Enhanced CAD-Based Quantifier Elimination With Multiple Equational Constraints – ***Preliminary Draft***

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Abstract

This paper presents two enhancements to cylindrical algebraic decomposition (CAD) based quantifier elimination (QE) for cases in which multiple equational constraints are present in the given input formula ϕ^*. The first enhancement provides more detail in the output when there is a conceptual partition of the set of variables of ϕ^* into parameters and unknowns. In such cases, we describe how to partition the parameter space so that: (1) in each open set of the partition the number ν of associated unknowns is a finite constant or is infinite; and (2) for each such open set for which ν is finite, an expression for the unknowns in terms of the parameters is provided. The second enhancement is an efficiency gain achievable in certain situations. Indeed, when certain conditions are met, the second CAD equational projection step can be reduced more significantly than is supported by the prior existing theory. Relevant theorems and worked examples for both enhancements are provided. Application areas include approximation theory, cuspidal manipulator classification, and biological/chemical systems.

1 Introduction

Quantifier elimination (QE) by cylindrical algebraic decomposition (CAD) is a general purpose tool for solving algebraic and geometric problems. While its many improvements have led to some success with its use in practical applications, there is room for further development of the method. Improvements which better address the requirements of special types of problems and take advantage of any favourable features of such problem types to reduce the amount of computation needed may be particularly beneficial. In this paper we present two enhancements to CAD based QE for when multiple equational constraints (ECs) are present in the given input formula.

The first enhancement consists of providing more detail in the output, when appropriate, as follows. In classical QE for Tarski algebra, when given a quantified formula ϕ^* in prenex form, the output is usually expected to be a quantifier-free formula ϕ', involving the free variables of ϕ^* only, equivalent to ϕ^*. However, it is sometimes the case that the given ϕ^* is itself the result of a careful formulation (or reformulation) of a problem of practical interest [Col96, Jir97]. Practical mathematical questions often seek the values of, or information about, desired *unknowns*, given a set of data constituting the *parameters* of the problem. Particularly useful is some kind of closed-form expression for each unknown in terms of the parameters. Thus, in the case that a QE input ϕ^* comes with a clear partition of its set of variables into parameters and unknowns, we aim to provide more than a simple quantifier-free formula ϕ' describing the truth set of ϕ^*. Indeed, we would like to provide a partition $\mathcal{P}$ of the parameter space, such that in each element of $\mathcal{P}$ the number of solutions (that is, the number of associated vectors of unknowns) is a finite constant or is infinite. Ideally then, for each element of $\mathcal{P}$ having a finite positive constant number of solutions, we would like to find an expression for the solutions in terms of the parameters. This could be regarded as extending the scope of the work reported in [LR07], in which the given parametric polynomial systems are assumed to be quantifier-free and to satisfy other restrictions. Problems of the kind we treat are found in application areas such as approximation theory [Col96], cuspidal manipulator classification [LR07], control systems design [Jir97], and biological and chemical systems [SE22].

Our second enhancement to CAD-based QE with multiple equational constraints (ECs) consists of identifying efficiency gains achievable in specific situations. Our aim here is to exploit the special structure of certain problem classes of genuine practical interest to "*push back the doubly exponential wall*" of CAD-based QE. The metaphor is chosen to reflect that while the fundamental complexity of the general QE problem means such a wall exists, optimisations like those we present move the wall back bringing many new problems into the scope of practical solutions. The tantalizing, yet elusive, goal of identifying interesting sub-problem classes for which a major improvement to the upper complexity bound for CAD-based QE could be demonstrated also motivates us.

1.1 Notation

We use the following notation throughout, unless explicitly varied.

ϕ^*: The given input prenex formula of Tarski algebra.

n: The total number of variables, free and bound, in ϕ^*. The variables we generally use are $x_1, \ldots, x_n$. CAD requires a fixed ordering of the variables, $\prec$, which determines the order in which they are eliminated: we assume an ordering consistent with the variable indexing ($x_1 \prec \cdots \prec x_n$) with x_n the first variable to be eliminated. We may then say a polynomial is of level k if the highest ordered variable it contains is x_k.

In small examples, we may use simple letters without subscripts. For instance, when $n = 3$, we may use x, y, z with z the first to be eliminated.

k: The number of variables in ϕ^* that we consider to be bound (by quantifiers). We assume these variables are $x_{n-k+1}, \ldots, x_n$, consistent with the assumption that ϕ^* is a prenex formula. We denote the number of free variables of ϕ^* by $s = n - k$.

d: The maximum degree (*in any individual variable*) of the polynomials in ϕ^*.

$\mathbf{f} := (f_1, \ldots, f_t)$ **:** The polynomials in ϕ^* which are used in equational constraints for $x_1, \ldots, x_n$. We assume that $t \geq 1$; in fact, since we are primarily interested in the case of "multiple" equational constraints, we will usually assume that $t \geq k$. Note that no f_i should lie in $\mathbf{Z}[x_1, \ldots, x_s]$, i.e. every equational constraint should involve a bound variable.

In summary, we may express ϕ^* in the form

$$(Q_{n-k+1}x_{n-k+1}) \ldots (Q_n x_n)\phi(x_1, \ldots, x_n)$$

where ϕ is $f_1 = 0 \wedge \cdots \wedge f_t = 0 \wedge \psi$, for some quantifier-free formula ψ. We will further assume that $(Q_j x_j) = (\exists x_j)$, for every j. While we are primarily interested in enhancing QE for such formulas of Tarski algebra, we may sometimes consider analogous QE for constructible sets over the complex field $\mathbf{C}$.

1.2 Examples

Example 1. *Let ϕ be $(x-1)^2+(y-2)^2 = 0$. Then ϕ is equivalent to $x = 1 \wedge y = 2$ over the reals $\mathbf{R}$. So, if we put $\phi^* \equiv \phi$ (that is, we apply no quantifiers to ϕ) as input, we may find output $\phi' \equiv x = 1 \wedge y = 2$. If we put $\phi^* \equiv (\exists y)\phi$ as input, we may find output $\phi' \equiv x = 1$.*

We remark that over $\mathbf{C}$ the situation is different. For example, if we put $\phi^* \equiv (\exists y)\phi$ as input, then we may find output $\phi' \equiv 0 = 0$.

Example 2. *We consider an interesting problem based on [FGT01, Example 3.11]. The original example presents a Gröbner basis under the variable ordering $x \succ y \succ z \succ a \succ b \succ c$ for the ideal I of $K[c, b, a, z, y, x]$ generated by*

$$\{ax^2 + x + y,\ bx + z,\ cy + y - z\},$$

where K is an arbitrary field. The context is the study of systems of parametric polynomial equations, in which c, b, a are the parameters and z, y, x are the unknowns. Accordingly, let us put $K = \mathbf{R}$, and define

$$\phi^* \equiv (\exists z)(\exists y)(\exists x)[ax^2 + x + y = 0 \wedge bx + z = 0 \wedge cy + y - z = 0].$$

For now, we make some simple observations concerning this problem, which we will discuss in more detail in later sections. First, it is not difficult to see quickly that ϕ^* is equivalent to $0 = 0$; that is, the truth set of ϕ^* in real c, b, a-space is the whole space. The reason is that, for all $c, b, a \in \mathbf{R}$, $x = y = z = 0$ is a solution of the given system. However, it is reasonable to ask for a description of all the solutions, as the parameter triple (c, b, a) varies in $\mathbf{R}^3$. For example, *when we put $c = b = a = 0$, we easily see that $(x, y, z) = (0, 0, 0)$ is the unique solution. Alternatively, suppose we put $c = -1$ and $b = 0$, and take an arbitrary real $a > 0$. Then our system becomes $ax^2 + x + y = 0 \wedge z = 0$, which has infinitely many real solutions. Indeed, the set of all real solution triples is $\{(x, y, 0) \in \mathbf{R}^3 \mid ax^2 + x + y = 0\}$; that is, the parabola $y = -ax^2 - x$ lying in the plane $z = 0$.*

Subsequent sections will detail an enhancement to CAD-based QE for such parametric systems. This will allow a generic solution description for this example. Comparison with ideal theoretic methods applicable primarily for the case $K = \mathbf{C}$, but with relevance to the real case also, is anticipated in the final version of this paper.

Example 3. *For this example, n does not denote the number of variables as normal. Solotareff's approximation problem [Col96] asks one to find the best approximation, in the sense of the uniform norm on the interval $[-1, 1]$, to a real polynomial of degree n by a real polynomial of degree at most $n - 2$. The best approximation is known to be unique. The given polynomial may be taken to be $x^n + rx^{n-1}$, where r is an arbitrary non-negative real number. Theory in [Ach56] provides explicit, computational expressions for the coefficients of the best approximation when $0 \le r \le S_n = n(\tan(\pi/(2n)))^2$. For $r > S_n$ the question may be advantageously formulated as a QE problem using some further theory in [Ach56], as explained in [Col96].*

We assume $r > S_n$ henceforth. The case $n = 2$ is easy to solve by hand, while the paper [Col96] describes computer-assisted solutions to this problem for the cases $n = 3$, 4, and a partial solution in the case $n = 5$. Here we consider the QE formulation of the problem in the case $n = 3$ presented in that paper. That is, noting that $S_3 = 1$, we wish to find the best approximation $ax + b$ to $x^3 + rx^2$, where $r > 1$. Using theory we deduce $a = 1$, and we obtain the following quantified formula which implicitly expresses b in terms of r:

$$\begin{aligned} \phi^* \equiv (\exists u)[r > 1 \wedge -1 < u \wedge u < 1 \wedge 3u^2 + 2ru - 1 = 0 \\ \wedge\, u^3 + ru^2 - u + r - 2b = 0]. \end{aligned}$$

With the help of the QE program used in [Col96] the quantifier-free formula

$$\begin{aligned} \phi' \equiv [27b^2 - 2r^3 b - 36rb + r^4 + 11r^2 - 1 = 0 \\ \wedge\, 27b - r^3 - 18r < 0 \wedge r > 1] \end{aligned}$$

was found as a solution. Insightful examination of the program's trace yields a still more explicit expression of b in terms of $r > 1$: b is the lesser of the two real roots of the polynomial on the left-hand side of the first conjunct in ϕ'.

