A CONCAVITY INEQUALITY FOR HESSIAN QUOTIENT EQUATIONS

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Abstract. We prove a concavity inequality for the Hessian quotient operator σ_k/σ_{k-1} using a change of basis for symmetric polynomials. As a consequence, the interior C^2 estimate for convex solutions of the σ_k/σ_{k-1} equation holds without the additional structural concavity assumption imposed in previous work. The method also yields several useful inequalities for elementary symmetric polynomials.

1. Introduction

Concavity inequalities for elementary symmetric functions play an important role in second-order estimates for Hessian and curvature equations. Guan–Ren–Wang [2] used such an inequality implicitly in their global C^2 estimate for convex solutions of curvature equations. Chu [1, Lemma 3.3] later stated it explicitly in a form close to the one here. Related concavity inequalities were developed by Ren and Wang for the $(n-1)$- and $(n-2)$-Hessian equations on the corresponding Gårding cones; see [10, 11, 12]. Lu [7] obtained a related inequality for semi-convex solutions of Hessian equations. More recently, Zhang [14, Lemma 1.1] proved a closely related inequality for k-Hessian equations under a semi-convexity assumption.

For Hessian quotient equations, a stronger concavity inequality with an additional positive term is needed. Guan and Sroka [3] proved such an inequality for positive Hessian quotient operators. This inequality gives the key Jacobi inequality used in Lu's interior C^2 estimate for Hessian quotient equations [6]. In particular, Lu studied

$$\frac{\sigma_n}{\sigma_l}(D^2u) = f$$

and obtained interior C^2 estimates for $l = n-1, n-2$.

In our previous work [8], we considered the general Hessian quotient equation

$$\frac{\sigma_k}{\sigma_l}(D^2u) = f, \qquad l = k-1,\ k-2,$$

and proved an interior C^2 estimate for convex solutions under an additional structural concavity assumption. The main result of this paper establishes this concavity inequality when $l = k-1$. Consequently, the interior C^2 estimate for the σ_k/σ_{k-1} equation holds without this additional assumption.

Let

$$F = \frac{\sigma_k}{\sigma_{k-1}}(\lambda), \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0, \qquad 2 \leq k \leq n-1.$$

We prove the following.

Lemma 1.1. *There exists $\varepsilon > 0$ depending only on n and k, and $M = M(\varepsilon, n, k, F)$, such that if $\lambda_1 > M$, then, for any $\xi \in \mathbb{R}^n$,*

$$\frac{1}{F}\left(\sum_i F^{ii}\xi_i\right)^2 - \sum_{i,j} F^{ii,jj}\xi_i\xi_j + 2\sum_{i>1}\frac{F^{ii}\xi_i^2}{\lambda_1} - \frac{F^{11}\xi_1^2}{\lambda_1} \geq \varepsilon\frac{F^{11}\xi_1^2}{\lambda_1},$$

where

$$F^{ii} = \frac{\partial F}{\partial \lambda_i}, \qquad F^{ii,jj} = \frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}.$$

Thus, the structural assumption in [8] holds for $l = k-1$. Together with [8], Lemma 1.1 gives the interior C^2 estimate for convex solutions of the σ_k/σ_{k-1} equation without this additional assumption.

Remark 1.2. *After this work was completed, Li and Wu* [4] *independently obtained a related concavity inequality under different conditions. Their different approach also applies to σ_k/σ_{k-2}.*

We now give a brief outline of the proof. A determinant computation reduces Lemma 1.1 to a scalar inequality involving elementary symmetric polynomials. The endpoint cases $k = 2$ and $k = n-1$ are proved directly. For $3 \leq k \leq n-2$, we further reduce the problem to the positivity of a cubic polynomial

$$N(x) = A_3x^3 + A_2x^2 + A_1x + A_0.$$

Using Rolle's theorem and reciprocal variables, we reduce the terms involving $e_{k-1}, \dots, e_{k-4}$ to the first four elementary symmetric polynomials $e_1, \dots, e_4$.

The main idea is to *change the basis* of symmetric polynomials. This gives useful inequalities for elementary symmetric polynomials that are not transparent from the Newton–Maclaurin inequalities alone. The coefficients A_i also have a multi-affine structure when expressed in terms of certain ratios of elementary symmetric polynomials. Together, these facts reduce the positivity problem to checking values at the vertices of suitable rectangles, giving a direct proof of the desired concavity inequality.

We also give an independent computational verification using SageMath [13]. Expanding the coefficients A_i in the monomial basis reduces their positivity to coefficient positivity together with a few elementary comparisons.

The paper is organized as follows. Section 2 gives some preliminary formulas. Section 3 proves the symmetric polynomial inequalities used later. Section 4 reduces the main estimate to a scalar inequality, and Section 5 treats the endpoint cases. Section 6 reduces the remaining cases to a cubic polynomial. Section 7 gives an independent computational verification using the monomial basis. Finally, Section 8 gives a direct proof using the multi-affine structure and the inequalities from Section 3.

2. Preliminaries

We collect some basic properties of the Hessian and Hessian quotient operators.

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, we denote

$$(\lambda|i) = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n) \in \mathbb{R}^{n-1}.$$

Thus, $(\lambda|i)$ is obtained from λ by deleting its i-th component. Similarly, $(\lambda|ij)$ is obtained by deleting its i-th and j-th components.

We recall some basic formulas for the elementary symmetric functions; see, for instance, [2, 5].

