

State convertibility and fluctuation theorems from a dynamical reference: majorization meets martingales

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State convertibility represents a fundamental concept used to determine whether a transformation is possible given a specific set of resources. Within the field of Thermodynamics, where physical process are required to preserve a reference state typically in microcanonical or canonical form, this translates into the notions of majorization and thermo-majorization—criteria that require constructing and comparing state-dependent Lorenz curves. In this work, we firstly unify and extend these notions to an arbitrary and possibly time-dependent reference distribution $g(t)$, introducing the concept of $g(t)$ -majorization; we then introduce a dual picture whereby state convertibility is turned into a one-dimensional convex-order problem, which allows us to demonstrate that a transition is admissible if and only if the associated real-valued distributions of relative populations $k_j(t)/g_j(t)$ are connected by a martingale. Building on it, we then derive an exact fluctuation theorem for a reference-relative entropy production whose average violation certifies, through a χ^2 -divergence bound, the mismatch between an assumed and the true reference evolution—a model-independent diagnostic that requires no independent characterization of the latter and turns an observed breakdown of the fluctuation relation into a certified lower bound on the reference error.

Introduction.—The cornerstone of the entire statistical mechanics, the probabilistic and microscopic framework of Thermodynamics, was laid by Ludwig Boltzmann in 1877 [1]. In this pioneering paper, he showed that for N particles at fixed total energy, a given energy-distribution profile $k = (k_1, \dots, k_m)$, with $k_j = n_j/N$, is compatible with $W(k) = N! / \prod_j (Nk_j)!$ microscopic configurations. If the particles are then drawn from a uniform probability distribution, Boltzmann showed that the occurrence of any energy-distribution is observed with probability proportional to $W(k)$ and introduced the notion of statistical entropy as the logarithm of this count. By then identifying equilibrium as simply the most probable configuration, Boltzmann turned Thermodynamics into a counting problem and gave the Second Law its statistical meaning. In terms of the occupation fractions $k_j = n_j/N$, Stirling’s formula renders the counting entropy as $\frac{1}{N} \ln W \rightarrow -\sum_j k_j \ln k_j$, the form later central to Gibbs and, in information theory, to Shannon [2]. The object at the center of the argument is the *population vector* $k = (k_1, \dots, k_m)$, a normalized distribution over microstates.

Despite its fundamental importance, Boltzmann’s original construction was still subject to limitation: it singled out the unique most probable macrostate relative to a fixed, uniform count, and it quantified only the average of the underlying fluctuating quantity, not its distribution, nor the transformations between macrostates that a physical process may actually perform. The century that followed resolved these limitations, but in separate directions, each generalizing the population vector in its own way and each leaving a characteristic blind spot. Large deviation theory recognized Boltzmann’s counting as the first instance of a large-deviation principle [3, 4] and,

through the work of Kullback and Leibler in statistics [5] and Sanov in probability [6], generalized the entropy to the relative entropy $D_{\text{KL}}(k||g) = \sum_j k_j \ln(k_j/g_j)$ against an arbitrary reference g , identifying it as the exponential rate at which the empirical distribution of samples drawn from g deviates to k . Information theory, in parallel, recognized that this same relative entropy coincides with the negative of Boltzmann’s entropy and with Shannon’s information entropy [2, 7], and, following Jaynes [8], recast equilibrium itself as inference against a reference—supplying the bridge between thermodynamics and information, but not the dynamical question of which state transformations are physically allowed. That question is the leitmotif of the resource-theoretical approach to Thermodynamics, which established that convertibility is governed by majorization when the allowed operations preserve a uniform distribution, and by thermo-majorization when they preserve a Gibbs state [9–12]; yet these criteria are tied to a *fixed*, equilibrium reference and are decided by constructing and comparing state-dependent Lorenz curves, an operationally demanding procedure. A fourth thread, martingale theory, meanwhile turned the Second Law into a family of exact fluctuation relations for the entropy produced along individual trajectories [13–15] and revealed that the exponential of the entropy production is a martingale [16–18]—but, again, against the equilibrium or steady-state reference of standard stochastic thermodynamics.

Each of these developments takes a reference distribution as the object against which irreversibility is measured, exactly as Boltzmann took his count; and each treats a different facet—the rate of a fluctuation, its informational meaning, the convertibility it permits, its trajectory statistics—in a formalism of its own. What

has been missing is the common structure that would let a single, arbitrary, and possibly time-dependent reference $g(t)$ carry all four at once.

Our work precisely provides that. Associating to each population vector $k(t)$ the real-valued distribution of its relative populations $\kappa_j = k_j(t)/g_j(t)$ —the *parent distribution*—collapses the separate questions into one. State convertibility, classically decided by comparing Lorenz curves, becomes a one-dimensional convex-order comparison of parent distributions, equivalent to the existence of a martingale connecting them. The point-wise quantity $-\ln \kappa_j$ —the reference-relative surprisal of information theory—defines a *reference-relative entropy production* Δs whose mean is Boltzmann’s relative entropy and which obeys an exact integral fluctuation theorem, placing the fluctuation relations of stochastic thermodynamics within martingale theory for an arbitrary dynamical reference. In addition, we also show that when the assumed reference departs from the true one, the average violation of that theorem certifies, through a χ^2 divergence, a model-independent lower bound on the mismatch; we illustrate this last result on a driven two-level system, where an apparent breakdown of the second law becomes a quantitative measure of nonadiabatic driving. Large deviation theory, information theory, resource theories, and stochastic fluctuation relations thus meet on the single terrain that Boltzmann first surveyed: the distribution of a system’s populations relative to a reference.

State convertibility.—Consider a generic set of microstates $\{\mathcal{S}_j\}_{j=1}^m$ (not necessarily associated with particles’ energies) and let $g(t) = (g_1(t), \dots, g_m(t))$ be a reference distribution of their populations whose time-evolution is known a priori. Moreover, let $k(t)$ be another distribution absolutely continuous with respect to $g(t)$, i.e. $g_j = 0 \Rightarrow k_j = 0$ (automatic for $g_j > 0$). We ask whether, among the stochastic processes compatible with the evolution of g , there exists one carrying $k(t_1)$ into a *target* $k(t_2)$: that is, whether there is a column-stochastic *transition matrix* T such that

$$T g(t_1) = g(t_2) \quad \wedge \quad T k(t_1) = k(t_2). \quad (1)$$

The first condition fixes T to be a legitimate evolution for the reference itself; the second demands that the same T reproduce the desired transition. Convertibility is thus a statement about compatibility with the reference dynamics, not about $k(t_1) \rightarrow k(t_2)$ in isolation.