The publication of [Col96] preceded the rigorous, partial validation of the equational constraint projection reduction technique used in that paper. (See Section 2.1 for more details about this technique.) The theory in [McC99] fully justified such reduction in the case of three variables; hence the solution of

Solotareff in the case $n = 3$ reviewed in this example is confirmed. Solotareff's problem in the case $n = 4$ will be examined in detail in Section 5, following presentation of progress on equational projection validation in Section 4.

1.3 Outline of the paper

We outline the structure of the paper, and our contributions. Section 2 provides a summary of relevant background material: we will assume that the reader is familiar with the CAD algorithm, its theory, and its application to QE; but will offer a summary of work published to date on the use of equational constraints, when present, to reduce the size of projection sets and to simplify the lifting process. So far, only a *semi-restricted* equational projection operation has been validated in general [McC01, EBD20].

In Section 3, we describe a new enhancement to traditional "single-step projection" CAD-based QE with many equational constraints. We describe an algorithmic framework for finding generic solutions of such QE problems representing parametric systems in which the output is more informative than that supplied by existing CAD-based QE tools. In Section 4 we report progress on our efforts to validate Collins' *original* fully reduced equational projection operation, in the case that multiple equational constraints are present which satisfy certain conditions. Section 5 contains detailed discussion of two examples, in which we apply the work reported in Sections 3 and 4. Section 6 concludes the present paper, looking ahead to planned future contributions.

2 Background material

A *cylindrical algebraic decomposition* (CAD) is a decomposition of $\mathbf{R}^n$ into cells (connected subsets), each of which is semi-algebraic, that are arranged cylindrically: the projection of any two cells is either identical or disjoint, i.e. the cells stack up in cylinders above an induced CAD of $\mathbf{R}^{n-1}$. The original CAD algorithm produced a CAD relative to a set of input polynomials such that each polynomial had constant sign $(-/0/+)$ within each cell (*sign invariance*), with subsequent algorithms developed to take logical formulae as input and produce corresponding *truth-invariant* CADs. The main idea in traditional CAD algorithms is to identify a set of projection polynomials and use these to build the decomposition: with each cell either a section (where a projection polynomial vanishes) or a sector (the space between two sections).

Descriptions of the CAD algorithm, its variants and associated theory can be found in [ACM84, Col75, CH91, McC88, McC98]. We will make use of terminology and results from [McC88, McC98]. The definitions of key technical terms from those works (such as submanifold, analytic delineability and order-invariance) we use are summarised in [McC99].

2.1 Using equational constraints to reduce projection

An *equational constraint (EC)* is an equation logically implied by the quantifier-free matrix of the prenex input formula ϕ^*). The idea of using an EC to reduce the projection operations and lifting process in CAD-based QE originated with Collins [Col96, Col98]. The basic idea is that, if $A \subset \mathbf{Z}[x_1, \ldots, x_n]$ is the set of polynomials in ϕ^* and $f(x_1, \ldots, x_n) = 0$ is an EC (with f assumed to be squarefree for simplicity), then the projection of A may consist of only the discriminant of f, (a sufficient subset of) the coefficients of f, and the resultant of f and g, for each polynomial $g \in A \setminus \{f\}$ (where each g is assumed to be prime to f, for simplicity). Adjustment of the basic idea to take account of the factorization of f and the other polynomials g into irreducible polynomials can easily be made. While this description of the idea focuses on the *first* projection (that is, elimination of the last variables x_n), Collins implied that a similar reduction could be made for subsequent projection steps, provided that suitable ECs are available either explicitly or implicitly. However, while the key claim of these initial works on the topic is true in the special case $n = 2$ (proved below in Proposition 1), the validity of the claim for $n > 2$ was not addressed in those original works: in short, the improvement was at that stage intuitive and conjectural. The conference papers [McC99, McC01] endeavoured to justify the key claim rigorously, but succeeded in providing only partial justification for the original approach for $n > 3$ (as well its complete verification for $n = 3$). The reader is referred to the comprehensive work [EBD20] for a detailed synthetic review of this subject.

We recall the main fundamental lemma from [McC99] for the partial validation of the reduced equational projection.

Theorem 1 (Theorem 2.2 in [McC99])**.** *Let $n \geq 2$, let $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ be real polynomials of positive degrees in the main variable x_n, let $R(x_1, \ldots, x_{n-1})$ be the resultant of f and g, and suppose that $R \neq 0$. Let S be a connected subset of $\mathbf{R}^{n-1}$ on which f is delineable and in which R is order-invariant. Then g is sign-invariant in each section of f over S.*

The above theorem underpins the validity argument for Collins' original, fully reduced, equational projection scheme for the *first* projection step (the different invariance properties of the hypothesis and conclusion block its repeated application for subsequent steps). Theorem 2.3 of [McC99] formalizes this validation argument. In general, subsequent projection steps can use only a semi-restricted projection scheme, when suitable equational constraints (explicit or implicit) are available [McC01, EBD20]. Section 4 below presents certain extensions of the above theorem, as well as some other new results, which are useful in validating the original, fully reduced equational projection for both the first and second projection steps, under some special assumptions.

For the reader's convenience, we summarize below the definitions of the fully- and semi-reduced equational projection sets from [McC99, McC01, EBD20]. Here, A denotes an irreducible basis in $\mathbf{Z}[x_1, \ldots, x_n]$, $E \subset A$, and $P(A)$ denotes the projection set defined in [Bro01] (the "Brown-McCallum projection"):

$$P_E(A) = P(E) \cup \{\mathrm{res}_{x_n}(f, g) \mid f \in E \wedge g \in A \setminus E\};$$
$$P_E^*(A) = P_E(A) \cup \mathrm{discr}(A \setminus E) \cup \mathrm{ldcf}(A \setminus E).$$

Remark 1. *The sets* $\mathrm{discr}(A \setminus E)$ *and* $\mathrm{ldcf}(A \setminus E)$ *represent sets of discriminants and leading coefficients, respectively, with respect to* x_n *of elements of* $A \setminus E$.

Remark 2. *When* $A \subset \mathbf{Z}[x_1, \ldots, x_n]$ *is arbitrary,* $P_E(A)$ *is defined to be* $\mathrm{cont}(A) \cup P_F(B)$*, where* $\mathrm{cont}(A)$ *denotes the set of non-zero contents with respect to* x_n *of the elements of* A*,* B *is the finest squarefree basis for the set* $\mathrm{prim}(A)$ *of all primitive parts with respect to* x_n *of the elements of* A *which have positive degree in* x_n*, and* F *is the finest squarefree basis for* $\mathrm{prim}(E)$*. For* A *arbitrary,* $P_E^*(A)$ *is defined analogously.*

Remark 3. *The semi-reduced equational projection set* $P_E^*(A)$ *was first defined in [McC01], but a suitable coefficient set for* $A \setminus E$ *was erroneously omitted; this was corrected in [EBD20].*

Remark 4. *In the sources cited for the above definitions,* $P(A)$ *denoted the McCallum projection set. However, the use of the Brown-McCallum projection set instead is likely to be more efficient. It remains valid for CAD-based QE where the originally defined equational projection sets are valid, provided that the set of polynomials in the input formula is well-oriented [McC98], the finitely many points over which projection factors vanish identically are computed, and such points are "added" during the lifting phase of CAD construction [Bro01].*

It was stated in [McC99] (and recalled above) in relation to Collins' original, fully reduced, equational projection scheme that "the validity of the scheme is certainly clear in the bivariate case". However, no general proof of this claim was provided in that work, nor has it been supplied in any subsequent paper (as far as we know). So, for completeness, we offer here the following slight strengthening of Theorem 2.3 of [McC99] in the bivariate case. For convenience, we use the notion of an *irreducible basis* in a polynomial ring [MPP19].

Proposition 1. *Let* A *be an irreducible basis in* $\mathbf{Z}[x, y]$*. Let* $E \subseteq A$ *and let* S *be an open interval in* $\mathbf{R}^1$*. Suppose each element of* $P_E(A)$ *is order-invariant in* S*. Then each element of* E *is analytic delineable on* S*, the sections over* S *of the elements of* E *are pairwise disjoint, and each element of* A *is order-invariant in every such section.*

Proof. Let $f_i(x, y) \in E$. By definition of $P_E(A)$, the leading coefficient $l_i(x)$ of f_i belongs to $P_E(A)$. Since S is an open interval of the real line and $l_i(x)$ is order-invariant in S, by hypotheses, it follows that $l_i(x)$ vanishes nowhere in S. Also, by hypothesis, the discriminant $d_i(x)$ of $f_i(x, y)$ is order-invariant in S; and the resultant $r_{i,j}(x)$ of f_i and every other $f_j \in E$ is order-invariant in S. Let $f(x, y)$ denote the product of all the $f_i(x, y)$ in E. Then the leading coefficient $l(x)$ of $f(x, y)$, which is the product of all the $l_i(x)$, vanishes nowhere in S. Therefore

