rather than to the algebra; $\mathbf{L}$ is the part of the construction that is canonical.

Lemma 4.2. *Let $v = b_{-n_1} b_{-n_2} \cdots b_{-n_s} v_\varphi$ be a PBW monomial with $n_1 \geq n_2 \geq \cdots \geq n_s \geq 1$ and write $n := n_1 + n_2 + \cdots + n_s$. Then, for every weight φ ,*

$$(13) \quad \mathbf{L} \cdot v = qn \cdot v.$$

If moreover φ is Shapovalov-regular, so that $q \neq 0$ and $L_0 = q^{-1}\mathbf{L}$ is defined (12), then

$$(14) \quad L_0 \cdot v = n \cdot v.$$

Consequently $\mathbf{L}$ acts on $M(\varphi)[-n]$ by the scalar qn and L_0 by the scalar n , and the spectrum of L_0 on $M(\varphi)$ is $\{0, 1, 2, \dots\}$ with multiplicity $p(n)$ at level n , where $p(n)$ is the partition function. Note that the eigenvalue of $\mathbf{L}$ carries the factor q , which is exactly what the normalisation in (12) removes.

Proof. We prove (13); the normalised form (14) follows by dividing by q , which is legitimate exactly when φ is Shapovalov-regular. We proceed by induction on the length s . For $s = 0$ ($v = v_\varphi$), all $b_k v_\varphi = 0$ for $k \geq 1$, so $\mathbf{L} v_\varphi = 0 = 0 \cdot v_\varphi$. For $s \geq 1$, write $v = b_{-n_1} w$ with $w = b_{-n_2} \cdots b_{-n_s} v_\varphi$ of weight $-(n - n_1)$. Using $[b_k, b_{-n_1}] = \delta_{k, n_1} n_1 \omega_1 c$ (the genus-zero coefficients (3), for which $\psi_{k, -k} = k \omega_1$), we compute

$$\begin{aligned} L_0 \cdot v &= \frac{1}{q} \sum_{k \geq 1} b_{-k} b_k b_{-n_1} w \\ &= \frac{1}{q} \sum_{k \geq 1} b_{-k} (b_{-n_1} b_k + \delta_{k, n_1} n_1 \omega_1 c) w \\ &= b_{-n_1} \left(\frac{1}{q} \sum_{k \geq 1} b_{-k} b_k \right) w + \frac{n_1 q}{q} b_{-n_1} w \\ &= b_{-n_1} (L_0 w) + n_1 v = (n - n_1) v + n_1 v = n v, \end{aligned}$$

where we used the inductive hypothesis $L_0 w = (n - n_1) w$ in the last line, and c acts as the scalar $\varphi(c)$ on $M(\varphi)$ throughout. This proves (13).

The level- n weight space $M(\varphi)[-n]$ is spanned by all such PBW monomials with $n_1 + \cdots + n_s = n$; the count of such monomials is $p(n)$ by the PBW theorem applied to $\mathcal{H}_2^-$. Each such monomial is a L_0 -eigenvector with eigenvalue n , so L_0 acts as the scalar n on $M(\varphi)[-n]$. $\square$

4.2. The Casimir operator Ω .

Definition 4.3. The *Casimir operator* of $M(\varphi)$ is the linear operator

$$(15) \quad \Omega := -L_0(L_0 + \text{Id}) = -\frac{1}{q^2} \mathbf{L}^2 - \frac{1}{q} \mathbf{L} \quad \text{on } M(\varphi),$$

where L_0 is as in (12).

Lemma 4.4. For every $v \in M(\varphi)[-n]$ ($n \geq 0$),

$$(16) \quad \Omega \cdot v = -n(n+1)v.$$

In particular, Ω commutes with the $\mathcal{H}_2$ -action restricted to each weight space (since both are scalar there) but does not commute with the full $\mathcal{H}_2$ -action: the operators $b_{\pm k}$ shift weight, and $-n(n+1)$ depends on n . This is the source of the non-trivial intertwining content of Section 4.5.

Proof. By Lemma 4.2, $L_0 \cdot v = nv$, hence $\Omega \cdot v = -L_0(L_0 + \text{Id})v = -L_0(n+1)v = -n(n+1)v$. $\square$

4.3. The polynomial quotient ψ .

Definition 4.5. Let $\mathbb{C}[\zeta]$ be the polynomial algebra in one indeterminate ζ , regarded as a $\mathbb{Z}_{\geq 0}$ -graded vector space with $\deg \zeta = 1$. The letter ζ is used here to keep the indeterminate of this quotient distinct from the parameter a of the curve. Define a linear map

$$\psi: M(\varphi) \longrightarrow \mathbb{C}[\zeta]$$

by

$$(17) \quad \psi(b_{-n_1} b_{-n_2} \cdots b_{-n_s} v_\varphi) := \zeta^{n_1+n_2+\cdots+n_s}$$

on each PBW monomial, and extended linearly. Equivalently, ψ sends every level- n PBW monomial to the single monomial ζ^n , and is therefore the unique linear map sending each $M(\varphi)[-n]$ surjectively onto $\mathbb{C} \cdot \zeta^n$.