The central object is the vector of *relative populations*

$$\kappa_j(t) := \frac{k_j(t)}{g_j(t)}, \quad (2)$$

which measures, microstate by microstate, how far k deviates from the reference: $\kappa_j = 1$ signals a microstate populated exactly as g predicts, $\kappa_j \geq 1$ over- or under-population. A first characterization of admissibility can

be formulated directly in terms of the parent distribution, through a Lorenz-type integral inequality that unifies and generalizes majorization and thermo-majorization to an arbitrary dynamical reference, which we therefore term *$g(t)$ -majorization*. While important for the complete picture, it is not needed for the main results that follow and therefore deferred to the End Matter [see Eq. (16)].

Here we proceed directly to the transparent probabilistic criterion. To each $\kappa(t)$ we associate the real-valued *parent distribution*

$$p_\kappa(x; t) = \sum_j g_j(t) \delta(x - \kappa_j(t)), \quad (3)$$

the law of the relative population obtained by drawing a microstate from the reference ensemble $g(t)$: it tracks the probability, under g , of drawing a given amount of deviation from the reference. Because g and k are each normalized, p_κ has unit mass and unit first moment, $\mathbb{E}_{x \sim p_\kappa}[1] = \sum_j g_j = 1$ and $\mathbb{E}_{x \sim p_\kappa}[x] = \sum_j g_j \kappa_j = \sum_j k_j = 1$; it is thus a genuine probability distribution, but one over relative populations rather than over microscopic configurations.

The equivalence rests on a coupling. A martingale coupling of the two parent distributions is a joint law $P(n, j) := P(x_1 = \kappa_n(t_1), x_2 = \kappa_j(t_1))$, with $x_1 \sim p_\kappa(\cdot; t_1)$ and $x_2 \sim p_\kappa(\cdot; t_2)$, whose marginals are the reference weights, $\sum_j P(n, j) = g_n(t_1)$ and $\sum_n P(n, j) = g_j(t_2)$, and which satisfies the martingale condition $\mathbb{E}[x_1 | x_2 = \kappa_j(t_2)] = \kappa_j(t_2)$. Writing the conditional expectation explicitly and using Bayes’ rule, the martingale condition reads $\sum_n \kappa_n(t_1) P(n, j) = \kappa_j(t_2) g_j(t_2) = k_j(t_2)$. Defining

$$T_{jn} := \frac{P(n, j)}{g_n(t_1)}, \quad (4)$$

the marginals and the martingale condition translate, respectively, into column-stochasticity of T , reference preservation $T g(t_1) = g(t_2)$, and the physical transition $T k(t_1) = k(t_2)$; the map (4) is a bijection, so the converse holds as well [19]. This establishes our first main result.

Theorem 1 (Convertibility criterion). *A transition $k(t_1) \rightarrow k(t_2)$ compatible with the reference dynamics of $g(t)$ exists if and only if the associated parent distributions are connected by a martingale. Namely*

$$\exists T | T g(t_1) = g(t_2) \wedge T k(t_1) = k(t_2) \quad (5)$$

iff

$$p_\kappa(\cdot; t_1) \xrightarrow{\text{mart.}} p_\kappa(\cdot; t_2).$$

Physically, the transition is possible precisely when the distribution of relative populations *concentrates* in time—when the population profile relaxes toward the

reference g , so that p_κ narrows; transitions that would require it to spread are forbidden. By Strassen's theorem [20], the existence of a martingale coupling is equivalent to the convex order of the two parent distributions, $p_\kappa(\cdot; t_1) \succeq^{\text{cx}} p_\kappa(\cdot; t_2)$, that is,

$$\mathbb{E}_{x \sim p_\kappa(\cdot; t_1)}[\phi(x)] \geq \mathbb{E}_{x \sim p_\kappa(\cdot; t_2)}[\phi(x)] \quad (6)$$

for every convex ϕ for which the expectations are defined. Since all parent distributions share unit mass and unit mean, equality holds automatically for affine ϕ , as the definition of convex order requires; the content of (6) is therefore the ordering on the genuinely convex part. Convex order thus plays, in the space of parent distributions, the role majorization plays in the space of population vectors—an equivalence classically known for a static, uniform reference [21, 22] and extended here, through the parent-distribution construction, to an arbitrary dynamical reference $g(t)$. The m -dimensional, sorting-based comparison of population vectors has become a one-dimensional convex-order comparison of real-valued distributions, decidable by the existence of a martingale. This distributional viewpoint is closely related to the correspondence between ranked samples and their associated order-statistics distributions developed in [23], where ranking information is likewise recast in terms of the statistics of an underlying real-valued distribution. That convertibility should be judged relative to a reference distribution is the organizing principle of the resource-theoretic view of thermodynamics [24, 25] and, classically, of d -majorization and relative majorization and their roots in the comparison of statistical experiments [26–29]; the present construction realizes that principle for a reference that is itself dynamical. The chain of equivalences (1) $\Leftrightarrow$ (16) $\Leftrightarrow$ martingale $\Leftrightarrow$ (6) is summarized in Fig. 1; the sorting-based construction of Γ_κ has been replaced by a single one-dimensional convex-order comparison, checkable through a martingale coupling.