Lemma 2.1. *For any $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, we have*

$$\sigma_k(\lambda) = \lambda_i \sigma_{k-1}(\lambda|i) + \sigma_k(\lambda|i),$$

$$\sum_i \lambda_i \sigma_{k-1}(\lambda|i) = k\sigma_k(\lambda), \qquad \sum_i \sigma_k(\lambda|i) = (n-k)\sigma_k(\lambda),$$

$$\sigma_{k-1}^2(\lambda|ij) - \sigma_k(\lambda|ij)\sigma_{k-2}(\lambda|ij) = \sigma_{k-1}(\lambda|i)\sigma_{k-1}(\lambda|j) - \sigma_k(\lambda)\sigma_{k-2}(\lambda|ij).$$

When there is no ambiguity, we write

$$\sigma_{k;i} = \sigma_k(\lambda|i), \qquad \sigma_{k;ij} = \sigma_k(\lambda|ij).$$

We also recall Newton's inequality

$$\sigma_{k-1}^2 \geq \frac{k(n-k+2)}{(k-1)(n-k+1)} \sigma_k \sigma_{k-2}.$$

Define the Gårding cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0,\ 1 \leq j \leq k\}.$$

If F is a symmetric function of the eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ of D^2u, we also regard F as a function of D^2u and write

$$F^{pq} = \frac{\partial F}{\partial u_{pq}}, \qquad F^{pq,rs} = \frac{\partial^2 F}{\partial u_{pq} \partial u_{rs}}.$$

We also recall the following formulas for the Hessian quotient operator; see [6].

Lemma 2.2. *Let $1 \leq k < \ell \leq n$, let $F = \frac{\sigma_\ell}{\sigma_k}$ and let u be a convex function. Suppose D^2u is diagonalized at x_0. Then at x_0, we have*

$$F^{pp} = \partial_{\lambda_p}\left(\frac{\sigma_\ell}{\sigma_k}\right) = \frac{\sigma_{\ell-1;p}}{\sigma_k} - \frac{\sigma_\ell \sigma_{k-1;p}}{\sigma_k^2},$$

$$F^{pp,qq} = \partial_{\lambda_q}\partial_{\lambda_p}\left(\frac{\sigma_\ell}{\sigma_k}\right) \begin{cases} -2\frac{\sigma_{\ell-1;p}\sigma_{k-1;p}}{\sigma_k^2} + 2\frac{\sigma_\ell\left(\sigma_{k-1;p}\right)^2}{\sigma_k^3}, & p = q. \\ \frac{\sigma_{\ell-2;pq}}{\sigma_k} - \frac{\sigma_{\ell-1;p}\sigma_{k-1;q}}{\sigma_k^2} - \frac{\sigma_{\ell-1;q}\sigma_{k-1;p}}{\sigma_k^2} - \frac{\sigma_\ell \sigma_{k-2;pq}}{\sigma_k^2} + 2\frac{\sigma_\ell \sigma_{k-1;p}\sigma_{k-1;q}}{\sigma_k^3}, & p \neq q, \end{cases}$$

Finally, recall that

$$F = \frac{\sigma_k}{\sigma_{k-1}}$$

is concave in Γ_k. In particular, when all eigenvalues are positive, we have

$$C_1(n,k,F) < \lambda_k < C_2(n,k,F). \tag{2.1}$$

3. Symmetric polynomial inequalities

In this section, we establish several inequalities for P, Q, R that will be used later in the proof of the main estimate. The key inequalities are obtained by changing between different bases of symmetric polynomials.

Let e_j denote the j-th elementary symmetric polynomial in k positive variables. Define

$$P = \frac{e_1 e_3}{e_2^2}, \qquad Q = \frac{e_2}{e_1^2}, \qquad R = \frac{e_2 e_4}{e_3^2}.$$

We use the standard change-of-basis formulas for symmetric functions; see [9, Chapter I]. The computations below were also verified using SageMath [13].

3.1. **Inequalities from power sums.** Let $p_j = \sum_i x_i^j$ be the j-th power sum. By Cauchy–Schwarz,

$$p_2^2 \le p_1 p_3. \tag{3.1}$$

Using the following expansions,

$$\begin{aligned} &p_1 = e_1, \qquad p_2 = e_1^2 - 2e_2, \qquad p_3 = e_1^3 - 3e_1e_2 + 3e_3, \\ &p_4 = e_1^4 - 4e_1^2e_2 + 4e_1e_3 + 2e_2^2 - 4e_4, \end{aligned}$$

we obtain from (3.1)

$$e_1^2e_2 + 3e_1e_3 - 4e_2^2 \ge 0. \tag{3.2}$$

Equivalently,

$$1 \ge (4 - 3P)Q.$$

We also use the inequality

$$p_4 \le p_2^2.$$

After substitution and simplification, this gives

$$2P^2QR \ge 2P - 1.$$

3.2. **Inequalities from Schur polynomials.** We next use the positivity of Schur polynomials. We use the expansions

$$s_{(2,2)} = \begin{vmatrix} e_2 & e_3 \\ e_1 & e_2 \end{vmatrix} \ge 0, \qquad s_{(3,3)} = \begin{vmatrix} e_2 & e_3 & e_4 \\ e_1 & e_2 & e_3 \\ e_0 & e_1 & e_2 \end{vmatrix} \ge 0.$$

The inequality $s_{(3,3)} \ge 0$ is equivalent to

$$-P^2QR + P^2Q + P^2R - 2P + 1 \ge 0.$$

3.3. **Inequalities from monomial symmetric polynomials.** Let m_λ denote the monomial symmetric polynomial associated with the partition λ. We use the expansions

$$\begin{aligned} m_{33} &= \sum_{\text{sym}} x_i^3x_j^3 = 3e_1^2e_4 - 3e_1e_2e_3 - 3e_1e_5 + e_2^3 - 3e_2e_4 + 3e_3^2 + 3e_6, \\ m_{222} &= \sum_{\text{sym}} x_i^2x_j^2x_k^2 = e_3^2 - 2e_2e_4 + 2e_1e_5 - 2e_6. \end{aligned}$$

Hence

$$-12P^2QR + 9P^2Q + 6P^2R - 6P + 2 = e_2^{-3}\left(2m_{33} + 3m_{222}\right) \ge 0.$$