Lemma 4.6. The map ψ of Definition 4.5 satisfies:

- (i) ψ is well-defined and surjective;
- (ii) ψ is graded: $\psi(M(\varphi)[-n]) = \mathbb{C} \cdot \zeta^n$ for each $n \geq 0$, with kernel $\ker(\psi) \cap M(\varphi)[-n]$ of dimension $p(n) - 1$;
- (iii) ψ intertwines the level operator L_0 on $M(\varphi)$ with the Euler operator $\zeta \partial_\zeta$ on $\mathbb{C}[\zeta]$: $\psi \circ L_0 = (\zeta \partial_\zeta) \circ \psi$;
- (iv) ψ intertwines Ω with the second-order operator $-(\zeta \partial_\zeta)((\zeta \partial_\zeta) + 1)$ on $\mathbb{C}[\zeta]$: $\psi \circ \Omega = -((\zeta \partial_\zeta)^2 + (\zeta \partial_\zeta)) \circ \psi$.

Proof. (i). The PBW monomials form a basis of $M(\varphi)$ (Section 2.3); defining a linear map on a basis is automatic and well-defined. Surjectivity is clear: $\psi(b_{-n} v_\varphi) = \zeta^n$.

(ii). The PBW basis of $M(\varphi)[-n]$ is indexed by partitions of n , hence has dimension $p(n)$, and ψ collapses all of them to ζ^n .

(iii). On a level- n PBW monomial v , $\psi(L_0 v) = \psi(nv) = n\zeta^n = (\zeta \partial_\zeta)(\zeta^n) = (\zeta \partial_\zeta)\psi(v)$ by Lemma 4.2.

(iv). Apply (iii) twice: $\psi \circ \Omega = -\psi \circ L_0 \circ (L_0 + \text{Id}) = -(\zeta \partial_\zeta) \circ \psi \circ (L_0 + \text{Id}) = -(\zeta \partial_\zeta)((\zeta \partial_\zeta) + 1) \circ \psi$. $\square$

4.4. The Legendre identification Ψ .

Definition 4.7. Let $\{P_n(x)\}_{n \geq 0}$ denote the standard Legendre polynomials on $[-1, 1]$, normalised by $P_n(1) = 1$. Define a linear isomorphism of graded vector spaces

$$\Psi: \mathbb{C}[\zeta] \xrightarrow{\sim} \mathbb{C}[x], \quad \Psi(\zeta^n) := P_n(x).$$

Both sides are $\mathbb{Z}_{\geq 0}$ -graded by polynomial degree (with $\deg \zeta = \deg x = 1$); Ψ preserves the grading because $\deg P_n = n$. The inverse $\Psi^{-1}: \mathbb{C}[x] \rightarrow \mathbb{C}[\zeta]$ exists because $\{P_n(x)\}$ is a basis of $\mathbb{C}[x]$.

Definition 4.8. Let

$$\Phi := \Psi \circ \psi: M(\varphi) \longrightarrow \mathbb{C}[x].$$

By construction, Φ sends every level- n PBW monomial to $P_n(x)$.

4.5. The Sugawara–Legendre intertwining. The next theorem is the main result of this section.

Theorem 4.9. *Let $L_0, \Omega, \psi, \Psi, \Phi = \Psi \circ \psi$ be as in Definitions 4.1, 4.3, 4.5, 4.7, 4.8, and let $L = (1 - x^2)\partial_x^2 - 2x\partial_x$ denote the classical Legendre differential operator. Then:*

- (i) $\psi: M(\varphi) \rightarrow \mathbb{C}[\zeta]$ is a well-defined surjective linear map sending each level- n weight space to the line $\mathbb{C} \cdot \zeta^n$.
- (ii) $\Phi = \Psi \circ \psi$ sends each level- n weight space to the line $\mathbb{C} \cdot P_n(x) \subset \mathbb{C}[x]$.
- (iii) Φ intertwines the action of Ω on $M(\varphi)$ with the action of L on $\mathbb{C}[x]$:

$$(18) \quad \Phi(\Omega \cdot v) = L(\Phi(v)) \quad \text{for all } v \in M(\varphi).$$

(iv) Equivalently, for every $n \geq 0$ and every $v \in M(\varphi)[-n]$,

$$(19) \quad L(\Phi(v)) = -n(n+1)\Phi(v),$$

which is the classical Legendre eigenvalue identity for $P_n(x)$.

Proof. Step 1: (i). This is Lemma 4.6 ((i))–((ii)).

Step 2: (ii). By Definition 4.7, $\Psi(\zeta^n) = P_n(x)$. Combining with Step 1, $\Phi(M(\varphi)[-n]) = \Psi(\mathbb{C} \cdot \zeta^n) = \mathbb{C} \cdot P_n(x)$.

Step 3: pushforward of Ω along ψ . By Lemma 4.6 ((iv)), $\psi \circ \Omega = -(\zeta \partial_\zeta)((\zeta \partial_\zeta) + 1) \circ \psi$.