Fluctuation relation.—Beyond convertibility, the same reference-state construction yields a natural fluctuation relation. For each microstate S_j define the reference-relative surprisal

$$s_j(t) := -\ln \kappa_j(t) = -\ln \frac{k_j(t)}{g_j(t)}, \quad (7)$$

so that $e^{-s_j(t)} = \kappa_j(t)$. This is the difference between the Shannon information content $-\ln k_j$ of microstate j and that assigned to it by the reference, $-\ln g_j$ [2, 7]; sampled according to the population vector, its mean is precisely the relative entropy, $\sum_j k_j s_j = D_{\text{KL}}(k \| g)$, recovering Boltzmann's and Sanov's entropy. For a process T which now is assumed to evolve the population vector $k(t_1)$ into the population vector $k(t_2) = T k(t_1)$, we introduce the *reference-relative entropy production* as

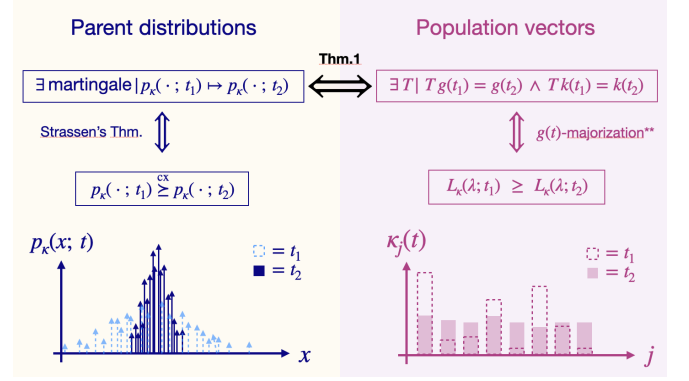


FIG. 1. Schematic of the framework equivalence demonstrated in this paper. A population vector $k(t)$, compared against a reference $g(t)$, is mapped to the parent distribution $p_\kappa(\cdot; t)$ of its relative populations $\kappa_j(t) = k_j(t)/g_j(t)$. State convertibility (1) and the $g(t)$ -majorization criterion (**see End Matter (16)) on the right side (purple background) are shown to be fully equivalent to each other and to the existence of a martingale coupling between the respective parent distributions (Theorem 1) and finally, by virtue of Strassen's Theorem, to convex order relation (6) (left hand side).

$\Delta s := s_{j_2}(t_2) - s_{j_1}(t_1)$, and its average is taken over the physical joint law of (j_1, j_2) ,

$$\Pi(j_1, j_2) = k_{j_1}(t_1) T_{j_2 j_1}, \quad (8)$$

the initial population followed by the forward transition. This is the law of where the system actually is at the two times, and it must be distinguished from the reference-weighted coupling of the previous section. This yields our second main result.

Theorem 2 (Reference-generalized fluctuation theorem). *For any transition matrix T compatible with the reference dynamics, $Tg(t_1) = g(t_2)$, the reference-relative entropy production satisfies the integral fluctuation theorem*

$$\mathbb{E}_\Pi[e^{-\Delta s}] = 1, \quad \text{whence} \quad \mathbb{E}_\Pi[\Delta s] \geq 0. \quad (9)$$

The proof is immediate: using $k_{j_1}(t_1) = g_{j_1}(t_1) \kappa_{j_1}(t_1)$ and then reference preservation $Tg(t_1) = g(t_2)$,

$$\begin{aligned} \mathbb{E}_\Pi[e^{-\Delta s}] &= \sum_{j_1, j_2} k_{j_1}(t_1) T_{j_2 j_1} \frac{\kappa_{j_2}(t_2)}{\kappa_{j_1}(t_1)} \\ &= \sum_{j_2} \kappa_{j_2}(t_2) g_{j_2}(t_2) = \sum_{j_2} k_{j_2}(t_2) = 1, \end{aligned} \quad (10)$$

and $\mathbb{E}_\Pi[\Delta s] \geq 0$ follows from Jensen's inequality applied to the convex function e^{-x} .

Equation (9) is the martingale-driven generalization, to an arbitrary and possibly time-dependent reference $g(t)$, of the integral fluctuation theorems of stochastic thermodynamics, ordinarily stated relative to a Gibbs reference [13, 14], and it is consistent with the

exponential-martingale structure of entropy production [16–18, 30]. The second-law inequality $\mathbb{E}_\Pi[\Delta s] \geq 0$ it entails is a statement about irreversibility measured purely with respect to the chosen reference: no admissible process decreases the reference-relative entropy on average.

Certifying reference mismatch.—Equation (9) presumes the evolution of $g(t)$ is known exactly. In practice it often is not. Consider a system driven quasi-adiabatically by a time-dependent Hamiltonian $H(t)$ in contact with a bath at inverse temperature β ; initialized in the Gibbs state $g(t_1) \propto e^{-\beta H(t_1)}$, its state at t_2 is only approximated by the instantaneous Gibbs state $g(t_2) \propto e^{-\beta H(t_2)}$, the approximation becoming exact only in the adiabatic limit. More generally the assumed reference $g(t_2)$ may differ from the true evolution $g^{\text{true}}(t_2)$ of $g(t_1)$, and one wishes to quantify the discrepancy.

Suppose, then, that the physical process is compatible with the true reference, $Tg(t_1) = g^{\text{true}}(t_2)$, while the entropy production is evaluated with the assumed reference $g(t_2)$. Repeating the derivation of (9) with this single change, the inner sum collapses to $g^{\text{true}}(t_2)$ rather than $g(t_2)$, and, subtracting unity,

$$\mathbb{E}_\Pi[e^{-\Delta s}] - 1 = \sum_j k_j(t_2) \frac{g_j^{\text{true}}(t_2) - g_j(t_2)}{g_j(t_2)}. \quad (11)$$

Writing the right-hand side as an inner product of the vectors $(g_j^{\text{true}} - g_j)/\sqrt{g_j}$ and $k_j(t_2)/\sqrt{g_j}$ and applying the Cauchy–Schwarz inequality yields our third main result.

Theorem 3 (Reference-mismatch bound). *The χ^2 divergence between the true and the assumed reference at t_2 is lower-bounded by the observed violation of the fluctuation theorem,*

$$D_{\chi^2}(g^{\text{true}}(t_2) \parallel g(t_2)) \geq \frac{(\mathbb{E}_\Pi[e^{-\Delta s}] - 1)^2}{\sum_j k_j^2(t_2)/g_j(t_2)}, \quad (12)$$

with $D_{\chi^2}(x \parallel y) := \sum_j (x_j - y_j)^2/y_j$, and requires no independent knowledge of the true reference g^{true} .