A further inequality, which is important later, follows from the monomial expansion

$$\begin{aligned} &e_2e_3^2 + e_1e_3e_4 - 2e_4e_2^2 \\ &\quad = m_{332} + 2m_{3221} + m_{32111} + 4m_{2222} + 6m_{22211} + 6m_{221111} + 5m_{2111111} \ge 0. \end{aligned} \tag{3.3}$$

Therefore,

$$e_2e_3^2(1 + PR - 2R) \ge 0, \qquad \text{or equivalently} \qquad 1 \ge (2 - P)R.$$

Remark 3.1. *The nonnegativity in* (3.2) *and* (3.3) *does not follow directly from the Newton–Maclaurin inequalities. The change-of-basis formulas above allow us to use positivity in the power-sum and monomial bases.*

3.4. **Collection of inequalities.** The Newton inequalities also give

$$0 < P \leq \frac{2(k-2)}{3(k-1)}, \qquad 0 < Q \leq \frac{k-1}{2k}, \qquad 0 < R \leq \frac{3(k-3)}{4(k-2)}.$$

We collect the above inequalities for later reference.

Lemma 3.2. *Suppose that all the variables are positive. Then the following inequalities hold:*

(1) $1 \geq (4-3P)Q$.
(2) $1 \geq (2-P)R$.
(3) $2P^2QR \geq 2P-1$.
(4) $-P^2QR + P^2Q + P^2R - 2P + 1 \geq 0$.
(5) $-12P^2QR + 9P^2Q + 6P^2R - 6P + 2 \geq 0$.
(6)
$$0 < P \leq \frac{2(k-2)}{3(k-1)}, \qquad 0 < Q \leq \frac{k-1}{2k}, \qquad 0 < R \leq \frac{3(k-3)}{4(k-2)}.$$

4. Reduction of the main estimate

We first reduce the main estimate to a simpler algebraic inequality. Our goal is to prove

$$M^{ij}\xi_i\xi_j = \frac{1}{F}\left(\sum_i F^{ii}\xi_i\right)^2 - \sum_{i,j} F^{ii,jj}\xi_i\xi_j + 2\sum_{i>1}\frac{F^{ii}\xi_i^2}{\lambda_1} - (1+\varepsilon)\frac{F^{11}\xi_1^2}{\lambda_1} \geq 0.$$

4.1. **Reduction to a determinant.** Since $M = (M^{ij})$ has at most one negative eigenvalue, it suffices to show that $\det M \geq 0$. We compare M with the simpler matrix

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \det N = \det D\,(A - BD^{-1}C),$$

where

$$\begin{aligned}
A &= -F^{11,11} + \frac{1}{F}(F^{11})^2 - (1+\varepsilon)\frac{F^{11}}{\lambda_1}, \\
B &= C^T = \left[-F^{11,jj} + \frac{1}{F}F^{11}F^{jj}\right]_{2\leq j\leq n}, \\
D &= \operatorname{diag}\left(\frac{2}{\lambda_1}\mathbf{v}\right) + \frac{1}{F}\mathbf{v}\mathbf{v}^T, \\
\mathbf{v} &= (F^{22}, \ldots, F^{nn})^T.
\end{aligned}$$

We note that M and N only differ at the lower-right block.

$$M = N + \begin{pmatrix} 0 & 0 \\ 0 & D_2 \end{pmatrix}, \qquad D_2 = \left[-F^{ii,jj}\right]_{2\leq i,j\leq n} \geq 0.$$

Thus, it is enough to prove $N \geq 0$. Since $D > 0$, this is equivalent to

$$A - BD^{-1}C \geq 0.$$

To simplify the formulas, set

$$\begin{aligned} a_j &= \frac{1}{\sigma_k^2}\left(\sigma_{k-1;1j}^2 - \sigma_{k;1j}\sigma_{k-2;1j}\right), \\ b_j &= \frac{1}{\sigma_{k-1}^2}\left(\sigma_{k-2;1j}^2 - \sigma_{k-1;1j}\sigma_{k-3;1j}\right), \\ m &= n-k. \end{aligned}$$

4.2. **Estimates for the matrix entries.** Throughout this section, the constants implicit in the $O(\cdot)$ notation may depend on n, k, and F, but are independent of λ_1.
The derivatives F^{ii}. By Lemma 2.2, we have

$$F^{ii} = \frac{\sigma_{k-1;i}}{\sigma_{k-1}} - \frac{\sigma_k\sigma_{k-2;i}}{\sigma_{k-1}^2} = \frac{\sigma_{k-1;i}^2 - \sigma_{k;i}\sigma_{k-2;i}}{\sigma_{k-1}^2}.$$

Since λ_k is bounded, the Newton inequality gives, for $j \geq 2$,

$$\begin{aligned} F^{jj} &= \frac{(\lambda_1\sigma_{k-2;1j} + \sigma_{k-1;1j})^2 - (\lambda_1\sigma_{k-1;1j} + \sigma_{k;1j})(\lambda_1\sigma_{k-3;1j} + \sigma_{k-2;1j})}{\sigma_{k-1}^2} \\ &= \lambda_1^2 b_j\left(1 + O\left(\frac{1}{\lambda_1}\right)\right). \end{aligned} \tag{4.1}$$

Also, by Lemma 2.1,

$$\begin{aligned} \sum_{j=2}^n a_j &= \frac{1}{\sigma_k^2}\sum_{j=2}^n\left(\sigma_{k-1;1j}^2 - \sigma_{k;1j}\sigma_{k-2;1j}\right) = \frac{1}{\sigma_k^2}\sum_{j=2}^n\left(\sigma_{k-1;1}\sigma_{k-1;j} - \sigma_k\sigma_{k-2;1j}\right) \\ &= \frac{1}{\sigma_k^2}\left(m\sigma_{k-1;1}^2 - (m+1)\sigma_{k;1}\sigma_{k-2;1}\right). \end{aligned} \tag{4.2}$$