Step 4: pushforward of $-(\zeta \partial_\zeta)((\zeta \partial_\zeta) + 1)$ along Ψ . This is the classical fact that the Legendre differential operator $L = (1 - x^2)\partial_x^2 - 2x\partial_x$ satisfies the eigenvalue identity $L P_n(x) = -n(n+1) P_n(x)$ for all $n \geq 0$ (see, e.g., [24, Ch. 12, eq. 12.3.6]). On the basis $\{\zeta^n\}$, both $-(\zeta \partial_\zeta)((\zeta \partial_\zeta) + 1)$ acting by $-n(n+1)$ on ζ^n , and L acting by $-n(n+1)$ on $P_n(x)$, agree under Ψ :

$$\Psi\left(-(\zeta \partial_\zeta)((\zeta \partial_\zeta) + 1) \zeta^n\right) = \Psi(-n(n+1) \zeta^n) = -n(n+1) P_n(x) = L P_n(x) = L \Psi(\zeta^n).$$

By linearity, this extends to all of $\mathbb{C}[\zeta]$: $\Psi \circ (-(\zeta \partial_\zeta)((\zeta \partial_\zeta) + 1)) = L \circ \Psi$.

Step 5: (iii). Combine Steps 3 and 4:

$$\Phi \circ \Omega = \Psi \circ \psi \circ \Omega = \Psi \circ (-(\zeta \partial_\zeta)((\zeta \partial_\zeta) + 1)) \circ \psi = L \circ \Psi \circ \psi = L \circ \Phi.$$

This is (18).

Step 6: (iv). For $v \in M(\varphi)[-n]$, Lemma 4.4 gives $\Omega \cdot v = -n(n+1)v$. Then (18) yields $L \Phi(v) = \Phi(\Omega \cdot v) = \Phi(-n(n+1)v) = -n(n+1)\Phi(v)$, which is (19). $\square$

4.6. The Casimir tower and the generating function. The Casimir tower and the Legendre generating-function identity now follow as immediate corollaries of Theorem 4.9.

Definition 4.10. We call the family $\{\Omega^r\}_{r \geq 1}$ of iterates of the Casimir operator Ω of Definition 4.3 the *Casimir tower* generated by Ω . Each Ω^r is a well-defined linear operator on $M(\varphi)$ acting diagonally on the level- n weight space; the structural properties of the family are recorded in Corollary 4.11 below.

Corollary 4.11. *For each integer $r \geq 1$:*

- (i) Ω^r is a well-defined linear operator on $M(\varphi)$, acting on each level- n weight space as the scalar $(-n(n+1))^r$.
- (ii) Φ intertwines Ω^r with L^r : $\Phi \circ \Omega^r = L^r \circ \Phi$.

In particular, the family $\{\Omega^r\}_{r \geq 1}$ pushes forward under Φ to the family $\{L^r\}_{r \geq 1}$ of iterated Legendre operators; this is the Casimir tower referred to in the introduction.

Proof. (i) Lemma 4.4 shows Ω is a level-dependent scalar $-n(n+1)$ on $M(\varphi)[-n]$; hence Ω^r is the scalar $(-n(n+1))^r$ on the same weight space.

(ii) By Theorem 4.9 (iii), $\Phi \circ \Omega = L \circ \Phi$. Iterating gives, for $r \geq 2$, $\Phi \circ \Omega^r = (\Phi \circ \Omega) \circ \Omega^{r-1} = L \circ (\Phi \circ \Omega^{r-1}) = L^r \circ \Phi$ by induction on r . $\square$

For $r = 2$, Ω^2 acts on $M(\varphi)[-n]$ as $n^2(n+1)^2$, and Corollary 4.11 shows that Φ identifies Ω^2 with L^2 .

Corollary 4.12. *Let $G(z, x) = (1 - 2xz + z^2)^{-1/2} = \sum_{n \geq 0} P_n(x) z^n$ denote the Legendre generating function, and let $\mathcal{E} := z^2 \partial_z^2 + 2z \partial_z = z \partial_z (z \partial_z + 1)$ be the associated Euler-type operator in the z -variable. Then for every $r \geq 1$:*

$$(20) \quad L^r_{(x)} G(z, x) = (-1)^r \mathcal{E}^r_{(z)} G(z, x).$$

Proof. For $r = 1$: a direct computation using $z^2 \partial_z^2(z^n) = n(n-1)z^n$ and $2z \partial_z(z^n) = 2n z^n$ gives $\mathcal{E} G = \sum_{n \geq 0} (n(n-1) + 2n) P_n(x) z^n = \sum_{n \geq 0} n(n+1) P_n(x) z^n$. Combining with $L P_n = -n(n+1) P_n$ yields $L_{(x)} G = -\mathcal{E}_{(z)} G$. For $r \geq 2$, $L_{(x)}$ and $\mathcal{E}_{(z)}$ act on different variables and hence commute; iterating gives $L^r_{(x)} G = (-1)^r \mathcal{E}^r_{(z)} G$. $\square$

On the space $\{G_f(z, x) = \sum_n f_n P_n(x) z^n\}$ of Legendre-expanded series, Corollary 4.12 gives $L_{(x)} = -\mathcal{E}_{(z)} = -(z^2 \partial_z^2 + 2z \partial_z)$: the Legendre differential operator in x is interchangeable with the second-order Euler–Cauchy operator in z , and at every order $r \geq 1$ their iterates satisfy $L^r_{(x)} = (-\mathcal{E}_{(z)})^r$.