This result runs opposite to the usual role of the χ^2 divergence in Monte Carlo and importance sampling, where a *known* $\chi^2(x \parallel y)$ upper-bounds the degradation of an estimator built from a mismatched proposal [31]. Equation (12) instead turns an *observed* violation of (9) into a certified *lower* bound on an otherwise inaccessible divergence, with no independent knowledge of g^{true} .

The bound bypasses entirely the need to physically prepare the reference state, an operational advantage on platforms such as driven mesoscopic devices, quantum processors, or complex biophysical systems, where the target reference ensemble can be inaccurate [32], costly to prepare [33], or simply inaccessible [34, 35].

Instead, it enables the mismatch between the assumed reference and its true dynamical evolution to be quantified from measurements performed on arbitrary, experimentally accessible states $k(t)$. The probe state is moreover a tunable diagnostic. Its denominator obeys the identity $\sum_j k_j^2(t_2)/g_j(t_2) = 1 + D_{\chi^2}(k(t_2) \parallel g(t_2))$, so that (12) reads $D_{\chi^2}(g^{\text{true}} \parallel g) \geq (\mathbb{E}_\Pi[e^{-\Delta s}] - 1)^2/[1 + D_{\chi^2}(k(t_2) \parallel g(t_2))]$. The r.h.s. of (11) is a $k(t_2)$ -weighted average of the local reference mismatch $(g_j^{\text{true}} - g_j)/g_j$: spreading the probe uniformly dilutes a localized discrepancy against microstates where the reference is accurate, whereas concentrating $k(t_2)$, through the choice of the initial probe and the assumed dynamics, on microstates suspected of control errors or unmodeled noise sharpens the numerator. Because concentrating the probe modifies the denominator, the tightest bound optimizes the ratio rather than the numerator alone—a genuine trade-off that a judiciously engineered probe resolves, exposing localized discrepancies that a uniformly spread probe would average away.

Finally, although D_{χ^2} is not a distance, it induces one in the small-deviation limit. Writing $g^{\text{true}} - g =: \epsilon v$ with $\sum_j v_j = 0$ and ϵ small, the χ^2 divergence reduces to $D_{\chi^2}(g^{\text{true}} \parallel g) = \epsilon^2 D_{\text{RF}}(v, v)$, with $D_{\text{RF}}(v, v) := \sum_j v_j^2/g_j$ the Rao–Fisher metric evaluated at g [36, 37]; for the χ^2 divergence this quadratic form is exact along the ray $g + \epsilon v$. In this limit (12) certifies a lower bound on the Rao–Fisher distance between the assumed and true reference trajectories, making explicit a feature that runs through the whole construction: the reference g is an active geometric object, setting the metric with which nearby departures from the assumed dynamics are measured [38].

Example: a driven two-level system.—The simplest setting in which a genuinely time-dependent reference arises is a classical two-level system—the dissipative counterpart of Boltzmann’s two-level unit—with energy gap $\epsilon(t) \equiv E_+(t) - E_-(t)$ swept in time and in contact with a thermal bath. Its populations obey a Pauli master equation (a classical Markov-jump, or Liouville, generator) whose transition rates satisfy detailed balance at inverse temperature β , so that at each instant the unique stationary state is the instantaneous Gibbs distribution $\mathcal{G}(t) := (1 - \mathcal{G}_+(t), \mathcal{G}_+(t))$, with $\mathcal{G}_+(t) = [1 + e^{\beta E_+(t)}]^{-1}$. In this model, the excited-state population of $k(t)$ relaxes according to $\dot{k}_+(t) = -\Gamma[k_+(t) - \mathcal{G}_+(t)]$, and a large relaxation rate Γ relative to the sweep speed makes the state track $\mathcal{G}(t)$ adiabatically. Preparing the system at t_1 in the stationary state and reading it out at t_2 , we form the reference-relative entropy production and average it over the physical joint law (8).

Two choices of reference make the framework’s content explicit (Fig. 2). Taking $g(t_2)$ to be the *true* reference $g^{\text{true}}(t_2)$ obtained by propagating $g(t_1) = \mathcal{G}(t_1)$ under the same channel obtained by solving the above master

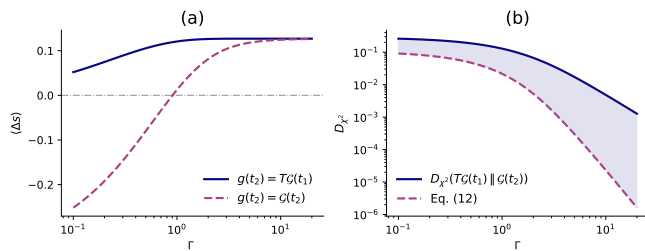


FIG. 2. Driven two-level system, with linearly decreasing energy gap $\epsilon(t) = \epsilon_0 + (\epsilon_1 - \epsilon_0)t$, $\epsilon_0 = 2$, $\epsilon_1 = 1/2$. Furthermore, we choose the temperature $\beta = 1$ and the initial distribution $k(t_1) = (1, 0)$, i.e. the ground state; finally, without loss of generality (one could always rescale the rate Γ to ensure this condition), we simply set the initial time $t_1 = 0$ and the final time $t_2 = 1$. a) Mean reference-relative entropy production $\langle \Delta s \rangle$ versus relaxation rate Γ (adiabaticity increasing to the right), for the true propagated reference $g(t_2) = T\mathcal{G}(t_1)$ (solid) and for the instantaneous Gibbs reference $\mathcal{G}(t_2)$ (dashed). The former obeys the second law exactly; the latter shows an apparent violation, $\langle \Delta s \rangle < 0$, that measures the non-adiabatic lag. (b) The true reference mismatch $D_{\chi^2}(T\mathcal{G}(t_1) \parallel \mathcal{G}(t_2))$ (solid) and the lower bound certified by the fluctuation-theorem violation through Theorem 3 (dashed); the bound holds at every driving speed.

equation, Theorem 2 holds exactly, and with it the second law $\mathbb{E}_{\Pi}[\Delta s] \geq 0$, at every driving speed [Fig. 2(a), solid]. Taking instead the *instantaneous* Gibbs state $\mathcal{G}(t)$ to be reference, i.e. $g(t_2) = \mathcal{G}(t_2)$, the true propagated reference $g_{\text{true}}(t)$ lags behind for any finite-speed driving, Theorem 2 is violated, and $\mathbb{E}_{\Pi}[\Delta s]$ turns *negative* [Fig. 2(a), dashed], an apparent breakdown of the Second Law whose magnitude grows as the drive is made faster and vanishes in the adiabatic limit. This is not a paradox but a diagnostic: by Theorem 3, the measured violation lower-bounds the χ^2 divergence between the true and the assumed reference [Fig. 2(b)], certifying the nonadiabatic lag with no independent characterization of the true dynamics. Dropping the external time dependence returns $\mathcal{G}$ to the fixed Gibbs state and recovers the standard entropy production and its ordinary fluctuation relation as the stationary limit of the construction.