$f(x, y)$ is degree-invariant in S, of positive degree. Also, the discriminant $d(x)$ of $f(x, y)$, which is the product of all the $d_i(x)$ and the squares of all the $r_{i,j}(x)$ (where all the $d_i(x)$ and all the $r_{i,j}(x)$ are elements of $P_E(A)$), is order-invariant in S. Hence, by Theorem 2 of [McC98], f is analytic delineable on S and is order-invariant in each of its sections over S. The first two conclusions of the proposition follow by Lemma A.7 of [McC88]. The order-invariance of each element of E in every section of f over S follows by Lemma A.3 of [McC88]. Finally, consider an arbitrary element $g \in A \setminus E$, and again let $f_i \in E$. By definition of $P_E(A)$, we know $\mathrm{res}_y(f_i, g) \in P_E(A)$. Therefore, by the order-invariance hypothesis, this polynomial vanishes nowhere in S. It follows that g vanishes at no point of a section of f_i over S; hence g is order-invariant (of order 0) in every section of f_i over S. $\square$

3 Framework for enhanced CAD-based QE for parametric systems with many ECs

Recall that we assume throughout this paper that our given input formula ϕ^* has the form

$$(\exists x_{n-k+1}) \cdots (\exists x_n)\phi(x_1, \ldots, x_n),$$

where $\phi \equiv [f_1 = 0 \wedge \cdots \wedge f_t = 0] \wedge \psi$, for some quantifier-free formula ψ (see Section 1.1). Recall also that we denote the free variables of ϕ^* by $x_1, \ldots, x_s$, where $s = n - k$, and let us suppose $t \geq k$. For this subsection we shall further assume that an initial segment of our list of n variables is partitioned into p *parameters* and u *unknowns* thus: $x_1, \ldots, x_p, x_{p+1}, \ldots, x_{p+u}$. In general, the only restrictions on p and u are that $p \leq s$ and $p + u \leq n$. However, for simplicity, we shall consider only the case $p = s \geq 1$, $u \geq 1$ and $p + u = n$. Adjustment of our framework described below for other cases which may arise in practice would be straightforward.

We will present our proposed CAD-based method for finding generic solutions of a parametric system given by a quantified formula ϕ^* as described above. We first formalize our terminology relating to solutions.

Definition 1. *Let $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbf{R}^s$ be a specific real parameter vector. We shall call any specific vector of real unknowns $\beta = (\beta_{s+1}, \ldots, \beta_n) \in \mathbf{R}^k$ for which $\phi(\alpha, \beta)$ is true a* solution vector for ϕ associated with α.

Definition 2. *A* system of generic solutions *for ϕ^* consists of a list $\mathcal{O}$ of pairwise disjoint connected open subsets of parameter space $\mathbf{R}^s$ such that:*

1. *the closure of $\cup\mathcal{O}$ is $\mathbf{R}^s$;*

2. *for every subset $S \in \mathcal{O}$, we have that for any $\alpha \in S$, the number of distinct solution vectors for ϕ associated with α is a constant, ν_S of $\mathbf{N} \cup \{\infty\}$, as α varies within S; and*

3. *for each subset $S \in \mathcal{O}$ for which ν_S is finite and positive, the ν_S distinct solution vectors for S are expressible as continuous functions of the parameters in S.*

Our method for finding a description of a system of generic solutions for a given ϕ^* is described formally in Algorithms 1 and 2, the latter being a sub-algorithm of the former. Our method is essentially a synthesis of existing methods and concepts (such as QE by partial CAD, equational constraint projection, etc.), together with an extension to the stack construction phase of partial CAD. A key notion for the method is that of a *pivot* constraint [Col98]: during the projection phase, when more than one equational constraint is available at a given level, one such must be chosen and designated as the pivot constraint at that level. The next projection set computed is then semi-restricted with respect to the (set of irreducible factors of the) pivot, as reviewed in Section 2.1. As a prelude to our description of Algorithm 1 we present a foundational result that underpins it.

Proposition 2. *Let $\mathcal{D}$ be a truth-invariant CAD of $\mathbf{R}^n$ for ϕ and let $\mathcal{D}_s$ denote the CAD of $\mathbf{R}^s$ induced by $\mathcal{D}$. Let c be a cell of $\mathcal{D}_s$, and let $\alpha \in c$. Then the total number $\nu_{c,\alpha}$ of distinct solution vectors $\beta \in \mathbf{R}^k$ for ϕ over α is an element of $\mathbf{N} \cup \{\infty\}$, constant as α varies in c: hence we may define ν_c, the cardinality of the set of solution vectors associated with the points of c, to be this number. Moreover, in the case that $0 < \nu_c < \infty$, the ν_c distinct solution vectors for c are expressible as continuous functions of the parameters in c.*

Proof. Recall that $1 \le k \le n-1$ and $s = n-k$ (from the preamble). Define the proposition $\mathcal{P}(k)$ as follows: for every cell $c \in \mathcal{D}_{n-k}$ and for every $\alpha \in c$, the number $\nu_{c,\alpha}$ is a constant element of $\mathbf{N} \cup \{\infty\}$, as α varies in c. We shall prove that $\mathcal{P}(k)$ is true for all k in the range $1 \le k \le n-1$ by induction on k.

The induction base is proved as follows. Let $c \in \mathcal{D}_{n-1}$. First suppose that the cylinder over c contains some sector-cell $\hat{c}$ of $\mathcal{D}$ for which ϕ is true throughout $\hat{c}$. Take any $\alpha \in c$. Then, by the definition of sector-cell, there are infinitely many $\beta \in \mathbf{R}$ such that (α, β) lies in $\hat{c}$, hence for which $\phi(\alpha, \beta)$ is true. Therefore, $\nu_{c,\alpha} = \infty$. We have shown that $\nu_{c,\alpha}$ is constant (and infinite) for all $\alpha \in c$. Now suppose, on the other hand, that the cylinder over c contains no sector-cell for which ϕ is true. Take any $\alpha \in c$: then $\nu_{c,\alpha}$ equals the total number of section-cells of $\mathcal{D}$ over c for which ϕ is true, a number clearly independent of α. Again, we have shown that $\nu_{c,\alpha}$ is constant for all $\alpha \in c$ (but in this case, it is finite). The induction base $\mathcal{P}(1)$ is now established.

For the induction step, let $k > 1$ and assume $\mathcal{P}(k-1)$ is true (as induction hypothesis). We must show that $\mathcal{P}(k)$ holds. Let $c \in \mathcal{D}_{n-k}$. Suppose that the cylinder over c contains some cell $\hat{c} \in \mathcal{D}_{n-k+1}$ for which $\nu_{\hat{c}}$ is infinite (noting that the notation $\nu_{\hat{c}}$ is justified by the induction hypothesis). Then the number $\nu_{c,\alpha}$, for points $\alpha \in c$, is constant and infinite also. So, we may assume henceforth that the cylinder over c contains no cell $\hat{c}$ in $\mathcal{D}_{n-k+1}$ for which $\nu_{\hat{c}}$ is infinite. Two cases remain to be considered. Firstly, consider the case in which the cylinder over c contains some sector-cell $\hat{c} \in \mathcal{D}_{n-k+1}$ for which $\nu_{\hat{c}}$ is positive (noting again that the notation is justified by the induction hypothesis). Then the number $\nu_{c,\alpha}$, for points $\alpha \in c$, is constant and infinite. The reason is that, with $\alpha \in c$ arbitrary, for each of the infinitely many $\beta_{s+1} \in \mathbf{R}$ such that $(\alpha, \beta_{s+1}) \in \hat{c}$, there is a solution vector $(\beta_{s+2}, \ldots, \beta_n)$ for ϕ associated with (α, β_{s+1}). Secondly, consider the case in which the cylinder over c contains no such sector-cell. In this case, the number $\nu_{c,\alpha}$, for points $\alpha \in c$, equals the sum over the sections $\hat{c}$ of $\mathcal{D}_{n-k+1}$ of the finite quantities $\nu_{\hat{c}}$ (noting again the use of the induction hypothesis), a sum which is independent of α. This completes the proof of the induction step, establishing the truth of $\mathcal{P}(k)$ for all k, with $1 \le k \le n-1$.

Define the proposition $\mathcal{Q}(k)$ as follows: for every cell $c \in \mathcal{D}_{n-k}$, for which $0 < \nu_c < \infty$, the ν_c distinct solution vectors for c are expressible as continuous functions of the parameters in c. Then $\mathcal{Q}(k)$ may be proved true for all k in the range $1 \le k \le n-1$ by induction on k, by analogy with (and suitable modification of) the proof for proposition $\mathcal{P}$ provided above.

$\square$

Algorithm 1: Generic Solutions for a Parametric System

$(\mathcal{I}, \mathcal{S}, \mathcal{N}, \mathcal{F}) \leftarrow \mathrm{GSPS}(\phi^*)$

Input: ϕ^* is a formula of Tarski algebra of the form $\phi^* \equiv (\exists x_{n-k+1}) \cdots (\exists x_n)\phi(x_1, \ldots, x_n)$, where $\phi \equiv [f_1 = 0 \wedge \cdots \wedge f_t = 0] \wedge \psi$, for some quantifier-free formula ψ, and $t \geq k \geq 1$. With $s = n - k \geq 1$, ϕ^* is considered to be a parametric system in which $x_1, \ldots, x_s$ are parameters and $x_{s+1}, \ldots, x_n$ unknowns.

Output: $\mathcal{I}$ and $\mathcal{S}$ are lists of indices and sample points, respectively, for the s-cells of the CAD $\mathcal{D}_s$ of $\mathbf{R}^s$ induced by a truth-invariant CAD $\mathcal{D}$ of $\mathbf{R}^n$ for ϕ. $\mathcal{N}$ is the list of numbers ν_c, for each cell c of $\mathcal{D}_s$ (where ν_c is defined by the above proposition). $\mathcal{F}$ is a list $\{F_c \mid c \in \mathcal{D} \wedge 0 < \nu_c < \infty\}$ such that every F_c is a list of suitable functional expressions of the ν_c associated unknown vectors in terms of the parameter vector $x \in c$.