5. FOCK SPACE REALIZATION

This section constructs the Fock space realization of the Verma module $M(\varphi)$ for the four-point Heisenberg algebra $\mathcal{H}_2$. The main result (Theorem 5.2) is that $M(\varphi)$ is canonically isomorphic to a standard Fock space $\mathcal{F}_\varphi$. The polynomial vectors $\{\tilde{P}_n\}_{n \geq 0}$ appear as the Gram–Schmidt basis of $\mathcal{F}_\varphi$ with respect to the Fock inner product. This gives a direct realization of the Shapovalov orthogonality as an inner product of many-body states.

5.1. The Fock module.

Definition 5.1. Fix a Shapovalov-regular linear functional $\varphi: (\mathcal{H}_2)_0 \rightarrow \mathbb{C}$ (Definition 3.4); by Theorem 3.8 this is exactly the condition $\varphi(c) \neq 0$. The *Fock vacuum* is the vector $v_\varphi \in M(\varphi)$ characterized by:

$$(21) \quad b_k v_\varphi = 0 \quad \text{for all } k \geq 1,$$

$$(22) \quad b_0 v_\varphi = \varphi(b_0) v_\varphi,$$

$$(23) \quad c v_\varphi = \varphi(c) v_\varphi.$$

The *creation operators* are $a_k^\dagger := b_{-k}$ for $k \geq 1$. The *Fock module* is

$$\mathcal{F}_\varphi := \mathbb{C}[a_1^\dagger, a_2^\dagger, a_3^\dagger, \dots] \cdot v_\varphi = \text{Span}_{\mathbb{C}} \{ (a_{n_1}^\dagger)^{k_1} (a_{n_2}^\dagger)^{k_2} \dots v_\varphi \mid n_1 > n_2 > \dots \geq 1, k_i \geq 0 \}.$$

The Fock module $\mathcal{F}_\varphi$ is graded by *level* (the total mode number): a monomial $(a_{n_1}^\dagger)^{k_1} \dots v_\varphi$ has level $\ell = \sum_i n_i k_i$. The level- ℓ component $\mathcal{F}_\varphi[\ell]$ has dimension $p(\ell)$, the number of integer partitions of ℓ . In particular, $\mathcal{F}_\varphi[0] = \mathbb{C} v_\varphi$, $\mathcal{F}_\varphi[1] = \mathbb{C} a_1^\dagger v_\varphi$, and $\mathcal{F}_\varphi[2] = \mathbb{C} a_2^\dagger v_\varphi \oplus \mathbb{C} (a_1^\dagger)^2 v_\varphi$.

Theorem 5.2. *The $\mathcal{H}_2$ -module $M(\varphi)$ is isomorphic to $\mathcal{F}_\varphi$ as a $\mathcal{H}_2$ -module.*

Proof. The negative-mode subalgebra $\mathcal{H}_2^- = \text{Span}\{b_{-n} : n \geq 1\}$ is abelian: for $k, l > 0$, the cocycle satisfies $\psi_{-k, -l} = 0$ since $(-k) + (-l) < 0 \neq 0$, so $[b_{-k}, b_{-l}] = \psi_{-k, -l} c = 0$. By the PBW theorem, $U(\mathcal{H}_2^-)$ is therefore the polynomial algebra $\mathbb{C}[b_{-1}, b_{-2}, \dots]$. The Verma module $M(\varphi)$ is defined as the induced module $M(\varphi) = U(\mathcal{H}_2) \otimes_{U(\mathcal{H}_2^-)} \mathbb{C}_\varphi$, where $\mathbb{C}_\varphi$ is the one-dimensional module on which b_n ($n \geq 0$) and c act by $\varphi(b_n)$ and $\varphi(c)$ respectively. By PBW, $M(\varphi) \cong U(\mathcal{H}_2^-)$ as a vector space, with the generator $1 \otimes 1$ playing the role of v_φ . This is exactly $\mathcal{F}_\varphi$. $\square$

5.2. Fock inner product and Shapovalov form. The antiautomorphism $\sigma: \mathcal{H}_2 \rightarrow \mathcal{H}_2$, $\sigma(b_n) = b_{-n}$, $\sigma(c) = c$, defines the Shapovalov form (Section 3.2). Its restriction to $\mathcal{F}_\varphi$ is the *Fock inner product*:

Proposition 5.3. *For level-1 states and $k, l \geq 1$,*

$$(24) \quad \mathcal{S}_\varphi(a_k^\dagger v_\varphi, a_l^\dagger v_\varphi) = \delta_{k,l} \psi_{k,-k} \varphi(c),$$

where $\psi_{k,-k}$ is the coefficient of Definition 2.1. In the genus-zero case $\psi_{k,-k} = k \omega_1$ for every $k \geq 1$ by (3), so

$$(25) \quad \mathcal{S}_\varphi(a_k^\dagger v_\varphi, a_l^\dagger v_\varphi) = \delta_{k,l} k \omega_1 \varphi(c).$$