Conclusions.—In this work we have shown that state convertibility relative to an arbitrary, possibly time-dependent reference $g(t)$ is governed by $g(t)$ -majorization, which admits an exact dual description: each population vector maps to the parent distribution of its relative populations, and convertibility reduces to a one-dimensional convex-order condition, checkable through a martingale coupling rather than through the construction and comparison of Lorenz curves (Theorem 1). This unifies majorization and thermo-majorization as the uniform- and Gibbs-reference instances of a single ordering and extends both to references with no equilibrium or symmetry structure. Building on the same construction, the reference-relative sur-

prisal defines an entropy production obeying an exact fluctuation theorem (Theorem 2), whose average violation certifies, through a χ^2 -divergence bound, a lower bound on the mismatch between an assumed and the true reference (Theorem 3)—a diagnostic that needs no independent characterization of the true dynamics and that we demonstrated on a driven two-level system, where it converts an apparent violation of the second law into a measure of nonadiabatic driving.

Two directions follow concretely. First, the fluctuation-theorem violation of Theorem 3 is directly measurable: for a driven two-level system or a Markov-jump network whose stationary state is only approximately known, a single ensemble of forward trajectories fixes $\mathbb{E}_{\Pi}[e^{-\Delta s}]$ and hence a certified lower bound on the reference error, a nonequilibrium alternative to preparing and tomographing the stationary state. Second, because the denominator of Theorem 3 equals $1 + D_{\chi^2}(k(t_2) \parallel g(t_2))$, the probe that maximizes the certified bound is itself determined by the reference, defining an optimization—concentrating population where the reference is least trusted—that is worth characterizing in full.

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END MATTER

The Lorenz-curve criterion.—In this End Matter we want to explicitly show how state convertibility admits an exact characterization directly in terms of the population vector, in the classical language of majorization. Introduce the non-increasing step function

$$\Gamma_\kappa(\lambda; t) := \sum_{j=1}^m \kappa_m^\downarrow(t) \mathbb{1}_{\{\sum_{i=1}^{m-1} g_i^\downarrow(t) \leq \lambda < \sum_{i=1}^m g_i^\downarrow(t)\}}, \quad (13)$$

with $\lambda \in [0, 1]$. Here, the superscript $\downarrow$ indicates that the components are reordered according to the non-increasing ordering of the population vector $\kappa(t)$. More precisely, let π_t be a permutation such that

$$\kappa_{\pi_t(1)}(t) \geq \kappa_{\pi_t(2)}(t) \geq \cdots \geq \kappa_{\pi_t(d)}(t). \quad (14)$$

Then, for any vector v , we define

$$v_m^\downarrow(t) := v_{\pi_t(m)}(t). \quad (15)$$

Thus, both $\kappa^\downarrow(t)$ and $g^\downarrow(t)$ are expressed in the ordering induced by $\kappa(t)$: the components of $\kappa(t)$ are arranged in non-increasing order, and the components of $g(t)$ are rearranged according to the same permutation. Its running integral $L_\kappa(\Lambda; t) := \int_0^\Lambda \Gamma_\kappa(\lambda; t) d\lambda$ is the Lorenz curve of k taken relative to g : concave, non-decreasing, and pinned at $L(0; t) = 0$, $L(1; t) = 1$. A transition matrix

satisfying the admissibility conditions (1) exists if and only if [19]

$$L_\kappa(\Lambda; t_1) \geq L_\kappa(\Lambda; t_2) \quad \forall \Lambda \in [0, 1], \quad (16)$$

that is, the earlier reference Lorenz curve lies everywhere above the later one. This criterion is the exact statement of *majorization relative to the reference g* —the classical notion of d -majorization or relative majorization [26, 27, 29], here made time-dependent. It unifies and generalizes the two familiar cases in a single inequality. When g is uniform (the microcanonical case, microstates of equal energy), the slabs have equal width and (16) reduces at the breakpoints to the standard majorization condition $\sum_{i \leq \ell} \kappa_i^\downarrow(t_1) \geq \sum_{i \leq \ell} \kappa_i^\downarrow(t_2)$. When g is a fixed Gibbs state (the canonical case), it becomes the thermo-majorization criterion of the resource theory of thermodynamics [11, 39–42], in its exact, non-approximated form. For a general and possibly time-dependent $g(t)$, (16) extends both to references with no equilibrium or symmetry structure. The criterion is equivalent to Theorem 1: sorting-based Lorenz dominance, $g(t)$ -majorization, the convex order of parent distributions, and the existence of a martingale coupling are one and the same relation. The equivalence is proved in full detail in the Supplemental Material [19], where the staircase construction of (13) and the reference Lorenz curves are also illustrated. Its practical drawback, and the reason it is not the criterion we adopt in the main text, is that evaluating Γ_κ requires sorting the relative populations at each time.

Supplementary Material

In this Supplementary Material we provide all the detailed proofs of the necessity and sufficiency of the Lorenz Curve criterion, and of Theorems 1 and 3 of the main text.

A — Proof of the Necessity and Sufficiency of the Lorenz-Curve Criterion

We hereby prove the following statement: there exists a stochastic matrix T with

$$Tg(t_1) = g(t_2) \wedge Tk(t_1) = k(t_2) \quad (S1)$$

if and only if

$$L_\kappa(\Lambda; t_1) \geq L_\kappa(\Lambda; t_2) \text{ for all } \Lambda \in [0, 1], \quad (S2)$$

starting from the following two lemmas.