1. **Initialization.** Set $A \leftarrow$ the set of polynomials which occur in ϕ and $E \leftarrow \{f_1, \ldots, f_t\}$. Set $J_n \leftarrow A$ and $C_n \leftarrow$ the set of level n polynomials in E. If $C_n \neq \emptyset$ then set $e_n \leftarrow$ to an element of C_n and $E_n \leftarrow \{e_n\}$ else set $e_n \leftarrow 0$. [C_n is the candidate set of level n EC polynomials, and e_n is the level n pivot if an EC exists.]

2. **Compute remaining pivots.** For $j \leftarrow n-1$ down to $n-k+1$: set $C_j \leftarrow$ the candidate set of level j EC polynomials; if $C_j \neq \emptyset$ then set $e_j \leftarrow$ an element of C_j and $E_j \leftarrow \{e_j\}$ else set $e_j \leftarrow 0$. [C_{n-1} is the union of the set of all level $n-1$ elements of E and the set of level $n-1$ resultants with respect to x_n of pairs of distinct elements of C_n, for which one element of each pair is e_n. General definition of C_j is needed.]

3. **Compute projection sets.** For $j \leftarrow n$ down to $n-k+1$: if $e_j \neq 0$ then set $J_{j-1} \leftarrow P^*_{E_j}(J_j)$; else set $J_{j-1} \leftarrow P(J_j)$. [Where possible, in the case that $e_j \neq 0$, replace $P^*_{E_j}(J_j)$ by the fully restricted projection set $P_{E_j}(J_j)$. See Section 2.1 for a review of the definitions of these projection sets. Note, in particular, that construction of each set J_{j-1} entails computation of $\mathrm{cont}(J_j)$, $\mathrm{prim}(J_j)$, and irreducible basis for the latter.]

4. **Analyse projection factor coefficient sets.** For each projection factor of level 3 upwards, examine zeros of the coefficient system. If a positive dimensional solution set is found then report A as not well-oriented and exit. Otherwise solve the coefficient system and retain solutions for lifting [Bro01].

Algorithm 1: continued

5. **Compute open cells of a J_s-invariant CAD of parameter space $\mathbf{R}^s$ together with solution vector cardinality information.** Compute lists $\mathcal{I}$ and $\mathcal{S}$ for the s-cells of a CAD $\mathcal{D}_s$ of $\mathbf{R}^s$ such that every element of J_s is order-invariant in each cell constructed. [See [McC93] or [LR07] for an efficient algorithm for this task.] For each cell c constructed, with rational sample point $\alpha_c \in \mathcal{S}$, construct stacks which, together with c, form a tree $\mathcal{T}_c$ of cells rooted at c sufficient to determine the cardinality ν_c of the set of associated solution vectors in $\mathbf{R}^k$.

6. **Construct $\mathcal{F}$.** For each s-cell c of $\mathcal{D}_s$ with sample point $\alpha_c \in \mathbf{Q}^s$, and with $0 < \nu_c < \infty$, use sub-algorithm CFEL to construct a list F_c of functional expressions of the ν_c associated solution vectors in $\mathbf{R}^k$ in terms of the parameters $x \in c$. Exit.

Some remarks are in order. First, Algorithm 1 is based upon the equational variant of algorithm QEPCAD (Quantifier Elimination by Partial CAD) which is sketched in [McC01]. It is also influenced by [LR07]: the specification of our GSPS is in accord with the framework and concepts of [LR07]; and our use of "open CAD" in step 5 of GSPS recalls its mention in [LR07]. As in [LR07], the focus is on generic solutions of a system, since these are likely to be of greatest practical interest and can be computed more efficiently. Straightforward changes can be made to the above algorithms to yield full solutions, but at greater computational cost. The validity of Algorithm 1 follows from the correctness of the equational variant of QEPCAD which is proved in [McC01], the validity of the Brown-McCallum projection operation [Bro01], the correctness of the "open CAD" algorithm [McC93, LR07], and Proposition 2.

Second, it must be admitted that the more informative output specified by GSPS may take much longer to compute than the relatively simpler output provided by QEPCAD. The reason is that, for each cell $c \in \mathcal{D}$, QEPCAD can abort the stack construction process based at c and its descendants as soon as the truth value of ϕ^* on c is determined. However, it may require many more stack constructions based at c and its descendants to compute the desired number ν_c, rather than just determining its sign. The construction of the list $\mathcal{F}$ is likely to require a similar amount of additional work.

Third, in view of the additional work likely to be needed for complex problem instances, its full application may be useful only in relatively simple or special situations. For example, problems with a small number of unknown variables, or those having several unknowns but for which theory predicts a small number of solutions for each parameter vector, may be treatable by our algorithmic framework. Section 5 revisits Examples 1 and 2, demonstrating some concrete benefits of GSPS applied to these specific problems.

Fourth, even if one is faced with a complex problem, our algorithmic frame-

Algorithm 2: Construct Functional Expression List

$F \leftarrow \mathrm{CFEL}(\alpha, \mathcal{B}, B)$

Input: α is the sample point of a cell $c \in \mathbf{R}^s$, which is the root of a tree $\mathcal{T}_c$ of cells whose non-root nodes are partitioned into stacks. [We say c is at level s, its children are at level $s+1$, etc.] $\mathcal{B}$ is a non-empty list of vectors $\beta \in \mathbf{R}^k$ such that the set of sample points for the cells of $\mathcal{T}_c$ at level n is $\{(\alpha, \beta) \mid \beta \in \mathcal{B}\}$. B is a list $(B_{s+1}, \ldots, B_n)$, in which each B_i is an irreducible basis at level i such that $\prod_{p_i \in B_i} p_i$ is analytic delineable over every cell of T at level $i-1$.

Output: F is a list of functional expressions for the cells of $\mathcal{T}_c$ at level n over c. That is, for every cell c_n of $\mathcal{T}_c$ at level n, F contains the description of an analytic function $\theta : c \to \mathbf{R}^k$ whose graph is c_n.

1. **Initialization.** Set F to be the empty list.

2. **Construct every needed functional expression.** Remove a vector β from the list $\mathcal{B}$. For $i \leftarrow s+1$ to n perform the following actions. (Let $p_i(x, x_{s+1}, \ldots, x_i) \in B_i$ be such that $p_i(\alpha, \beta_{s+1}, \ldots, \beta_i) = 0$. Suppose β_i is the k_ith real root of $p_i(\alpha, \beta_{s+1}, \ldots, \beta_{i-1}, x_i)$ and that real valued functions $\theta_{s+1}, \ldots, \theta_{i-1}$, each with domain c, have been defined. Define $\theta_i : c \to \mathbf{R}$ by letting $\theta_i(x)$ be the k_ith real root of $p_i(x, \theta_{s+1}(x), \ldots, \theta_{i-1}(x), x_i)$.) Set $\theta \leftarrow (\theta_{s+1}, \cdots, \theta_n)$. Append θ to the list F.

3. **Is $\mathcal{B}$ non-empty?** If $\mathcal{B}$ is non-empty go back to Step 2; else Exit.

work applied to the problem in a partial way may yield valuable insight into the solutions.

Fifth, we are hopeful that further progress on the theory of equational projection set reduction may yield more efficiencies for the method. The next section reports on some limited progress in that direction.

We conclude this section with some simple examples.

Example 4. *Let $n = 2$ and $\phi^* \equiv (\exists y)[y^2 - x = 0]$. Given this input, algorithm* GSPS *yields descriptions of the two 1-cells $c_1 = (-\infty, 0)$ and $c_2 = (0, \infty)$ in $\mathcal{D}_1$. For c_1, $\nu = 0$; while for c_2, $\nu = 2$. Put $c = c_2$, for simplicity. Algorithm* CFEL *then produces a list $F_c = (\theta, \psi)$, where $\theta : c \to \mathbf{R}$ is given by $\theta(x) = -\sqrt{x}$ and $\psi : c \to \mathbf{R}$ is given by $\psi(x) = \sqrt{x}$. (Strictly speaking, $\theta(x)$ is expressed as "the first (smaller) real root of $y^2 - x$", etc.) Informally, for the first solution over c, the unknown y is expressed in terms of the parameter x by $y = \theta(x)$; and for the second solution over c, $y = \psi(x)$.*

Example 5. *Let $n = 3$ and $\phi^* \equiv (\exists y)(\exists z)[z^2 + y^2 + x^2 - 1 = 0 \wedge z - y = 0]$.*

Given this input, algorithm GSPS *yields descriptions of the three 1-cells:* $c_1 = (-\infty, -1)$, $c_2 = (-1, 1)$ *and* $c_3 = (1, \infty)$. *For* c_1 *and* c_3, $\nu = 0$; *while for* c_2, $\nu = 2$. *Put* $c = c_2$, *for simplicity. Algorithm* CFEL *then produces a list* $F_c = (\theta, \psi)$. *We have* $\theta = (\theta_2, \theta_3)$, *where* $\theta_2(x) = -\sqrt{(1-x^2)/2}$ *and* $\theta_3(x) = \theta_2(x)$. *We have* $\psi = (\psi_2, \psi_3)$, *where* $\psi_2(x) = \sqrt{(1-x^2)/2}$ *and* $\psi_3(x) = \psi_2(x)$. *(Again, strictly speaking,* $\theta_2(x)$ *is expressed as "the first (smaller) real root of* $2y^2 + x^2 - 1$*", etc.) Informally, for the first solution over* c, *the unknowns* y *and* z *are expressed in terms of the parameter* x *by* $y = \theta_2(x)$ *and* $z = \theta_3(x)$; *and for the second solution over* c, $y = \psi_2(x)$ *and* $z = \psi_3(x)$.

4 Progress on validating use of fully reduced equational projection

In this section we present variations of existing theorems relating to delineability. These are of help in justifying the use of the fully reduced equational projection operation beyond the first projection step in certain special situations, consistent with the framework described in the previous section.