For general multi-mode states the form is computed from the contravariance relation

$$(26) \quad \mathcal{S}_\varphi(b_{-k} v, w) = \mathcal{S}_\varphi(v, b_k w), \quad v, w \in \mathcal{F}_\varphi, k \geq 1.$$

Proof. For (24), with $k, l \geq 1$,

$$\begin{aligned} \mathcal{S}_\varphi(b_{-k} v_\varphi, b_{-l} v_\varphi) &= \mathcal{S}_\varphi(v_\varphi, b_k b_{-l} v_\varphi) && \text{by (26)} \\ &= \mathcal{S}_\varphi(v_\varphi, b_{-l} b_k v_\varphi + [b_k, b_{-l}] v_\varphi) \\ &= \mathcal{S}_\varphi(v_\varphi, \delta_{k,l} \psi_{k,-k} c v_\varphi) && \text{since } b_k v_\varphi = 0 \\ &= \delta_{k,l} \psi_{k,-k} \varphi(c) \mathcal{S}_\varphi(v_\varphi, v_\varphi), \end{aligned}$$

the third line using $[b_k, b_{-l}] = \delta_{k,l} \psi_{k,-k} c$ from Definition 2.1. Normalising by $\mathcal{S}_\varphi(v_\varphi, v_\varphi) = 1$ gives (24), and substituting $\psi_{k,-k} = k \omega_1$ gives (25). $\square$

5.3. Low-degree polynomial vectors in the Fock basis. We now identify the polynomial vectors $\tilde{P}_n \in \mathcal{F}_\varphi$ at low levels.

Level 0 ($n = 0$): $\mathcal{F}_\varphi[0] = \mathbb{C} v_\varphi$ is one-dimensional.

$$\tilde{P}_0 = v_\varphi.$$

Level 1 ($n = 1$): $\mathcal{F}_\varphi[1] = \mathbb{C} a_1^\dagger v_\varphi$ is one-dimensional. By Proposition 5.3, $\mathcal{S}_\varphi(a_1^\dagger v_\varphi, a_1^\dagger v_\varphi) = \omega_1 \varphi(c)$. Assume $q = \omega_1 \varphi(c) \in \mathbb{R}_{>0}$, so that the square roots below are real and positive (Theorem 3.7 (iv)). Since $\mathcal{S}_\varphi(\tilde{P}_1, \tilde{P}_1) = h_1 = \frac{2}{3}$, we normalise:

$$(27) \quad \tilde{P}_1 = \sqrt{\frac{h_1}{\omega_1 \varphi(c)}} a_1^\dagger v_\varphi = \sqrt{\frac{2}{3 \omega_1 \varphi(c)}} b_{-1} v_\varphi.$$

With the normalisation $\omega_1 = 1$, $\varphi(c) = 1$: $\tilde{P}_1 = \sqrt{2/3} b_{-1} v_\varphi$.

Level 2 ($n = 2$): $\mathcal{F}_\varphi[2]$ is two-dimensional, spanned by $\{a_2^\dagger v_\varphi, (a_1^\dagger)^2 v_\varphi\} = \{b_{-2} v_\varphi, b_{-1}^2 v_\varphi\}$. The Shapovalov form on this basis (with $\omega_1 = \varphi(c) = 1$) is:

$$(28) \quad \mathcal{S}_\varphi(b_{-2} v_\varphi, b_{-2} v_\varphi) = 2,$$

$$(29) \quad \mathcal{S}_\varphi(b_{-2} v_\varphi, b_{-1}^2 v_\varphi) = 0,$$

$$(30) \quad \mathcal{S}_\varphi(b_{-1}^2 v_\varphi, b_{-1}^2 v_\varphi) = 2.$$

Equation (29) follows from $\mathcal{S}_\varphi(b_{-2} v_\varphi, b_{-1}^2 v_\varphi) = \mathcal{S}_\varphi(v_\varphi, b_2 b_{-1}^2 v_\varphi)$, and $[b_2, b_{-1}^2] = 0$ (since $[b_2, b_{-1}] = 0$ for $2 \neq 1$ in the genus-zero cocycle), giving $b_2 b_{-1}^2 v_\varphi = 0$. Equation (30) follows from the repeated application of the contravariance relation, using $[b_1, b_{-1}] = \omega_1 c$ and $b_1 v_\varphi = 0$.

The vector $\tilde{P}_2$ is the one given by Definition 3.6: the representative, with respect to $\mathcal{S}_\varphi$, of the functional taking the value 1 on each of $b_{-2} v_\varphi$ and $b_{-1}^2 v_\varphi$. Since the Gram matrix (28)–(30) is diagonal with equal entries, that representative is proportional to the sum of the two basis vectors, and the normalisation of Theorem 3.7 (iv) fixes the scalar. The proposition below records the result and identifies the complementary direction.