Lemma 1. *Let $P_t \in \mathbb{R}^{m \times m}$ be the permutation matrix with $(P_t v)_j = v_{\pi_t(j)}$ for every vector v . If there exists a stochastic $\tilde{T}$ with*

$$\tilde{T} g^\downarrow(t_1) = g^\downarrow(t_2), \quad \tilde{T} k^\downarrow(t_1) = k^\downarrow(t_2),$$

then $T := P_{t_2}^{-1} \tilde{T} P_{t_1}$ is stochastic and satisfies $Tg(t_1) = g(t_2)$, $Tk(t_1) = k(t_2)$.

Proof. Permutation matrices have exactly one 1 per row and column, hence are stochastic, and $P_{t_2}^{-1} = P_{t_2}^\top$ is again a permutation matrix, hence stochastic. If A, B are stochastic then so is AB :

$$\sum_i (AB)_{ij} = \sum_i \sum_\ell A_{i\ell} B_{\ell j} = \sum_\ell B_{\ell j} \left(\sum_i A_{i\ell} \right) = \sum_\ell B_{\ell j} = 1.$$

Hence $T = P_{t_2}^{-1} \tilde{T} P_{t_1}$ is stochastic. Moreover $P_{t_1} g(t_1) = g^\downarrow(t_1)$ by definition of P_{t_1} , so

$$Tg(t_1) = P_{t_2}^{-1} \tilde{T} P_{t_1} g(t_1) = P_{t_2}^{-1} \tilde{T} g^\downarrow(t_1) = P_{t_2}^{-1} g^\downarrow(t_2) = g(t_2),$$

using $P_{t_2}^{-1} g^\downarrow(t_2) = g(t_2)$ (the inverse rearrangement). The identity $Tk(t_1) = k(t_2)$ follows identically. $\square$

By Lemma 1 it suffices to prove the theorem under the standing assumption that $g(t_1), k(t_1), g(t_2), k(t_2)$ are *already* arranged in the order induced by their own κ , i.e. $\kappa_1(t) \geq \dots \geq \kappa_m(t)$ for $t = t_1, t_2$; we adopt this assumption for the remainder of the proof and drop the $\downarrow$ superscripts.

Lemma 2. *Condition (S2) holds for all $\Lambda \in [0, 1]$ if and only if it holds at every point of the common refinement*

$$\left\{ \sum_{j=1}^r g_j(t_1) \right\}_{r=0}^m \cup \left\{ \sum_{j=1}^r g_j(t_2) \right\}_{r=0}^m.$$

Proof. Necessity is trivial. For sufficiency: on each interval between two consecutive points of the combined breakpoint set, both $L_\kappa(\cdot; t_1)$ and $L_\kappa(\cdot; t_2)$ are linear (each is linear between its *own* breakpoints, and the combined set refines both), so their difference is affine on that interval. An affine function that is ≥ 0 at both endpoints of an interval is ≥ 0 throughout the interval, since it attains its extrema at the endpoints. Hence non-negativity at the finitely many combined breakpoints implies non-negativity everywhere. $\square$

We proceed in two steps: first, we prove that the existence of a stochastic matrix satisfying Eq. (S1) implies the Lorenz-curve condition (S2); subsequently, we prove the converse implication.

Step 1: Proving (S1) $\Rightarrow$ (S2)

Lemma 3 (Variational form of L_κ). *For every $t \in \{t_1, t_2\}$ and $\Lambda \in [0, 1]$,*

$$L_\kappa(\Lambda; t) = \max \left\{ \sum_{j=1}^m \theta_j k_j(t) : \theta \in [0, 1]^m, \sum_{j=1}^m \theta_j g_j(t) \leq \Lambda \right\}. \quad (\text{S3})$$

Proof. This is the classical fractional knapsack problem, with item j having weight $g_j(t)$ and value $k_j(t) = \kappa_j(t)g_j(t)$. Since the ratios of value to weight are ordered as

$$\kappa_1(t) \geq \dots \geq \kappa_m(t),$$

the optimal strategy is to select the items in decreasing order of $\kappa_j(t)$, taking each item fully until the capacity Λ is reached and, if necessary, taking a fractional amount of the final item. This is precisely the construction of $L_\kappa(\Lambda; t)$. $\square$

Proposition 1. *If a stochastic matrix T exists such that*

$$Tg(t_1) = g(t_2), \quad Tk(t_1) = k(t_2),$$

then (S2) holds for all $\Lambda \in [0, 1]$.

Proof. Fix $\Lambda \in [0, 1]$, and let $\theta^* \in [0, 1]^m$ be a maximizer in (S3) for $L_\kappa(\Lambda; t_2)$, so that

$$L_\kappa(\Lambda; t_2) = \sum_{i=1}^m \theta_i^* k_i(t_2), \quad \sum_{i=1}^m \theta_i^* g_i(t_2) \leq \Lambda. \quad (\text{S4})$$

Define

$$\phi_j := \sum_{i=1}^m \theta_i^* T_{ij}, \quad j \in [m]. \quad (\text{S5})$$

Since $T_{ij} \geq 0$, $\sum_i T_{ij} = 1$, and $\theta_i^* \in [0, 1]$, each ϕ_j is a convex combination of the θ_i^* and therefore $\phi \in [0, 1]^m$.

Using $Tk(t_1) = k(t_2)$, we obtain

$$\sum_{i=1}^m \theta_i^* k_i(t_2) = \sum_{i=1}^m \theta_i^* \sum_{j=1}^m T_{ij} k_j(t_1) = \sum_{j=1}^m \phi_j k_j(t_1). \quad (\text{S6})$$

Similarly, using $Tg(t_1) = g(t_2)$,

$$\sum_{j=1}^m \phi_j g_j(t_1) = \sum_{i=1}^m \theta_i^* \sum_{j=1}^m T_{ij} g_j(t_1) = \sum_{i=1}^m \theta_i^* g_i(t_2) \leq \Lambda. \quad (\text{S7})$$

Hence ϕ is feasible for the maximization problem (S3) defining $L_\kappa(\Lambda; t_1)$. Therefore,

$$L_\kappa(\Lambda; t_2) = \sum_{j=1}^m \phi_j k_j(t_1) \leq L_\kappa(\Lambda; t_1). \quad (\text{S8})$$