Therefore,

$$F^{11} > \frac{F^2}{m}\sum_{j=2}^n a_j. \tag{4.3}$$

Moreover,

$$F^{11} = O\left(\frac{(\lambda_2 \ldots \lambda_k)^2}{(\lambda_1 \ldots \lambda_{k-1})^2}\right) = O\left(\frac{1}{\lambda_1^2}\right), \tag{4.4}$$

$$F^{nn} = O\left(\frac{(\lambda_1 \ldots \lambda_{k-1})^2}{(\lambda_1 \ldots \lambda_{k-1})^2}\right) = O(1). \tag{4.5}$$

The term A. We have

$$\begin{aligned} F^{11,11} &= -2\frac{\sigma_{k-2;1}}{\sigma_{k-1}}F^{11} = \frac{-2\sigma_{k-2;1}}{\lambda_1\sigma_{k-2;1} + \sigma_{k-1;1}}F^{11} \\ &= -2\frac{F^{11}}{\lambda_1}\left(1 + O\left(\frac{1}{\lambda_1}\right)\right). \end{aligned}$$

Thus, by (4.4) and (4.3),

$$\begin{aligned} A &= -F^{11,11} + \frac{1}{F}\left(F^{11}\right)^2 - (1+\varepsilon)\frac{F^{11}}{\lambda_1} \\ &= (1-\varepsilon)\frac{F^{11}}{\lambda_1} + O\left(\frac{1}{\lambda_1^4}\right) \\ &> (1-\varepsilon)\frac{F^2}{m\lambda_1}\sum_{j=2}^n a_j + O\left(\frac{1}{\lambda_1^4}\right). \end{aligned} \tag{4.6}$$

The term B. A direct computation gives

$$B = -F^{11,jj} + \frac{1}{F}F^{11}F^{jj} = F(a_j - b_j). \tag{4.7}$$

The matrix D^{-1}. Let

$$\Delta = \operatorname{diag}\left(\frac{2}{\lambda_1}\mathbf{v}\right).$$

By the Sherman–Morrison formula,

$$D^{-1} = \left(\Delta + \frac{1}{F}\mathbf{v}\mathbf{v}^T\right)^{-1} = \Delta^{-1} - \frac{\Delta^{-1}\mathbf{v}\mathbf{v}^T\Delta^{-1}}{F + \mathbf{v}^T\Delta^{-1}\mathbf{v}}.$$

Using (4.5) and (4.1), we obtain

$$\begin{aligned} (D^{-1})_{ij} &= \frac{\lambda_1}{2F^{ii}}\delta_{ij} - \frac{\lambda_1^2}{4F + 2\lambda_1\sum_{k=2}^n F^{kk}} \\ &= \frac{1}{2\lambda_1 b_j}\delta_{ij}\left(1 + O\left(\frac{1}{\lambda_1}\right)\right) - \frac{1}{2\lambda_1\sum_{k=2}^n b_k}\left(1 + O\left(\frac{1}{\lambda_1}\right)\right). \end{aligned} \tag{4.8}$$

The term $-BD^{-1}C$. By (4.7) and (4.8),

$$\begin{aligned} -BD^{-1}C &= -\sum_{j=2}^n \frac{F^2(a_j-b_j)^2}{2\lambda_1 b_j} + \frac{F^2\left(\sum_{j=2}^n (a_j - b_j)\right)^2}{2\lambda_1\sum_{j=2}^n b_j} + O\left(\frac{1}{\lambda_1^4}\right) \\ &= \frac{F^2}{2\lambda_1}\left(-\sum_{j=2}^n \frac{a_j^2}{b_j} + \frac{\left(\sum_{j=2}^n a_j\right)^2}{\sum_{j=2}^n b_j}\right) + O\left(\frac{1}{\lambda_1^4}\right). \end{aligned} \tag{4.9}$$

4.3. **The key inequality.** Combining (4.6) and (4.9), we obtain

$$\begin{aligned} &A - BD^{-1}C \\ &> (1-\varepsilon)\frac{F^2}{m\lambda_1}\sum_{j=2}^n a_j \quad + \frac{F^2}{2\lambda_1}\left(-\sum_{j=2}^n \frac{a_j^2}{b_j} + \frac{\left(\sum_{j=2}^n a_j\right)^2}{\sum_{j=2}^n b_j}\right) + O\left(\frac{1}{\lambda_1^4}\right) \\ &= \frac{F^2}{2\lambda_1}\sum_{j=2}^n a_j\left((1-\varepsilon)\frac{2}{m} - \frac{a_j}{b_j} + \frac{\sum_{k=2}^n a_k}{\sum_{k=2}^n b_k}\right) + O\left(\frac{1}{\lambda_1^4}\right). \end{aligned} \tag{4.10}$$

Thus, the main estimate reduces to the following inequality.

Lemma 4.1. *Suppose there exists a constant $\varepsilon_0(n,k) > 0$ such that*

$$\frac{2}{m} - \frac{a_j}{b_j} + \frac{\sum_{k=2}^n a_k}{\sum_{k=2}^n b_k} > \varepsilon_0(n,k), \qquad 2 \le j \le n. \tag{4.11}$$

Then there exists $M = M(\varepsilon, n, k, F)$ such that

$$\det N \ge 0$$

whenever $\lambda_1 > M$.