Proposition 5.4. *In the Fock basis $\{b_{-2} v_\varphi, b_{-1}^2 v_\varphi\}$, and with $\omega_1 = \varphi(c) = 1$ (a value of q in $\mathbb{R}_{>0}$, so that Theorem 3.7(iv) applies), the polynomial vector $\tilde{P}_2$ is:*

$$(31) \quad \tilde{P}_2 = \sqrt{\frac{h_2}{4}} b_{-2} v_\varphi + \sqrt{\frac{h_2}{4}} b_{-1}^2 v_\varphi = \sqrt{\frac{1}{10}} (b_{-2} v_\varphi + b_{-1}^2 v_\varphi),$$

where $h_2 = \frac{2}{5}$. Under the composite map $\Phi = \Psi \circ \psi$ of Definition 4.8, $\tilde{P}_2$ pulls forward to $\Phi(\tilde{P}_2) = \sqrt{\frac{2}{5}} P_2(x) = \sqrt{h_2} P_2(x)$ in $\mathbb{C}[x]$, in agreement with the Legendre normalisation $P_2(1) = 1$ of Section 3. At the same level-2 weight space, the orthogonal complement of $\tilde{P}_2$ within $\mathcal{F}_\varphi[2]$ is the one-dimensional $\mathcal{S}_\varphi$ -orthogonal subspace $\mathbb{C} \cdot (b_{-2} v_\varphi - b_{-1}^2 v_\varphi)$, which coincides with $\ker(\psi)|_{\mathcal{F}_\varphi[2]}$ (of dimension $p(2) - 1 = 1$, cf. Lemma 4.6 (ii)); this is the “spare” direction implicit in the strict factor $p(n) - 1$ for general n in Remark 5.5.

Proof. At $q = 1$ the partitions of 2 are (2) and (1, 1), with $z_{(2)} = 2$, $z_{(1,1)} = 2$ and $\ell = 1, 2$ respectively, so (9) gives both diagonal entries equal to 2, in agreement with (28)–(30). By (7) the coefficients of $\tilde{P}_2$ are then proportional to $1/2$ and $1/2$, that is

equal, so $\tilde{P}_2 = \alpha(b_{-2}v_\varphi + b_{-1}^2v_\varphi)$ for a single scalar $\alpha > 0$. The norm (8) at $n = 2$, $q = 1$ equals 1, and (iv) rescales it to $h_2 = 2/5$; since $\mathcal{S}_\varphi(\alpha(b_{-2}v_\varphi + b_{-1}^2v_\varphi), \alpha(b_{-2}v_\varphi + b_{-1}^2v_\varphi)) = 4\alpha^2$, the condition $4\alpha^2 = 2/5$ gives $\alpha = 1/\sqrt{10}$. Applying Φ then gives $\Phi(\tilde{P}_2) = \Psi(2\alpha\zeta^2) = \sqrt{2/5}P_2(x)$. The complementary direction $b_{-2}v_\varphi - b_{-1}^2v_\varphi$ satisfies $\psi(b_{-2}v_\varphi - b_{-1}^2v_\varphi) = \zeta^2 - \zeta^2 = 0$ and is therefore the kernel direction at level 2. $\square$

Remark 5.5. For every n the level- n space $\mathcal{F}_\varphi[n]$ has dimension $p(n)$, the number of partitions of n , and one might expect the expansion of $\tilde{P}_n$ in the Fock basis to require a $p(n) \times p(n)$ orthogonalisation. It does not. By Theorem 3.7(i) the Gram matrix is diagonal in that basis, with entries $z_\lambda q^{\ell(\lambda)}$, so the orthogonalisation is a single division and

$$\tilde{P}_n = \left(\frac{2}{2n+1}\right)^{1/2} \binom{q^{-1} + n - 1}{n}^{-1/2} \sum_{\lambda \vdash n} \frac{1}{z_\lambda q^{\ell(\lambda)}} a_{\lambda_1}^\dagger \cdots a_{\lambda_s}^\dagger v_\varphi$$

for every $n \geq 0$, under the hypothesis $q \in \mathbb{R}_{>0}$ of (iv). For $n = 1, 2$ this reproduces (27) and (31). The corresponding statement for $m \geq 3$ is not available, since there the Gram matrix is not diagonal.

5.4. Summary: the Fock realization in low degree.

Corollary 5.6. *Let $m = 2$, $r = 1$ and let φ be Shapovalov-regular.*

- (i) *The Verma module $M(\varphi)$ is canonically isomorphic to the Fock space $\mathcal{F}_\varphi = \mathbb{C}[b_{-1}, b_{-2}, \dots]v_\varphi$ as $\mathcal{H}_2$ -modules.*
- (ii) *The Shapovalov form $\mathcal{S}_\varphi$ coincides with the Fock inner product defined by (25)–(26).*
- (iii) *The polynomial vectors $\tilde{P}_n$ are given in the Fock basis, for every $n \geq 0$, by Remark 5.5; the cases $n = 1, 2$ are (27) and Proposition 5.4.*
- (iv) *If in addition $q = \omega_1\varphi(c) \in \mathbb{R}_{>0}$, so that the normalisation of Theorem 3.7(iv) is available, the Gram matrix of the Fock inner product in the $\{\tilde{P}_n\}$ basis is $\mathcal{S}_\varphi(\tilde{P}_n, \tilde{P}_{n'}) = \frac{2}{2n+1}\delta_{nn'}$. Without that hypothesis the form is still diagonal in this basis, with the entries (8).*

6. EXAMPLES

6.1. The genus-zero case $m = 2$, $r = 1$. For the case worked out in this paper, let $p(t) = 1 - 2at + t^2$, so $A_2 = \mathbb{C}[t^{\pm 1}, u]/(u^2 - p(t))$. The four-point Heisenberg algebra $\mathcal{H}_2$ is generated by $\{b_n : n \in \mathbb{Z}\}$ with brackets $[b_k, b_l] = k\delta_{k+l,0}\omega_1 c$. For concreteness, take $\omega_1 = 1$ (the normalization is immaterial for the eigenvalues).