Since Λ was arbitrary, (S2) follows. □

Step 2: Proving (S2) $\Rightarrow$ (S1)

Assume (S2), let

$$\{0 = \lambda_0 < \lambda_1 < \dots < \lambda_N = 1\}$$

be the union of the breakpoints

$$\left\{ G_j^\downarrow(t_1) \right\}_{j=0}^m \cup \left\{ G_j^\downarrow(t_2) \right\}_{j=0}^m, \quad G_j^\downarrow(t) := \sum_{i=1}^j g_i^\downarrow(t),$$

and define

$$\ell_r := \lambda_r - \lambda_{r-1} > 0, \quad r = 1, \dots, N.$$

Thus $N \leq 2m - 1$ and $\sum_{r=1}^N \ell_r = 1$. For $\lambda \in (\lambda_{r-1}, \lambda_r)$, define

$$a_r := \Gamma_\kappa(\lambda; t_1), \quad b_r := \Gamma_\kappa(\lambda; t_2).$$

Since $\Gamma_\kappa(\cdot; t_1)$ and $\Gamma_\kappa(\cdot; t_2)$ are non-increasing, both $(a_r)_{r=1}^N$ and $(b_r)_{r=1}^N$ are non-increasing. Moreover, defining

$$J_j := \left\{ r : (\lambda_{r-1}, \lambda_r) \subset [G_{j-1}^\downarrow(t_1), G_j^\downarrow(t_1)) \right\},$$

and

$$\mathcal{I}_j := \left\{ r : (\lambda_{r-1}, \lambda_r) \subset [G_{j-1}^\downarrow(t_2), G_j^\downarrow(t_2)) \right\},$$

the families $\{J_j\}_{j=1}^m$ and $\{\mathcal{I}_j\}_{j=1}^m$ partition $[N]$, and

$$\sum_{r \in J_j} \ell_r = g_j(t_1), \quad \sum_{r \in \mathcal{I}_j} \ell_r = g_j(t_2), \quad (\text{S9})$$

$$a_r = \kappa_j(t_1) \quad (r \in J_j), \quad b_r = \kappa_j(t_2) \quad (r \in \mathcal{I}_j). \quad (\text{S10})$$

By Lemma 2, condition (S2) is equivalent to

$$\sum_{r=1}^q \ell_r b_r \leq \sum_{r=1}^q \ell_r a_r, \quad q = 1, \dots, N, \quad (\text{S11})$$

with equality for $q = N$. We now use the following weighted version of the Hardy–Littlewood–Pólya theorem.

Lemma 4 (Weighted Hardy–Littlewood–Pólya theorem). *Let $\ell_r > 0$ with $\sum_r \ell_r = 1$, and let $a, b \in \mathbb{R}^N$ be non-increasing. Then there exists a non-negative matrix $D \in \mathbb{R}^{N \times N}$ satisfying*

$$\sum_{r=1}^N D_{rs} = 1, \quad D\ell = \ell, \quad D(\ell \odot a) = \ell \odot b, \quad (\text{S12})$$

if and only if

$$\sum_{r=1}^q \ell_r b_r \leq \sum_{r=1}^q \ell_r a_r, \quad q = 1, \dots, N, \quad (\text{S13})$$

with equality for $q = N$.

The lemma is the standard weighted form of the Hardy–Littlewood–Pólya characterization of majorization: the inequalities above are precisely the condition that the weighted vector b is majorized by the weighted vector a . Applying Lemma 4 to the sequences constructed above, we obtain a non-negative matrix D such that

$$\sum_{r=1}^N D_{rs} = 1, \quad D\ell = \ell, \quad D(\ell \odot a) = \ell \odot b. \quad (\text{S14})$$

Define

$$T_{ij} := \frac{1}{g_j(t_1)} \sum_{r \in J_j} \sum_{s \in \mathcal{I}_i} \ell_r D_{sr}, \quad i, j \in [m]. \quad (\text{S15})$$

Such a matrix satisfies the following proposition.

Proposition 2. *The matrix T defined by (S15) is stochastic and satisfies*

$$Tg(t_1) = g(t_2), \quad Tk(t_1) = k(t_2).$$

Proof. Non-negativity is immediate. Moreover,

$$\begin{aligned} \sum_{i=1}^m T_{ij} &= \frac{1}{g_j(t_1)} \sum_{r \in J_j} \ell_r \sum_{i=1}^m \sum_{s \in \mathcal{I}_i} D_{sr} \\ &= \frac{1}{g_j(t_1)} \sum_{r \in J_j} \ell_r \sum_{s=1}^N D_{sr} = \frac{1}{g_j(t_1)} \sum_{r \in J_j} \ell_r = 1, \end{aligned} \quad (\text{S16})$$

where we used the column-stochasticity of D and $\sum_{r \in J_j} \ell_r = g_j(t_1)$. Hence T is stochastic. For the g -transport property,

$$\begin{aligned} \sum_{j=1}^m T_{ij} g_j(t_1) &= \sum_{j=1}^m \sum_{r \in J_j} \sum_{s \in \mathcal{I}_i} \ell_r D_{sr} \\ &= \sum_{s \in \mathcal{I}_i} \sum_{r=1}^N D_{sr} \ell_r = \sum_{s \in \mathcal{I}_i} (D\ell)_s = \sum_{s \in \mathcal{I}_i} \ell_s = g_i(t_2). \end{aligned} \quad (\text{S17})$$

Thus $Tg(t_1) = g(t_2)$. For the k -transport property, observe that for $r \in J_j$,

$$a_r = \kappa_j(t_1),$$

and hence

$$\ell_r a_r = \ell_r \kappa_j(t_1).$$

Using $k_j(t_1) = \kappa_j(t_1)g_j(t_1)$ and (S15),

$$\begin{aligned} \sum_{j=1}^m T_{ij} k_j(t_1) &= \sum_{j=1}^m \sum_{r \in J_j} \sum_{s \in \mathcal{I}_i} \ell_r a_r D_{sr} \\ &= \sum_{s \in \mathcal{I}_i} \sum_{r=1}^N D_{sr} \ell_r a_r = \sum_{s \in \mathcal{I}_i} [D(\ell \odot a)]_s \\ &= \sum_{s \in \mathcal{I}_i} \ell_s b_s. \end{aligned} \tag{S18}$$

Since $b_s = \kappa_i(t_2)$ for $s \in \mathcal{I}_i$,

$$\sum_{s \in \mathcal{I}_i} \ell_s b_s = \kappa_i(t_2) \sum_{s \in \mathcal{I}_i} \ell_s = \kappa_i(t_2) g_i(t_2) = k_i(t_2). \tag{S19}$$

Therefore $Tk(t_1) = k(t_2)$. □

This proves (S1), completing the proof of the sufficiency direction.