In the first subsection we review some existing published work in this direction [MNDS20], noting some flaws. In the second subsection, we present new theorems relating to delineability, which partially rectify the flaws noted in the existing work. In the third subsection, we describe how the new theorems justify the use of the fully reduced projection operation for the second projection step under suitable assumptions.

4.1 Review of existing work, with flaws described

Section IV of [MNDS20] (intended as a report on work in progress) presents a delineability criterion which has the aim of justifying a reduction in the second projection operation when constraints $f = 0$ and $e = 0$ are available at levels n and $n-1$, respectively. This criterion uses a proposed notion of pointwise intersection order of two coprime multivariate polynomials with complex number coefficients. The notion proposed is recalled as follows. When $f, g \in \mathbf{C}[x_1, \ldots, x_n]$ are coprime and $p \in \mathbf{C}^n$, an affine change of coordinates is first made, if needed, to ensure $p = 0$ and $f(0, \ldots, 0, x_n)$ is not identically zero; hence $f(0, \ldots, 0, x_n)$ vanishes at $x_n = 0$ to some finite order $m \geq 0$. Then Hensel's lemma or the Weierstrass preparation theorem is applied to find a so-called *distinguished polynomial* $h \in \mathbf{C}[[x_1, \ldots, x_{n-1}]][x_n]$ in x_n (that is, h is monic in x_n and $h(0, \ldots, 0, x_n) = x_n^m$) associated to f near the origin. Finally, the intersection order of f and g at p is defined to be the order of the resultant $\mathrm{res}_{x_n}(h, g)$. (This concludes the recollection of the notion proposed from [MNDS20].) However, our present author group, slightly different from that of [MNDS20], have subsequently noticed three issues with the work reported in that section of the article.

The first issue is that the definition recalled above of the intersection order (or intersection multiplicity/number) of two coprime multivariate polynomials *at a point* is nowhere to be found in the available standard texts on algebraic geometry, such as [CLO05, Har92, Har77, Sha13]. In fact, no other definition of such a pointwise intersection order is found in such texts. While this in itself does not invalidate the proposed notion, it is consistent with the more serious problem explained below. Of course, the case $n = 2$ is well understood and is discussed at length in some of the references mentioned and other works. But the case $n > 2$ is conspicuous by its absence. (However, the intersection number of n polynomials in n variables, having only finitely many common solutions, at a common solution point is developed. Also, the notion of the intersection multiplicity of two hypersurfaces, having no common component, *along an irreducible sub-variety of their intersection* is also defined.)

The second, more serious, issue is that the concept of the intersection order of f and g at p defined in [MNDS20] is tacitly assumed to be independent of the coordinate system in both the very definition of the concept given and the proof of the delineability criterion (Theorem 7 of [MNDS20]) which uses the concept. Now such coordinate independence is indeed true in the case $n = 2$. However, it is not true in general. A counter-example is furnished, with $n = 3$ and $(x_1, x_2, x_3) = (p, q, x)$, by setting $f(p, q, x) = x^3 + px + q$ and $g(p, q, x) = \partial f/\partial x = 3x^2 + p$. Then f is a distinguished polynomial in x, and we have $\mathrm{res}_x(f, g) = 27q^2 + 4p^3$, which has order 2 at the origin in the (p, q)-plane. However, g is a distinguished polynomial in p, and we have $\mathrm{res}_p(f, g) = 2x^3 - q$, which has order 1 at the origin in the (q, x)-plane. (It is intuitively clear that the intersection multiplicity of f and g along the curve C of intersection $f = g = 0$ is 1, since $\boldsymbol{\nabla} f = (x, 1, 3x^2 + p)$ and $\boldsymbol{\nabla} g = (1, 0, 6x)$ are linearly independent vectors at each point of C. So, it seems that the former resultant, namely $\mathrm{res}_x(f, g)$, misrepresents the intuitive idea of the intersection multiplicity of f and g at the origin.)

The third and relatively minor issue is that the first sentence of the statement of the delineability criterion (Theorem 7 of [MNDS20]) referred to above apparently has a typographical error: e should be a polynomial in $x_1, \dots, x_{n-1}$ (not in $x_1, \dots x_n$ as written in [MNDS20]).

The next subsection aims to correct, as far as possible, these flaws. The foundation is to replace the unsatisfactory notion of pointwise intersection order introduced in [MNDS20] by a somewhat different, simpler, coordinate-independent concept. The relevant theory is, of necessity, completely redeveloped using the new concept.

4.2 New theorems relating to delineability

The theory in this subsection relies upon the notion of the order of a real analytic function $g(x_1, \dots, x_n)$ relative to the real variety V_f of another real analytic function $f(x_1, \dots, x_n)$ at a non-singular point of V_f. We will now formulate this relative order notion in precise terms.

Definition 3. *Let f and g be real analytic functions defined in a neighbourhood of a point $p \in \mathbf{R}^n$. Let V_f denote the real hypersurface defined by $f = 0$ near p, and suppose that p is a non-singular point of V_f; that is, we suppose $f(p) = 0$, but some first-order partial derivative of f, say $\partial f/\partial x_i$ does not vanish at p. Then, by the implicit function theorem, V_f is represented near p by the graph of a real analytic function $x_i = \phi(x_1, \dots, \hat{x}_i, \dots, x_n)$ (where $\hat{x}_i$ denotes omission of the variable x_i). We say that g has order $k \in \mathbf{N} \cup \{\infty\}$ relative to V_f at p if the order of the real analytic function*

$$g(x_1, \dots, x_{i-1}, \phi(x_1, \dots, \hat{x}_i, \dots, x_n), x_{i+1}, \dots, x_n) \tag{1}$$

at $(p_1, \dots, \hat{p}_i, \dots, p_n)$ is k. We shall denote the order of g relative to V_f at p by $\mathrm{ord}_p g|_{f=0}$. *In this notation, the expression $g|_{f=0}$ is a standard designation for* *the* restriction *of the function g to the submanifold V_f, near p. This restriction is expressed conveniently by means of local coordinates in Equation (1).*

Remark 5. *The above definition is independent of the coordinates used to represent the submanifold V_f near p. In particular, if also $\partial f/\partial x_j(p) \neq 0$, for $j \neq i$, and we use the real analytic function $x_j = \psi(x_1, \ldots, \hat{x}_j, \ldots, x_n)$ to represent V_f near p, then we obtain the same value for $\mathrm{ord}_p g|_{f=0}$. Relevant details can be found in Appendix 2 of [Whi72] and Chapter 2 of [McC85].*

Remark 6. *A notion similar to, and more general than, the concept of relative order defined above was presented informally in a talk by C. W. Brown at SYNASC-2023 (in the Special Session in Honour of Professor James Davenport). Brown's notion, called restricted order, is a key component of on-going collaborative development of a new projection operator for CAD.*

The following straightforward result will be useful.

Proposition 3. *With the notation and assumptions of the above definition,* $\mathrm{ord}_p g|_{f=0} > 0$ *if and only if* $g(p) = 0$.

Proof. Observe that $p_i = \phi(p_1, \ldots, \hat{p}_i, \ldots, p_n)$, since $p \in V_f$ and V_f is the graph of ϕ, near p, by the assumptions within the definition. Suppose first that $g(p) = 0$. Therefore $g(p_1, \ldots, p_{i-1}, \phi(p_1, \ldots, \hat{p}_i, \ldots, p_n), p_{i+1}, \ldots, p_n) = 0$ by the above observation. Hence the order of the analytic function defined by Equation (1) is positive or infinite, from which $\mathrm{ord}_p g|_{f=0} > 0$ follows, by definition.

Conversely, suppose $\mathrm{ord}_p g|_{f=0} > 0$. Then, the real analytic function defined by Equation (1) vanishes at $(p_1, \ldots, \hat{p}_i, \ldots, p_n)$, by definition. But our initial observation says $p_i = \phi(p_1, \ldots, \hat{p}_i, \ldots, p_n)$. Therefore, $g(p) = 0$. $\square$

We now proceed to state and prove a trilogy of theorems relating to delineability which use the concept of relative order introduced above. It is suggested that, for a first reading, the reader may like to closely examine the statements of these theorems, and to leave the study of the detailed proofs for another reading. In reading the theorem statements, it may be helpful to keep in mind the desired application to reduction of the second projection step. Our first significant result is a variation and consequence of the key lifting theorem (Theorem 2) from [McC98]. Its wording is close to that of Theorem 7 of [MNDS20], keeping in mind that the key hypothesis that "d and e are intersection order invariant in σ" in Theorem 7 of [MNDS20] is replaced by the assumption that "the order of $d|_{e=0}$ is invariant in σ" in the theorem below.

Theorem 2. *Let $f(x_1, \ldots, x_n) \in \mathbf{R}[x_1, \ldots, x_n]$ be a real polynomial of positive degree in x_n, and let $e(x_1, \ldots, x_{n-1}) \in \mathbf{R}[x_1, \ldots, x_{n-1}]$ be a real polynomial of positive degree in x_{n-1}. Let $d(x_1, \ldots, x_{n-1})$ and $a(x_1, \ldots, x_{n-1})$ be the discriminant and the leading coefficient of f, respectively, both with respect to x_n, suppose that d is a non-zero polynomial, and suppose that e and d are relatively prime. Let σ be a connected submanifold of $\mathbf{R}^{n-1}$ contained in the real hypersurface (variety) defined by $e = 0$ in which a is sign-invariant and f vanishes* *identically at no point. Suppose that σ contains no singular point of the real hypersurface defined by $e = 0$ and that the order of $d|_{e=0}$ is invariant in σ.*

Then f is analytic delineable on σ.