Indeed, choose $\varepsilon > 0$ sufficiently small so that (4.11) implies

$$(1-\varepsilon)\frac{2}{m} - \frac{a_j}{b_j} + \frac{\sum_{k=2}^n a_k}{\sum_{k=2}^n b_k} \ge \frac{\varepsilon_0}{2}, \qquad 2 \le j \le n.$$

Moreover,

$$\sum_{j=2}^n a_j \ge a_n \ge \frac{c(n,k,F)}{\lambda_1^2}.$$

It follows from (4.10) that

$$A - BD^{-1}C \ge \frac{c(n,k,F)}{\lambda_1^3} - O\left(\frac{1}{\lambda_1^4}\right) > 0$$

when λ_1 is sufficiently large. Therefore $\det N \ge 0$.

Remark 4.2. *In fact, we only require $j = n$ in the inequality (4.11), since a_j/b_j is increasing in j. Indeed, differentiating the quotient $\left(\sigma_{k-1}^2 - \sigma_k\sigma_{k-2}\right) / \left(\sigma_{k-2}^2 - \sigma_{k-1}\sigma_{k-3}\right)$ with respect to λ_i and using Lemma 3.2(4) shows that the derivative is positive. Also, we remark that (4.11) can also be written as a weighted variance.*

5. Proof of the key inequality: endpoint cases

It remains to prove the key inequality (4.11) used in the previous section. We first introduce some notation that will be used throughout the proof. Set

$$e_l = \sigma_{l;1j}, \qquad x = \lambda_j. \tag{5.1}$$

Thus, e_l is the l-th elementary symmetric polynomial in the remaining $n-2$ variables. We will also use

$$\frac{\sigma_{k-1}^2}{\sigma_k^2} = \left(\frac{\sigma_{k-2;1}}{\sigma_{k-1;1}}\right)^2 \left(1 + O\left(\frac{1}{\lambda_1}\right)\right). \tag{5.2}$$

Since all the terms involved are uniformly bounded, the error introduced by (5.2) is $O(\lambda_1^{-1})$. Hence, for λ_1 sufficiently large, it can be absorbed into the small constant ε_0 in (4.11).

We first prove (4.11) directly in the endpoint cases $k = 2$ and $k = n-1$. The remaining cases $3 \le k \le n-2$ are treated in the following sections.

5.1. **The case $k = 2$.** Using (5.1) and (5.2), the left-hand side of (4.11) for $k = 2$ becomes

$$\begin{aligned} & \frac{2}{m} - \frac{e_1^2 - e_2}{(x+e_1)^2} + \frac{1}{(x+e_1)^2} \frac{m\,(x+e_1)^2 - (m+1)\,(xe_1+e_2)}{(m+1)} \\ & = \frac{\left(m^2+2m+2\right)x^2 + \left(m^2+3m+4\right)e_1 x + (m+2)\,e_1^2}{m\,(m+1)\,(x+e_1)^2}. \end{aligned} \tag{5.3}$$

All the coefficients in the numerator are strictly positive. Hence, for some $\varepsilon_0 = \varepsilon_0(n,k) > 0$, they remain strictly positive after replacing $\frac{2}{m}$ by $\frac{2-\varepsilon_0}{m}$. Therefore, (4.11) holds for $k = 2$.

5.2. **The case $k = n-1$.** It suffices to consider the first two terms in (4.11). We have

$$\begin{aligned} & \frac{2}{m} - \left(\frac{\sigma_{k-2;1}}{\sigma_{k-1;1}}\right)^2 \frac{\sigma_{k-1;1j}^2 - \sigma_{k;1j}\sigma_{k-2;1j}}{\sigma_{k-2;1j}^2 - \sigma_{k-1;1j}\sigma_{k-3;1j}} \\ & = 2 - \left(\frac{xe_{n-4}+e_{n-3}}{xe_{n-3}+e_{n-2}}\right)^2 \frac{e_{n-2}^2}{e_{n-3}^2 - e_{n-2}e_{n-4}} \\ & = \frac{N(x)}{(xe_{n-3}+e_{n-2})^2\left(e_{n-3}^2 - e_{n-2}e_{n-4}\right)}, \end{aligned}$$

where

$$\begin{aligned} N(x) = & \left(2e_{n-3}^4 - 2e_{n-3}^2 e_{n-2}e_{n-4} - e_{n-2}^2 e_{n-4}^2\right)x^2 + \\ & \left(4e_{n-3}^3 e_{n-2} - 6e_{n-3}e_{n-2}^2 e_{n-4}\right)x + \left(e_{n-3}^2 e_{n-2}^2 - 2e_{n-2}^3 e_{n-4}\right). \end{aligned}$$

By the Newton inequality in $n-2$ variables, we have

$$e_{n-3}^2 \geq 2\frac{(n-2)}{(n-3)} e_{n-2}e_{n-4}.$$

Therefore, all the coefficients of $N(x)$ are strictly positive. Hence, for some $\varepsilon_0 = \varepsilon_0(n,k) > 0$, they remain strictly positive after replacing $\frac{2}{m}$ by $\frac{2-\varepsilon_0}{m}$. Thus, (4.11) holds for $k = n-1$.

Therefore, (4.11) holds in both endpoint cases. It remains to consider $3 \leq k \leq n-2$.

6. Reduction to a cubic polynomial

We now prove (4.11) for $3 \leq k \leq n-2$. We first reduce the inequality to the positivity of a cubic polynomial. By (4.2), the left-hand side of (4.11) can be written as

$$\frac{2}{m} - \frac{\sigma_{k-1}^2}{\sigma_k^2}\left(\frac{\sigma_{k-1;1j}^2 - \sigma_{k;1j}\sigma_{k-2;1j}}{\sigma_{k-2;1j}^2 - \sigma_{k-1;1j}\sigma_{k-3;1j}} - \frac{m\sigma_{k-1;1}^2 - (m+1)\sigma_{k;1}\sigma_{k-2;1}}{(m+1)\sigma_{k-2;1}^2 - (m+2)\sigma_{k-1;1}\sigma_{k-3;1}}\right). \tag{6.1}$$