The Verma module $M(\varphi)$ is generated from the highest-weight vector v_φ by the action of the universal enveloping algebra $U(\mathcal{H}_2)$ subject to $b_nv_\varphi = 0$ for $n > 0$, $b_0v_\varphi = \varphi(b_0)v_\varphi$ (where $\varphi(b_0)$ is a fixed scalar, say $\varphi(b_0) = 1$ for simplicity).

The Shapovalov form is defined by $\mathcal{S}_\varphi(v_\varphi, v_\varphi) = 1$ and the contravariance relation. For small degrees $n = 0, 1, 2, 3, 4$, the polynomial vectors $\tilde{P}_n$ correspond to the

Legendre polynomials:

$$\begin{aligned} P_0(a) &= 1, \\ P_1(a) &= a, \\ P_2(a) &= \frac{1}{2}(3a^2 - 1), \\ P_3(a) &= \frac{1}{2}(5a^3 - 3a), \\ P_4(a) &= \frac{1}{8}(35a^4 - 30a^2 + 3). \end{aligned}$$

The Gram matrix is diagonal: $\mathcal{S}_\varphi(\tilde{P}_n, \tilde{P}_{n'}) = h_n \delta_{nn'}$ with $h_n = \frac{2}{2n+1}$ for $n \geq 1$, the squared norms of the Legendre polynomials. Explicitly $h_1 = 2/3$, $h_2 = 2/5$, $h_3 = 2/7$, $h_4 = 2/9$; at level 0 the convention $\tilde{P}_0 = v_\varphi$ gives $h_0 = 1$ (Theorem 3.7 (iv)).

These values follow from Theorem 3.7 (iv): the closed form (7) and the diagonal entries (9) give the unnormalised norm (8), which the rescaling then sends to $2/(2n+1)$.

The Casimir operator Ω acts on the basis $\{\tilde{P}_n\}$ with eigenvalues $\lambda_n = -n(n+1)$:

$$\begin{aligned} \Omega \cdot \tilde{P}_0 &= 0 \cdot \tilde{P}_0, \\ \Omega \cdot \tilde{P}_1 &= -2 \cdot \tilde{P}_1, \\ \Omega \cdot \tilde{P}_2 &= -6 \cdot \tilde{P}_2, \\ \Omega \cdot \tilde{P}_3 &= -12 \cdot \tilde{P}_3, \\ \Omega \cdot \tilde{P}_4 &= -20 \cdot \tilde{P}_4. \end{aligned}$$

These values exactly match the eigenvalues $-n(n+1)$ of the Legendre differential operator $L = (1-x^2)\partial_x^2 - 2x\partial_x$ of Section 4, under the identification Ψ .

These are not separate computations: by Lemma 4.4 the element Ω acts on the whole of $M(\varphi)[-n]$ by the scalar $-n(n+1)$, so the table records the action on one chosen vector of each level rather than a property distinguishing $\tilde{P}_n$ within its level. What distinguishes $\tilde{P}_n$ is Definition 3.6.

One feature of the construction is visible already here. By Definition 2.1 the zero mode b_0 is central in $\mathcal{H}_2$, so the module $M(\varphi)$, the form $\mathcal{S}_\varphi$ and the vectors $\tilde{P}_n$ depend on the weight φ only through the central charge $\varphi(c)$: two weights differing in $\varphi(b_0)$ give the same polynomial family. In this sense the family is determined by the cocycle and by a , not by the choice of φ . For $m \geq 3$ the zero modes are no longer central and this independence should not be expected.

6.2. An illustration of the obstruction at $m = 4$. The following is not a worked example but an indication of what changes beyond the genus-zero case. Consider the curve $u^4 = 1 - 2at + t^2$, so $m = 4$, $r = 1$.

The Heisenberg subalgebra $\mathcal{H}_4$ is generated by two families: $\{b_n^{(1)} : n \in \mathbb{Z}\}$ and $\{b_n^{(2)} : n \in \mathbb{Z}\}$, with cocycle coefficients $\psi_{kl}^{(ij)}(a)$ computed in [3]. By [3, Thm. 1.1] the space $\Omega_{A_4}^1/dA_4$ has basis $t^{-1}dt$ together with $t^{-1}u^l dt$ and $t^{-2}u^l dt$ for $l = 1, 2, 3$, hence dimension $1 + 3 \cdot 2 = 7$. This agrees with the count $2g + N - 1$ of Section 2: the Riemann–Hurwitz formula gives $g = 1$, the covering is unramified over $t = 0$ (since

$p(0) = 1$) so four points lie there, and $\gcd(4, 2) = 2$ points lie over $t = \infty$, giving $N = 6$.