Proof. In the case that d vanishes nowhere in σ, the analytic delineability of f on σ follows immediately by Theorem 3.1 of [Bro01] (which yields the degree-invariance of f on σ) and Theorem 2 of [McC98] (which delivers the analytic delineability of f, using its degree-invariance and non-identically-vanishing on σ, and the order-invariance of d in σ). So, henceforth assume that d vanishes at some point of σ. Then, since the order of $d|_{e=0}$ is invariant in σ, by assumption, d must vanish everywhere in σ, by Proposition 3. Since σ is connected, it suffices to show that f is analytic delineable on σ near an arbitrary point p of σ.

Let s be the dimension of σ. The case $s = n-1$, in which σ is just an open subset of $\mathbf{R}^{n-1}$, cannot arise by our assumption that d is non-zero, and our previous conclusion that d vanishes everywhere in σ. By our assumption that e and d are relatively prime, our previous conclusion that d vanishes everywhere in σ, and the general form of Bezout's theorem ([Abh90], Lecture 30), we may deduce that the order of $d|_{e=0}$ at p is finite and positive, and that $s \neq n-2$. By excluding the trivial case $s = 0$, we therefore have $1 \leq s \leq n-3$.

Choose a coordinate system $\Phi : U \to U'$ about p, with U and U' suitable neighbourhoods of p and the origin, respectively, with Φ informally denoted by $(y_1, \ldots, y_{n-1})$, such that σ is defined near p by the equations $y_{s+1} = 0, \ldots, y_{n-1} = 0$ in the new coordinate system (by Theorem 2.2 of [McC88]). We remark that the choice of such a coordinate system entails $\Phi(p) = 0$.

We denote by $f'(y_1, \ldots, y_{n-1}, x_n)$, $d'(y_1, \ldots, y_{n-1})$, $a'(y_1, \ldots, y_{n-1})$ and $e'(y_1, \ldots, y_{n-1})$ the polynomials f, d, a and e expressed in the new coordinates. (We stress that the "dash" superscript does not denote derivative here.) Then d', a' and e' are real analytic functions defined near 0 (to be precise, defined in U'), and f' is a polynomial in x_n whose coefficients are real analytic functions defined near 0 (to be precise, defined in U'). Let T denote the s-dimensional linear subspace of $\mathbf{R}^{n-1}$ defined by $y_{s+1} = 0, \ldots, y_{n-1} = 0$. By the assumptions, T is contained in the real hypersurface defined by $e' = 0$, near 0, a' is sign-invariant in T, near 0, and f' vanishes identically at no point of T, near 0. Moreover, 0 is not a singular point of the real hypersurface $e' = 0$ and, by the proof of Theorem 2.2.2 of [McC85] and a previous conclusion, the order of $d'|_{e'=0}$ is invariant, finite and positive in T, near 0.

It will be helpful to introduce simplified notation for the variables. We shall put $x = (y_1, \ldots, y_s)$ and $y = (y_{s+1}, \ldots, y_{n-1})$. Our next task is to show that we may deduce $\partial e'/\partial y_j(0) \neq 0$, for some j, with $s+1 \leq j \leq n-1$. Suppose, for the sake of contradiction, that $\partial e'/\partial y_j(0) = 0$, for every j, with $s+1 \leq j \leq n-1$. Then, since 0 is a non-singular point of $e' = 0$, we must have $\partial e'/\partial y_i(0) \neq 0$, for some i, with $1 \leq i \leq s$. By continuity of $\partial e'/\partial y_i$, we may refine the neighbourhood U' of 0 as needed to ensure that $\partial e'/\partial y_i(q) \neq 0$, for all $q \in U'$. Put

$$T' = \{(x, y) \in U' \mid e'(x, y) = 0 \wedge y_{s+1} = 0 \wedge \cdots \wedge y_{n-1} = 0\}.$$

We claim that T' is an $(s-1)$-dimensional submanifold of $\mathbf{R}^{n-1}$. We prove our claim as follows. Take an arbitrary point $q \in T'$. Since the ith component of $de'(q)$ is non-zero (by the property of U' ensured above), and $dy_j(q)$ is the unit vector in the positive direction of the y_j-axis, for $s+1 \leq j \leq n-1$, it follows that the vectors $de'(q), dy_{s+1}(q), \ldots, dy_{n-1}(q)$ are linearly independent. Therefore, the set T' defined above is an $(s-1)$-dimensional submanifold of $\mathbf{R}^{n-1}$, by definition of submanifold [McC88, Whi72]. Our claim is proved. Yet our s-dimensional linear subspace T clearly satisfies $T \cap U' \subseteq T'$, by definition of T and the containment of $T \cap U'$ within the hypersurface $e' = 0$ noted above. In other words, we have one submanifold of $\mathbf{R}^{n-1}$, namely $T \cap U'$, contained in another, namely T'; yet the dimension of the former submanifold is greater than that of the latter. This contradicts a basic dimension property of submanifolds. We may conclude that indeed $\partial e'/\partial y_j(0) \neq 0$, for some j, with $s+1 \leq j \leq n-1$.

From here on, with slight abuse of notation, but hopefully no confusion, we drop the "dash" decoration on f, e and d. We denote the $(n-s-2)$-tuple of variables $(y_{s+1}, \ldots, \hat{y}_j, \ldots, y_{n-1})$ by η. (Here $\hat{y}_j$ denotes omission of y_j.) Recall that $e(x, y)$ is a real analytic function defined in U', $e(0, 0) = 0$, $\partial e/\partial y_j(0) \neq 0$, and the order of $d|_{e=0}$ is invariant, positive and finite in $T \cap U'$. By the implicit function theorem, there is an open box $B = \prod_{i=1}^{n-1} I_i$ centred at the origin in $\mathbf{R}^{n-1}$, and a function $\phi : (B^* = \prod_{i \neq j} I_i) \to I_j$ such that ϕ is analytic, $B \subseteq U'$, and for all $(x, y) \in B$, we have $e(x, y) = 0$ if and only if $y_j = \phi(x, \eta)$.

For $(x, \eta) \in B^*$, put

$$f^*(x, \eta, x_n) = f(x, y_{s+1}, \ldots, y_{j-1}, \phi(x, \eta), y_{j+1}, \ldots, y_{n-1}, x_n).$$

Then $f^*(x, \eta, x_n)$ is a polynomial in x_n whose coefficients are real analytic functions defined in B^*. Let us denote by $d^*(x, \eta)$ and $a^*(x, \eta)$ the discriminant and leading coefficient of $f^*(x, \eta, x_n)$, respectively, both with respect to x_n. It is apparent that

$$d^*(x, \eta) = d(x, y_{s+1}, \ldots, y_{j-1}, \phi(x, \eta), y_{j+1}, \ldots, y_{n-1}),$$

for all $(x, \eta) \in B^*$, and a similar property holds for a^*. Let us set T^* to be the s-dimensional linear subspace of $\mathbf{R}^{n-2}$ (the space of the coordinates (x, η)) defined by $\eta = 0$. It follows from the equation for d^* immediately above, together with the invariance and finiteness of the order of $d|_{e=0}$ in $T \cap U'$, that $d^*(x, \eta)$ is order-invariant, of finite order, in $T^* \cap B^*$. Moreover, a^* is sign-invariant in $T^* \cap B^*$, and f^* vanishes identically at no point of $T^* \cap B^*$. Therefore, by Theorem 7.1 of [McC01], which is a slight generalization of Theorem 2 of [McC98], and an analogous generalization of Theorem 3.1 of [Bro01], with the latter applied first to yield degree-invariance of f^*, f^* is analytic delineable on $T^* \cap B^*$. It follows that f is analytic delineable on σ near p. $\square$

Close inspection of the statement of Theorem 2 reveals that, to be useful for iterated equational projection, we need at least one more theorem which would yield as a conclusion the property related to a key hypothesis of Theorem 2 about the invariance of the order of $d|_{e=0}$ in the submanifold σ. In other words, we would like and aim to have a slight strengthening of Theorem 1, which would yield as conclusion, in the notation of that theorem, invariance of the order of g relative to the real hypersurface $f = 0$ in each section of f over S which contains no singular point of $f = 0$. However, despite some effort invested to find such a desired new theorem, we have succeeded in proving only a rather special fact in this direction (see Remark 7 below). In the statement of the next theorem, we have used notation similar to that of Theorem 1. For a second reading (of the following theorem statement), the reader is advised mentally to make the substitutions $n-1$ for n, e for f, d for g and ρ for r; this may help the reader to see the intended application of the following theorem in conjunction with Theorem 2.

Theorem 3. *Let f and g be real polynomials in $x_1, \ldots, x_n$, of positive degrees in x_n, let $r \in \mathbf{R}[x_1, \ldots, x_{n-1}]$ be the resultant of f and g with respect to x_n, and suppose that r is a non-zero polynomial. Let S be a connected submanifold of $\mathbf{R}^{n-1}$ on which f is analytic delineable and in which r is order-invariant. Suppose moreover that the codimension of S in $\mathbf{R}^{n-1}$ is not more than 1.*

Then, for every section σ of f over S which contains no singular point of the real hypersurface $f = 0$, the analytic function $g|_{f=0}$ is order-invariant in σ.

Remark 7. *The special hypothesis, which we conjecture could be removed, is that concerning the codimension of S. Since we do not (yet) see how to prove the theorem without this special hypothesis in force, we present the following proof which makes crucial use of this hypothesis, amongst the other ones.*

Proof. Let σ be a section of f over S determined by the analytic function $\theta : S \to \mathbf{R}$. By connectedness of σ, it suffices to show that $g_{f=0}$ is order-invariant in σ near an arbitrary point $(p, \theta(p))$ of σ. Let $(p, \theta(p))$ be such a point. That $g_{f=0}$ is order-invariant in σ, near $(p, \theta(p))$ follows easily by continuity of g in the case that $g(p, \theta(p)) \neq 0$. So, henceforth we will assume that $g(p, \theta(p)) = 0$. Therefore, $r(p) = 0$, and hence r vanishes everywhere in S, by the order-invariance assumption. Let s be the dimension of S. The case $s = n-1$ (that is, the case in which the codimension of S in $\mathbf{R}^{n-1}$ is 0) is one in which S is an open subset of $\mathbf{R}^{n-1}$. This case cannot arise by our assumption that r is a non-zero polynomial and our previous conclusion that r vanishes everywhere in S. Therefore, by our codimension assumption, we have $s = n-2$ (that is, S has codimension 1 in its space).