Using (5.1) and (5.2), we write (6.1) in terms of x and e_l. After taking a common denominator and canceling common factors, its numerator is a cubic polynomial in x:

$$N(x) = A_3x^3 + A_2x^2 + A_1x + A_0. \tag{6.2}$$

Set

$$\mathcal{P} = \frac{e_{k-3}e_{k-1}}{e_{k-2}^2}, \qquad \mathcal{Q} = \frac{e_{k-2}e_k}{e_{k-1}^2}, \qquad \mathcal{R} = \frac{e_{k-2}e_{k-4}}{e_{k-3}^2}.$$

Then the coefficients in (6.2) are given by

$$\begin{aligned}
A_0 &= \frac{\mathcal{P}e_{k-2}^6}{e_{k-3}}\Big(2(m+2)\mathcal{P}^2 - (m\mathcal{Q}+2m+6)\mathcal{P} + (m+2)\Big),\\
A_1 &= e_{k-2}^5\Big(2(m+2)\mathcal{R}\mathcal{P}^3 + \big(m^2(\mathcal{Q}+\mathcal{R}-\mathcal{Q}\mathcal{R}) - m(2\mathcal{Q}\mathcal{R}+\mathcal{Q}-4) + 4(1-\mathcal{R})\big)\mathcal{P}^2\\
&\qquad - (2m^2+5m+6)\mathcal{P} + (m^2+2m+2)\Big),\\
A_2 &= e_{k-3}e_{k-2}^4\Big(\big(2m^2(\mathcal{Q}+\mathcal{R}-\mathcal{Q}\mathcal{R}) + m(\mathcal{Q}+8\mathcal{R}-4\mathcal{Q}\mathcal{R}) + 8\mathcal{R}-2\big)\mathcal{P}^2\\
&\qquad - \big(4m^2+3m-2+4m\mathcal{R}+8\mathcal{R}\big)\mathcal{P} + 2m(m+1)\Big),\\
A_3 &= e_{k-3}^2e_{k-2}^3\Big(\big(m^2(\mathcal{Q}+\mathcal{R}-\mathcal{Q}\mathcal{R}) + m(\mathcal{Q}+2\mathcal{R}-2\mathcal{Q}\mathcal{R})\big)\mathcal{P}^2\\
&\qquad + \big(2m\mathcal{R}+4\mathcal{R}-2m^2-3m-2\big)\mathcal{P} + \big(m^2+2m+2-2m\mathcal{R}-4\mathcal{R}\big)\Big).
\end{aligned}$$

We next rewrite these coefficients in terms of the variables P, Q, R introduced in Section 3. Let E_j denote the normalized elementary symmetric polynomial. We use the following two standard identities. If $X = (x_1, \dots, x_n)$, then:

- By Rolle's theorem, there exists $Y = (y_1, \dots, y_{n-1})$ such that

$$E_j(X) = E_j(Y), \qquad 0 \le j \le |Y|. \tag{6.3}$$

- If $X^{-1} = \{1/a : a \in X\}$, then

$$E_j(X^{-1}) = \frac{E_{n-j}(X)}{E_n(X)}, \qquad 0 \le j \le n. \tag{6.4}$$

Applying (6.3) repeatedly, we reduce from $n-2$ to k variables while preserving the normalized elementary symmetric polynomials E_j, $0 \le j \le k$. We then pass to reciprocal variables using (6.4), which converts the high-degree quantities E_{k-j} into the low-degree quantities E_j.

Let

$$\lambda' = (\lambda'_1, \dots, \lambda'_{n-2}).$$

Then there exists $\mu = (\mu_1, \dots, \mu_k)$ such that

$$\begin{aligned}
\mathcal{P} &= \frac{e_{k-3}e_{k-1}}{e_{k-2}^2} = \frac{m}{m+1}\frac{k-2}{k-1}\frac{E_{k-3}(\lambda')E_{k-1}(\lambda')}{E_{k-2}^2(\lambda')} = \frac{m}{m+1}\frac{k-2}{k-1}\frac{E_{k-3}(\mu)E_{k-1}(\mu)}{E_{k-2}^2(\mu)}\\
&= \frac{m}{m+1}\frac{k-2}{k-1}\frac{E_3(\mu^{-1})E_1(\mu^{-1})}{E_2^2(\mu^{-1})} = \frac{3m}{2(m+1)}\frac{e_1e_3}{e_2^2}.
\end{aligned}$$

Similarly,

$$\mathcal{Q} = \frac{2(m-1)}{m}\frac{e_2}{e_1^2}, \qquad \mathcal{R} = \frac{4(m+1)}{3(m+2)}\frac{e_2e_4}{e_3^2}.$$

We still denote the elementary symmetric polynomials of μ^{-1} by e_j. Recall that

$$P = \frac{e_1e_3}{e_2^2}, \qquad Q = \frac{e_2}{e_1^2}, \qquad R = \frac{e_2e_4}{e_3^2}.$$

Therefore,

$$\mathcal{P} = \frac{3m}{2(m+1)}P, \qquad \mathcal{Q} = \frac{2(m-1)}{m}Q, \qquad \mathcal{R} = \frac{4(m+1)}{3(m+2)}R. \tag{6.5}$$

Thus the coefficients A_i can be expressed entirely in terms of P, Q, R and m.

Substituting these expressions into A_i, we obtain four polynomials in e_1, e_2, e_3, e_4. For example, up to a positive factor,

$$A_0 = 9m^2(m+2)e_1^3e_3^2 - 6m(m+1)(m+3)e_1^2e_2^2e_3 - 6m(m-1)(m+1)e_2^3e_3 + 2(m+1)^2(m+2)e_1e_2^4.$$

The Newton–Maclaurin inequalities alone do not seem sufficient to prove the positivity of A_0. The dominant term, $e_1e_2^4$, has a coefficient that is too small. Another positive term, $e_1^3e_3^2$, is dominated by the negative term $e_1^2e_2^2e_3$, while its comparison with $e_2^3e_3$ is unclear—it can be either larger or smaller. This suggests that the elementary symmetric polynomials may not be a suitable basis.