The Verma module $M(\varphi)$ and its Shapovalov form are defined as before. Theorem 3.7 does not apply here: it is proved for $m = 2$, $r = 1$. What one expects is that the polynomial vectors $\{\tilde{P}_n\}_{n \geq 0}$ associated with the superelliptic family of [3] again satisfy

$$\mathcal{S}_\varphi(\tilde{P}_n, \tilde{P}_{n'}) = h_n^{(4)} \delta_{nn'},$$

where the $h_n^{(4)}$ would be the squared norms of the superelliptic family. That this family exists, is orthogonal, and admits a positive normalisation is not established here.

No such relation is established here, and we are not aware of a verification of it in the literature.

The Sugawara side changes in the same way. A candidate Ω' would have to be built from both cocycle components ω_1 and ω_2 , and there is no reason to expect its image under an intertwiner to be a second-order operator; the natural guess is an operator of order m . Identifying it, even for $m = 4$, is open.

7. OUTLOOK

The observation that orthogonal polynomial families appear in the centres of universal central extensions attached to superelliptic curves is due to [3], where the centre relations are matched with three-term recurrences and the associated generating functions are shown to satisfy Sturm–Liouville equations. That correspondence is established there by explicit computation with the cocycle data.

What the present paper contributes is an account of the same phenomenon inside a module. The polynomial vectors are weight vectors of $M(\varphi)$, their orthogonality is a property of the contravariant form (Theorem 3.7), and the differential operator that annihilates them is the image of a Sugawara element under an explicit intertwiner (Theorem 4.9). Verma-type modules over Heisenberg subalgebras with a non-standard polarisation of the imaginary modes were studied by [6]; the modules used here carry the standard polarisation.

APPENDIX A. COMPUTATIONS

A.1. Cocycle formulas. For the genus-zero case $m = 2$, $r = 1$, the cocycle takes the simple form:

$$\psi_{kl}(a) = k \delta_{k+l,0} \omega_1, \quad \text{so that} \quad [b_k, b_l] = k \delta_{k+l,0} \omega_1 c,$$

where $\omega_1 > 0$ is a normalization constant. This identifies the four-point Heisenberg algebra with the standard infinite-dimensional Heisenberg algebra, after rescaling the central element by ω_1 .

For the superelliptic case $m \geq 3$, the cocycle has multiple components $\psi_{kl}^{(ij)}(a)$ indexed by $1 \leq i, j \leq \lfloor m/2 \rfloor$, each depending on the parameter a . We illustrate with the quartic case $m = 4$, $r = 1$ (i.e., $p(t) = 1 - 2at + t^2$, but now $u^4 = p(t)$), which has two families of generators $b_n^{(1)}$ and $b_n^{(2)}$, corresponding to the u and u^2 sectors. The

cocycle decomposes into three independent components:

$$\begin{aligned} [b_k^{(1)}, b_l^{(1)}] &= \psi_{kl}^{(11)}(a) \cdot c, \\ [b_k^{(1)}, b_l^{(2)}] &= \psi_{kl}^{(12)}(a) \cdot c, \\ [b_k^{(2)}, b_l^{(2)}] &= \psi_{kl}^{(22)}(a) \cdot c, \end{aligned}$$

where each $\psi_{kl}^{(ij)}(a)$ is a polynomial in a whose degree depends on $|k+l|$ and the sector indices i, j . The cross-sector component $\psi_{kl}^{(12)}(a)$ introduces genuine multi-component structure that is absent for $m = 2$.

Formulas for the sectors at branch degree $\deg p = 4$ are computed in [3]; the key structural observation is that the sector- $(1, 1)$ recurrence is always governed by Legendre polynomials, while the remaining sectors produce distinct (non-classical) families whose precise form depends on m .

A.2. Gram matrix entries for small n . For the genus-zero Legendre case ($m = 2$, $r = 1$), the Gram matrix of the Shapovalov form in the *normalised* family $\{\tilde{P}_n\}$ of Theorem 3.7 (iv) (so under the hypothesis $q = \omega_1 \varphi(c) \in \mathbb{R}_{>0}$) is diagonal:

n/n'	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	2/3	0	0	0	0
2	0	0	2/5	0	0	0
3	0	0	0	2/7	0	0
4	0	0	0	0	2/9	0
5	0	0	0	0	0	2/11

For $n \geq 1$ the diagonal entries are $h_n = \frac{2}{2n+1}$, the squared norms of the Legendre polynomials; the level-0 entry is 1, by the convention $\tilde{P}_0 = v_\varphi$. All off-diagonal entries vanish.

Before normalisation the same form gives the value (8) (Theorem 3.7 (iii)); the table above is the result of the rescaling in (iv) and therefore displays the classical Legendre norms by construction, not as an independent computation. The individual Gram entries in the monomial basis are $z_\lambda q^{\ell(\lambda)}$, by (9).

ACKNOWLEDGEMENTS

The author thanks the anonymous referee for a careful reading and for detailed comments that substantially improved the manuscript.

This work was financed, in part, by the São Paulo Research Foundation (FAPESP), grant 2024/14914-9.

CONFLICT OF INTEREST

The author declares that there is no conflict of interest.

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