By Theorem 2.2 of [McC88], we choose coordinates designated informally by $(y, y_n - 1) = (y_1, \ldots, y_{n-2}, y_{n-1})$ about p such that S is defined locally by the equation $y_{n-1} = 0$ in the new coordinate system, which maps p to 0, by definition. With slight abuse of notation, we denote by $f'(y, y_{n-1}, x_n)$, $g'(y, y_{n-1}, x_n)$ and $r'(y, y_{n-1})$ the polynomials f, g and r expressed in (y, y_{n-1})-coordinates. Then r' is a real analytic function defined near 0, and f' and g' are so-called pseudopolynomials in x_n near 0, that is, polynomials in x_n whose coefficients are real analytic. Moreover, by Theorem 2.1 of [McC88], r' is order-invariant in T near 0, where T is the hyperplane in $\mathbf{R}^{n-1}$ defined by $y_{n-1} = 0$. For simplicity, we hereafter drop the “dash” decoration on f, g and r, trusting that no confusion will arise. The analytic function θ, for its part, is now similarly considered to be defined in T (identified with $\mathbf{R}^{n-2}$), near 0, with graph denoted by σ; and we may assume that $\theta(0) = 0$, without loss of generality.

We claim that, in consequence of the assumption that σ contains no singular point of the real hypersurface $f = 0$, near the origin, we may conclude that $\partial f/\partial y_{n-1}(0) \neq 0$ or $\partial f/\partial x_n(0) \neq 0$. Suppose for the sake of contradiction that this is not the case. Then, by the non-singularity assumption, we must have $\partial f/\partial y_i(0) \neq 0$, for some i, with $1 \leq i \leq n-2$. Let σ_i denote the intersection of the real hypersurface $f = 0$ with all the hyperplanes $y_1 = 0, \ldots, y_{i-1} = 0, y_{i+1} = 0, \ldots, y_{n-1} = 0$, in $\mathbf{R}^n$, near the origin. Then $\sigma_i \subseteq \sigma$ is an analytic 1-submanifold of $\mathbf{R}^n$, near the origin, which lies in the plane of the y_i and x_n coordinates. The tangent space (line) of σ_i at 0 is the span of the vector $(0, \ldots, 0, 1)$, by Lemma 5C, Appendix II, of [Whi72]. This contradicts the analyticity at 0 of the function θ whose graph is σ. The claim is proved.

We shall focus attention on the zero set of f near the origin using an extension of Hensel's lemma ([Abh90], exercises on pages 92 and 95, with relaxation of monicity of f; or [McC99], Theorem 3.1, for the 3-variable case). Denote by d the degree of f with respect to x_n, and by m the multiplicity of the root 0 of $f(0, 0, x_n)$. (Recall that, in the argument list for $f(0, 0, x_n)$, the first 0 represents the first $n-2$ arguments, while the second represents $y_{n-1} = 0$.) By the extended Hensel's lemma just cited, there exists an open box $B \times (-\delta, \delta)$ centred at the origin in $\mathbf{R}^{n-1}$ and pseudopolynomials in x_n, $q^*(y, y_{n-1}, x_n)$ of degree $d - m$ and $h^*(y, y_{n-1}, x_n)$ of degree m, whose coefficients are analytic in $B \times (-\delta, \delta)$, such that $f = q^* h^*$, $h^*(0, 0, x_n) = x_n^m$, and $q^*(0, 0, 0) \neq 0$. Since $q^*(0, 0, 0) \neq 0$ and q^* is analytic, hence continuous, in $B \times (-\delta, \delta)$, we may refine the box $B \times (-\delta, \delta)$ and choose $\epsilon > 0$ such that $q^* \neq 0$ everywhere in $B \times (-\delta, \delta) \times (-\epsilon, \epsilon)$. By continuity of θ, after further refining B if necessary, we may moreover assume that for all $y \in B$, $\theta(y) \in (-\epsilon, \epsilon)$. It follows by delineability of f and the above properties of q^*, h^* and θ that, for all $y \in B$, $\theta(y)$ is a root of $h^*(y, 0, x_n)$ of multiplicity m, hence the *unique* root of $h^*(y, 0, x_n)$. By Theorem 3 of [Loo82] we have

$$r = \mathrm{res}_{x_n}(q^*, g)\mathrm{res}_{x_n}(h^*, g).$$

By hypothesis about the order-invariance of r in S (hence in $T \cap (B \times (-\delta, \delta))$, in the new coordinates), and Lemma A.3 of [McC88], $v(y, y_{n-1}) = \mathrm{res}_{x_n}(h^*, g)$ is order-invariant in $T \cap (B \times (-\delta, \delta))$. Let us put $\mu = \mathrm{ord}_{(0,0)} v$.

To arrive finally at our desired conclusion concerning the order of g relative to the real hypersurface $f = 0$ along σ we need to consider separately the two cases, at least one of which must occur as proved above, namely $\partial f/\partial x_n(0) \neq 0$ and $\partial f/\partial y_{n-1}(0) \neq 0$. For the former case the remainder of the proof is relatively short. In this case, we have $m = 1$ and $h^*(y, y_{n-1}, x_n) = x_n - \phi(y, y_{n-1})$, for some function ϕ analytic in $B \times (-\delta, \delta)$. Therefore, by Theorem 1 of [Loo82], $v(y, y_{n-1}) = g(y, y_{n-1}, \phi(y, y_{n-1}))$, for all $(y, y_{n-1}) \in B \times (-\delta, \delta)$. Hence, by definition, the order of $g|_{f=0}$ is invariant in σ, near the origin.

On the other hand, suppose that $\partial f/\partial y_{n-1}(0) \neq 0$. We aim to prove that the order of $g|_{f=0}$ is invariant along σ near 0, using this assumption. To prepare for this, it will be convenient to assume that the function θ is identically zero, which we may do without loss of generality by applying a suitable analytic change of coordinates ($y' = y$, $y'_{n-1} = y_{n-1}$, $x'_n = x_n - \theta(y)$). By the order-invariance of v in $T \cap (B \times (-\delta, \delta))$, and Lemma 4.4 of [MPP19] (using (3) implies (2)), after refining $B \times (-\delta, \delta)$ about 0 if necessary, we may find an analytic function u in $B \times (-\delta, \delta)$, with $u(y, y_{n-1}) \neq 0$ for all $(y, y_{n-1}) \in B \times (-\delta, \delta)$, such that $v(y, y_{n-1}) = y_{n-1}^{\mu} u(y, y_{n-1})$, for all $(y, y_{n-1}) \in B \times (-\delta, \delta)$ (where μ is the invariant order of v in $T \cap (B \times (-\delta, \delta))$, previously defined). Let b be an arbitrary element of B. Then, by (a slight extension of) the definition of the intersection multiplicity at the origin of two plane analytic curves in [Abh90] (page 126), we may deduce that the intersection multiplicity at $(0, 0)$ of the plane analytic curves $f(b, y_{n-1}, x_n) = 0$ and $g(b, y_{n-1}, x_n) = 0$ is μ. We wish to use the invariance of the intersection multiplicity at $(0, 0)$ of these two plane analytic curves under interchange of the coordinates, as discussed in [Abh90], Lecture 17. To this end, noting that neither plane analytic curve is necessarily polynomial in y_{n-1}, we apply the Weierstrass preparation theorem [Abh90], Lecture 16, to write

$$f(y, y_{n-1}, x_n) = q'(y, y_{n-1}, x_n) h'(y, y_{n-1}, x_n),$$

in $B \times (-\delta, \delta) \times (-\epsilon, \epsilon)$, and

$$g(b, y_{n-1}, x_n) = t'(y_{n-1}, x_n) x_n^{\gamma} k'(y_{n-1}, x_n), \tag{2}$$

in $(-\delta, \delta) \times (-\epsilon, \epsilon)$, where q', t' are everywhere non-vanishing in their respective domains, h', k' are distinguished polynomials in y_{n-1}, and x_n^{γ} is the highest power of x_n which divides $g(b, y_{n-1}, x_n)$. (For the application of Weierstrass to $f(y, y_{n-1}, x_n)$ we need the multivariate version of the theorem described in [Abh90], Lecture 16, Remark 5, page 120.) In fact, by our assumption that $\partial f / \partial y_{n-1}(0) \neq 0$, we have $h'(y, y_{n-1}, x_n) = y_{n-1} - \psi(y, x_n)$, for some analytic function ψ defined in $B \times (-\epsilon, \epsilon)$. Therefore, by the invariance of the intersection multiplicity at $(0, 0)$ of the two plane analytic curves $f(b, y_{n-1}, x_n) = 0$ and $g(b, y_{n-1}, x_n) = 0$ under interchange of the coordinates [Abh90] (Lecture 17, page 127), we have

$$\mu = \mathrm{ord}_{x_n} \mathrm{res}_{y_{n-1}} (h'(b, y_{n-1}, x_n), x_n^{\gamma} k'(y_{n-1}, x_n)).$$

By Equation (2), we have

$$g(b, \psi(b, x_n), x_n) = t'(\psi(b, x_n), x_n) x_n^{\gamma} k'(\psi(b, x_n), x_n),$$

and by Theorem 1 of [Loo82],

$$\mathrm{res}_{y_{n-1}} (h'(b, y_{n-1}, x_n), x_n^{\gamma} k'(y_{n-1}, x_n)) = x_n^{\gamma} k'(\psi(b, x_n), x_n).$$

Therefore, since $t' \neq 0$, $\mathrm{ord}_{x_n} g(b, \psi(b, x_n), x_n) = \mu$. Application of Lemma 4.4 of [MPP19](using (1) implies (3)) yields the desired invariance of the order of $g(y, \psi(y, x_n), x_n)$ in $B \times (-\epsilon, \epsilon)$, from which the invariance of the order of $g|_{f=0}$ along σ, near the origin, follows by definition. $\square$

We complete our trilogy of theorems relating to delineability with another variation of Theorem 1. The following variant is, like Theorem 3, also a slight strengthening of Theorem 1, achieved here by weakening the hypothesis a little.