We first give an independent computational verification of the positivity of A_i. A complete direct proof using the symmetric polynomial inequalities established earlier is given in Section 8.

7. A computational verification

We give an independent computational verification of the positivity of A_i. Using SageMath [13], we expand A_i in the monomial basis. All coefficients in the expansions of A_2 and A_3 are positive. For A_0 and A_1, only a few coefficients are negative, and they can be controlled by positive terms using Muirhead's inequality. For example, the only negative term in the expansion of A_1 is $m_{2^21^6}$, which is controlled by m_{31^7} since

$$7!m_{31^7} \geq 2!6!m_{2^21^6}.$$

This verifies the positivity of A_i. A complete direct proof is given in the next section.

8. A direct proof of positivity

We now give a direct proof of the positivity of the coefficients in (6.2). Instead of working in the elementary symmetric basis, we use the variables P, Q, R and the symmetric polynomial inequalities established in Section 3.

8.1. **Main idea.** The main observation is the *multi-affine* structure. For fixed $\mathcal{P}$, each coefficient A_i in (6.2) is multi-affine in $\mathcal{Q}$ and $\mathcal{R}$. Therefore, if $(\mathcal{Q}, \mathcal{R})$ is restricted to a suitable rectangle, the minimum of A_i is attained at one of its vertices.

The case of A_0 is direct. For A_1, A_2, A_3, finding a suitable rectangle is more difficult. We therefore write each A_i as a polynomial in m and use the symmetric polynomial inequalities from Section 3 to obtain suitable bounds for each coefficient.

Since only the sign of A_i matters, from now on we allow A_i to differ from its previous definition by a positive factor.

8.2. **The coefficient A_0.** In terms of P, Q, R, and up to a positive factor, A_0 can be written as

$$A_0 = e_1^9Q^4\, F_0(P, Q, R; m),$$

where

$$F_0 = 9m^2(m+2)P^2 - 6m(m+1)(m+3)P - 6m(m-1)(m+1)PQ + 2(m+1)^2(m+2).$$

Since F_0 is decreasing in Q, Lemma 3.2(1) shows that its minimum occurs at $Q = 1/(4-3P)$. Substituting this value, we obtain

$$F_0 = \frac{(2m+2-3Pm)}{4-3P}\left(9m\,(m+2)\,P^2 - 6\left(2m^2+5m+1\right)P + 4\,(m+1)\,(m+2)\right).$$

The discriminant of the quadratic polynomial in the second bracket is

$$-36\left(3m^2+6m-1\right).$$

Therefore

$$F_0 > C(n,k) > 0 \qquad \text{for} \qquad 0 \le P \le \frac{2(k-2)}{3(k-1)}.$$

Since F_0 is a polynomial and the admissible set is contained in a compact subset of $[0,\frac{2}{3}]\times[0,\frac{1}{2}]\times[0,\frac{3}{4}]$, this strict positivity is preserved under a sufficiently small perturbation of the coefficient $\frac{2}{m}$. Hence, for some $\varepsilon_0=\varepsilon_0(n,k)>0$, F_0 remains positive when $\frac{2}{m}$ is replaced by $\frac{2-\varepsilon_0}{m}$.

8.3. **The coefficients** A_1, A_2, A_3**.** We write the three coefficients in the following form.

$$A_1 = e_1^{10}Q^5\, F_1(P,Q,R;m),\;\; A_2 = e_1^{11}Q^6P\, F_2(P,Q,R;m),\; A_3 = e_1^{12}Q^7P^2\, F_3(P,Q,R;m),$$

where

$$\begin{aligned} F_1 &= \alpha m^4+\beta_1 m^3+\gamma_1 m^2+(12-18P)\,m+4.\\ F_2 &= 2\alpha m^3+\beta_2 m^2+\gamma_2 m-16PR+6P+4,\\ F_3 &= 3\alpha m^3+\beta_3 m^2+\gamma_3 m-16R+12. \end{aligned}$$

By Lemma 3.2(2,6), the sum of all terms in F_i other than α, β_i, and γ_i has a positive lower bound. Thus, it is enough to prove that α, β_i, and γ_i are nonnegative.

8.3.1. *The term* α*.*

$$\alpha = -12P^2QR+9P^2Q+6P^2R-6P+2,$$

which is nonnegative by Lemma 3.2(5).

8.3.2. *The terms* β_i*.* We will show that

$$\begin{aligned} \beta_1 &= 18P^3R-18P^2Q-6P^2R+18P^2-21P+8\ge 0,\\ \beta_2 &= -9P^2Q+36P^2R-16PR-21P+12\ge 0,\\ \beta_3 &= 36P^2QR-27P^2Q+24PR-27P-16R+18\ge 0. \end{aligned}$$

We first establish a lower bound for R. Since $\alpha\ge 0$, by Lemma 3.2(6),

$$\partial_Q\alpha = 3P^2(3-4R) > 0.$$

By Lemma 3.2(1), setting $Q=1/(4-3P)$ in α gives

$$\alpha\left(P,\frac{1}{4-3P},R\right) = \frac{2-3P}{4-3P}\left(6P^2R-9P+4\right)\ge 0.$$

Therefore,

$$R \ge \frac{9P-4}{6P^2}.$$

Combining this with Lemma 3.2(2), we obtain

$$\frac{1}{2-P}\ge R\ge \frac{9P-4}{6P^2}.$$

Next we use the above bounds to place the admissible region inside a rectangle. Since the functions are multi-affine in Q and R, their minimum on the rectangle is attained at a vertex. We consider the three terms separately.