Conclusion of the proof

By Proposition 1, the existence of a stochastic matrix T satisfying

$$Tg(t_1) = g(t_2), \quad Tk(t_1) = k(t_2)$$

implies

$$L_\kappa(\Lambda; t_1) \geq L_\kappa(\Lambda; t_2) \quad \text{for all } \Lambda \in [0, 1].$$

Conversely, assuming (S2), Lemma 2 reduces the Lorenz-curve condition to the finite set of atomic majorization inequalities. Lemma 4 then yields an ℓ -doubly-stochastic matrix D satisfying

$$D(\ell \odot a) = \ell \odot b,$$

and Proposition 2 folds this atomic transformation into a stochastic matrix T satisfying

$$Tg(t_1) = g(t_2), \quad Tk(t_1) = k(t_2)$$

in the sorted coordinates. Finally, Lemma 1 transports T back to the original coordinates while preserving both identities. Hence (S1) and (S2) are equivalent, proving Theorem 3.

B — Proof of Theorem 1

Let $P(j_1, j_2)$ denote a joint distribution of the pair (j_1, j_2) , with marginals

$$P(j_1) = \sum_n P(j_1, n), \quad P(j_2) = \sum_n P(n, j_2), \quad (\text{S20})$$

and define

$$T_{j_2 j_1} := \frac{P(j_1, j_2)}{g_{j_1}(t_1)}. \quad (\text{S21})$$

We establish the three properties required for T to be a stochastic transformation satisfying $Tg(t_1) = g(t_2)$ and $Tk(t_1) = k(t_2)$. First, summing over the first index gives

$$\sum_n T_{n j_1} = \sum_n \frac{P(j_1, n)}{g_{j_1}(t_1)} = \frac{P(j_1)}{g_{j_1}(t_1)}. \quad (\text{S22})$$

Therefore,

$$T \text{ is stochastic} \iff P(j_1) = g_{j_1}(t_1). \quad (\text{S23})$$

Second, we have

$$\sum_n T_{j_2 n} g_n(t_1) = \sum_n P(n, j_2) = P(j_2). \quad (\text{S24})$$

Hence,

$$Tg(t_1) = g(t_2) \iff P(j_2) = g_{j_2}(t_2). \quad (\text{S25})$$

Finally, using $k_n(t_1) = g_n(t_1)\kappa_n(t_1)$, we obtain

$$\sum_n T_{j_2 n} k_n(t_1) = \sum_n P(n, j_2) \kappa_n(t_1) \quad (\text{S26})$$

$$= P(j_2) \mathbb{E}[x_1 | x_2 = \kappa_{j_2}(t_2)]. \quad (\text{S27})$$

Consequently,

$$Tk(t_1) = k(t_2) \iff P(j_2) \mathbb{E}[x_1 | x_2 = \kappa_{j_2}(t_2)] = k_{j_2}(t_2). \quad (\text{S28})$$

Combining these three results, a stochastic transformation T satisfying

$$Tg(t_1) = g(t_2), \quad Tk(t_1) = k(t_2) \quad (\text{S29})$$

exists if and only if there exists a joint distribution $P(j_1, j_2)$ with marginals

$$P(j_1) = g_{j_1}(t_1), \quad P(j_2) = g_{j_2}(t_2), \quad (\text{S30})$$

such that

$$\mathbb{E}[x_1 | x_2 = \kappa_{j_2}(t_2)] = \kappa_{j_2}(t_2). \quad (\text{S31})$$

In other words, $P(j_1, j_2)$ defines a martingale coupling between the distributions $p_\kappa(x_1, t_1)$ and $p_\kappa(x_2, t_2)$.

C — Proof of Theorem 3

We begin by evaluating the exponential average of the entropy difference:

$$\mathbb{E}_{\Pi}[e^{-\Delta s}] = \sum_{j_1, j_2} k_{j_1}(t_1) T_{j_2 j_1} \frac{\kappa_{j_2}(t_2)}{\kappa_{j_1}(t_1)} \quad (\text{S32})$$

$$= \sum_{j_1, j_2} T_{j_2 j_1} g_{j_1}(t_1) \kappa_{j_2}(t_2) \quad (\text{S33})$$

$$= \sum_{j_2} g_{j_2}^{\text{true}}(t_2) \kappa_{j_2}(t_2) \quad (\text{S34})$$

$$= \sum_{j_2} k_{j_2}(t_2) \frac{g_{j_2}^{\text{true}}(t_2)}{g_{j_2}(t_2)}. \quad (\text{S35})$$

Consequently,

$$\mathbb{E}_{\Pi}[e^{-\Delta s}] - 1 = \sum_{j_2} \frac{k_{j_2}(t_2) [g_{j_2}^{\text{true}}(t_2) - g_{j_2}(t_2)]}{g_{j_2}(t_2)}. \quad (\text{S36})$$

Applying the Cauchy–Schwarz inequality then yields

$$(\mathbb{E}_{\Pi}[e^{-\Delta s}] - 1)^2 \leq \left[\sum_{j_2} \frac{k_{j_2}^2(t_2)}{g_{j_2}(t_2)} \right] \left[\sum_{j_2} \frac{(g_{j_2}^{\text{true}}(t_2) - g_{j_2}(t_2))^2}{g_{j_2}(t_2)} \right]. \quad (\text{S37})$$

Recognizing the second factor as the χ^2 -divergence,

$$D_{\chi^2}(g^{\text{true}}(t_2) \parallel g(t_2)) \geq \frac{(\mathbb{E}_{\Pi}[e^{-\Delta s}] - 1)^2}{\sum_j k_j^2(t_2)/g_j(t_2)}, \quad (\text{S38})$$

which proves the stated result.