For β_1. Observe that

$$\partial_Q\beta_1 = -18P^2 < 0, \qquad \partial_R\beta_1 = 6P^2(3P-1).$$

The sign of $\partial_R\beta_1$ changes at $P = 1/3$.

	Minimum vertex (Q,R)	Minimum value of β_1
$0 < P \le \frac{1}{3}$	$\left(\frac{1}{4-3P}, \frac{1}{2-P}\right)$	$\frac{(2-3P)(4-5P)(8-9P)}{(2-P)(4-3P)}$
$\frac{1}{3} < P \le \frac{2\,(k-2)}{3(k-1)}$	$\left(\frac{1}{4-3P}, \frac{9P-4}{6P^2}\right)$	$\frac{3(2-3P)^2(4-5P)}{4-3P}$

The minimum of β_1 on each interval is positive by direct calculus.

For β_2.

$$\partial_Q\beta_2 = -9P^2 < 0, \qquad \partial_R\beta_2 = 4P(9P-4).$$

$\partial_R\beta_2$ changes sign at $P = 4/9$.

	Minimum vertex (Q,R)	Minimum value of β_2
$0 < P \le \frac{4}{9}$	$\left(\frac{1}{4-3P}, \frac{1}{2-P}\right)$	$\frac{2(2-3P)(27P^2-52P+24)}{(2-P)(4-3P)}$
$\frac{4}{9} < P \le \frac{2\,(k-2)}{3(k-1)}$	$\left(\frac{1}{4-3P}, \frac{9P-4}{6P^2}\right)$	$\frac{4(3P-2)^2(9P-8)}{3P(3P-4)}$

The minimum of β_2 on each interval is positive by direct calculus.

For β_3.

$$\partial_Q\beta_3 = 9P^2(4R-3) < 0, \qquad \partial_R\beta_3|_{Q=\frac{1}{4-3P}} = \frac{4(9P^2-36P+16)}{3P-4}$$

$\partial_R\beta_3$ changes sign at $P_0 = 2 - \frac{2}{3}\sqrt{5}$.

	Minimum vertex (Q,R)	Minimum value of β_3
$0 < P \le P_0$	$\left(\frac{1}{4-3P}, \frac{1}{2-P}\right)$	$\frac{2(2-3P)(9P^2-33P+20)}{(2-P)(4-3P)}$
$P_0 < P \le \frac{2\,(k-2)}{3(k-1)}$	$\left(\frac{1}{4-3P}, \frac{9P-4}{6P^2}\right)$	$\frac{2(4-3P)(2-3P)^2}{3P^2}$

The minimum of β_3 on each interval is positive by direct calculus.

8.3.3. *The terms* γ_i. We will show that

$$\begin{aligned}
\gamma_1 &= 12P^2QR + 9P^2Q - 12P^2R + 18P^2 - 33P + 14 \ge 0,\\
\gamma_2 &= 24P^2QR - 9P^2Q + 24P^2R - 9P^2 - 32PR - 3P + 12 \ge 0,\\
\gamma_3 &= 24PR - 18P - 32R + 24 \ge 0.
\end{aligned}$$

We first establish a lower bound for Q. From Lemma 3.2(2,3),

$$0 \le 1 - 2P + 2P^2QR \le 1 - 2P + \frac{2P^2Q}{2-P},$$

we obtain

$$Q \geq Q_0 := \frac{(2P-1)(2-P)}{2P^2}.$$

Combining this with Lemma 3.2(1), we obtain

$$\frac{1}{4-3P} \geq Q \geq \max\{Q_0, 0\}.$$

We first consider γ_1.

$$\partial_R \gamma_1 = 12P^2(Q-1) < 0, \qquad \partial_Q \gamma_1|_{R=\frac{1}{2-P}} = \frac{3P^2(10-3P)}{2-P} > 0$$

Therefore,

$$\gamma_1 \geq \begin{cases} \gamma_1\left(P, 0, \frac{1}{2-P}\right) = \frac{(1-2P)(9P^2-24P+28)}{2-P}, & 0 < P \leq \frac{1}{2}, \\ \gamma_1\left(P, Q_0, \frac{1}{2-P}\right) = \frac{(2P-1)(2-3P)(3P+2)}{4-2P}, & \frac{1}{2} < P < \frac{2}{3}. \end{cases}$$

Hence $\gamma_1 \geq 0$.

Next, consider γ_2. By Lemma 3.2(6),

$$\begin{cases} \partial_R \gamma_2 = 24P^2(Q+1) - 32P < 36P^2 - 32P = 4P(9P-8) < 0, \\ \partial_Q \gamma_2|_{R=\frac{1}{2-P}} = \frac{3P^2(3P+2)}{2-P} > 0 \end{cases}$$

Therefore,

$$\gamma_2 \geq \gamma_2\left(P, Q_0, \frac{1}{2-P}\right) = \frac{(2-3P)(18-17P)}{2(2-P)}, \qquad 0 < P < \frac{2}{3}$$

Hence $\gamma_2 > 0$.

Finally, consider γ_3. Since

$$\partial_R \gamma_3 = 24P - 32 < 0,$$

we have

$$\gamma_3 \geq \gamma_3\left(P, \frac{1}{2-P}\right) = \frac{2(4-3P)(2-3P)}{2-P} > 0.$$

We have proved that α, β_i, and γ_i are all nonnegative. Since the remaining terms in F_i have a positive lower bound depending only on n and k, we obtain

$$F_i > C(n,k) > 0, \qquad 1 \leq i \leq 3.$$

Since each F_i is a polynomial, the same compactness argument as above shows that this positivity is preserved under a sufficiently small perturbation of $\frac{2}{m}$. After decreasing $\varepsilon_0(n,k) > 0$ if necessary, we may therefore replace $\frac{2}{m}$ by $\frac{2-\varepsilon_0}{m}$. This completes the direct proof.

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