Theorem 4. *Let f and g be elements of $\mathbf{R}[x_1, \ldots, x_n]$, of positive degrees in x_n, let r be the resultant of f and g with respect to x_n, and suppose $r \neq 0$. Let e be an element of $\mathbf{R}[x_1, \ldots, x_{n-1}]$, of positive degree in x_{n-1}, and suppose that e and r are relatively prime. Let $\sigma \subseteq \mathbf{R}^{n-1}$ be a connected submanifold contained in the real hypersurface $e = 0$ on which f is analytic delineable. Suppose that σ contains no singular point of $e = 0$ and that the order of $r|_{e=0}$ is invariant in σ.*

Then g is sign-invariant in each section of f over σ.

Proof. This proof will be modelled in part on that of Theorem 2, and in part on that of Theorem 3. Since many relevant details already appear in the proofs of those theorems, this proof will be more concise. Let $\hat{\sigma}$ be a section of f over σ determined by the analytic function $\theta : \sigma \to \mathbf{R}$. By analogy with the proof of Theorem 3, it suffices to show that g is sign-invariant in $\hat{\sigma}$ near an arbitrary point $(p, \theta(p))$ of $\hat{\sigma}$, and we need to consider further only the case in which r vanishes everywhere in σ. Moreover, where s denotes the dimension of σ, we may assume that $1 \leq s \leq n-2$.

By Theorem 2.2 of [McC88], we choose an analytic coordinate system Φ about p in $\mathbf{R}^{n-1}$, informally denoted by $(y_1, \ldots, y_{n-1})$, such that σ is defined locally by the equations $y_{s+1} = 0, \ldots, y_{n-1} = 0$ in the new coordinate system. Note that the choice of such a coordinate system entails $\Phi(p) = 0$. With slight abuse of notation, we denote by $f(y_1, \ldots, y_{n-1}, x_n)$, $g(y_1, \ldots, y_{n-1}, x_n)$, $r(y_1, \ldots, y_{n-1})$ and $e(y_1, \ldots, y_{n-1})$ the original polynomials f, g, r and e, respectively, expressed now in the new coordinates. We remind the reader that f and g, so expressed, are pseudopolynomials in x_n, near 0, and r and e, so expressed, are analytic near 0. Moreover, where T is the linear subspace of $\mathbf{R}^{n-1}$ defined by $y_{s+1} = 0, \ldots, y_{n-1} = 0$, the hypersurface $e = 0$ has no singular point in T, near 0, and the order of the analytic function $r|_{e=0}$ is invariant in T, near the origin. The analytic function θ, for its part, is now considered to be defined in T (identified with $\mathbf{R}^s$), near 0, with graph $\hat{\sigma}$, and we may assume that $\theta(0) = 0$, without loss of generality.

By analogy with the proof of Theorem 2, we introduce simplified notation for the variables: namely, we put $x = (y_1, \ldots, y_s)$ and $y = (y_{s+1}, \ldots, y_{n-1})$. It follows analogously, by the non-singularity assumption, that $\partial e / \partial y_j(0) \neq 0$, for some j, with $s+1 \leq j \leq n-1$. We denote the $(n-s-2)$-tuple of variables $(y_{s+1}, \ldots, \hat{y}_j, \ldots, y_{n-1})$ by η. (Here $\hat{y}_j$ denotes omission of y_j.) By the implicit function theorem, there is an open box $B = \prod_{i=1}^{n-1} I_i$ centred at the origin in $\mathbf{R}^{n-1}$ and a function $\phi : (B^* = \prod_{i \neq j} I_i) \to I_j$ such that ϕ is analytic, and for all $(x, y) \in B$, we have $e(x, y) = 0$ if and only if $y_j = \phi(x, \eta)$.

For $(x, \eta) \in B^*$, put

$$f^*(x, \eta, x_n) = f(x, y_{s+1}, \ldots, y_{j-1}, \phi(x, \eta), y_{j+1}, \ldots, y_{n-1}, x_n),$$

$$g^*(x, \eta, x_n) = g(x, y_{s+1}, \ldots, y_{j-1}, \phi(x, \eta), y_{j+1}, \ldots, y_{n-1}, x_n).$$

Then f^* and g^* are polynomials in x_n whose coefficients are real analytic functions defined in B^*. Let us denote by $r^*(x, \eta)$ the resultant of $f^*(x, \eta, x_n)$ and $g^*(x, \eta, x_n)$ with respect to x_n. It is apparent that

$$r^*(x, \eta) = r(x, y_{s+1}, \ldots, y_{j-1}, \phi(x, \eta), y_{j+1}, \ldots, y_{n-1}),$$

for all $(x, \eta) \in B^*$. Let us put T^* equal to the s-dimensional linear subspace of $\mathbf{R}^{n-2}$ (the space of the coordinates (x, η)) defined by $\eta = 0$. It follows from the equation for r^* immediately above, together with the invariance and finiteness of the order of $r|_{e=0}$ in T near the origin, that r^* is order-invariant, of finite order, in $T^* \cap B^*$ (after suitable refinement of B^*, if necessary). Clearly, f^* is analytic delineable on $T^* \cap B^*$, and $\hat{\sigma}$ (suitably restricted) may be considered to be a section of f^* over $T^* \cap B^*$. Therefore, by Theorem 7.2 of [McC01], g^* is sign-invariant in $\hat{\sigma}$, near the origin. Hence g is sign-invariant in $\hat{\sigma}$, near $(p, \theta(p))$. $\square$

4.3 Reducing the second projection step

Let $n \geq 3$ and let ϕ^* be an input formula as specified in Algorithm 1. Starting with $J_n = A$, the set of polynomials occurring in ϕ^*, Step 3 of this algorithm repeatedly computes semi-restricted equational projection sets $J_{j-1} = P^*_{E_j}(J_j)$; but it is remarked that, where possible, fully restricted projection sets $P_{E_j}(J_j)$ should be used instead, for increased efficiency. In fact, as recalled in Section 2.1, existing theory [McC99] already permits the use of the fully restricted projection for the first projection step; and an elementary argument shows that it can also be used for the "last" projection (of J_2, see Proposition 1). The present subsection proves a result, combining the theorems of the previous subsection, which states conditions under which the fully restricted projection operation can also be used for the second projection step.

We first recall some more of the notation of Algorithm 1, and make some basic assumptions. Suppose $f = 0$, with $f \in A$ assumed to be primitive and squarefree, is a level n constraint of the input formula ϕ^*, with $d = \mathrm{discr}_{x_n}(f)$ and $a = \mathrm{ldcf}_{x_n}(f)$ non-zero polynomials. We put $E_n = \{f\}$ and $J_{n-1} = P_{E_n}(J_n)$. Then, by definition, J_{n-1} is the union of $\mathrm{cont}(J_n)$ and $P_F(B)$, where B is the finest squarefree basis for $\mathrm{prim}(A)$ and F is the set of irreducible factors of f. Suppose that $e \in \mathbf{Z}[x_1, \ldots, x_{n-1}]$ is a product of distinct irreducible projection factors at level $n-1$, that $e = 0$ is a constraint of ϕ^*, and that e and a are coprime. (Recall that $e = 0$ may be either an explicit or implicit constraint.) Put $E_{n-1} = \{e\}$, let E denote the set of irreducible factors of e, and set $J_{n-2} = P_{E_{n-1}}(J_{n-1})$. We shall assume that f and e are *well-placed*: by this we mean that either $\mathrm{discr}_{x_n}(f)$ is a product of elements of E, or e and $\mathrm{discr}_{x_n}(f)$ are coprime.

In order to state our next result we require one more concept: we shall say that an element $g \in B \setminus F$ is *well-positioned* with respect to F and E if either $\mathrm{res}_{x_n}(f, g)$ is a product of elements of E, or e and $\mathrm{res}_{x_n}(f, g)$ are coprime.

Theorem 5. *Let $S \subseteq \mathbf{R}^{n-2}$ be a submanifold of codimension at most 1, suppose that each element of J_{n-2} is order-invariant in S, and suppose that S contains no point at which e vanishes identically.*

Then e is analytic delineable on S. Moreover, f is analytic delineable on each section σ of e over S which contains no singular point of $e = 0$ and no point at which f vanishes identically, and each element of B not in F which is well-positioned with respect to F and E is sign-invariant in every section of f over each such σ.

Proof. The fact that e is analytic delineable on S and order-invariant in each of its sections over S follows from the assumptions by Theorem 3.1 of [Bro01] and Theorem 2 of [McC98]. The detailed reasoning is as follows. By an assumption, each element of J_{n-2} is order-invariant in S. Since each of $\delta = \mathrm{discr}_{x_{n-1}}(e)$ and $\alpha = \mathrm{ldcf}(e)$ is a product of certain elements of J_{n-2}, by definition of this projection set, δ is order-invariant in S (by Lemma A.3 of [McC88]) and α is sign-invariant in S. Another assumption states that e vanishes identically at no point of S. Therefore, by Theorem 3.1 of [Bro01], e is degree-invariant on S. Hence, by Theorem 2 of [McC98], e is analytic delineable on S and is order-invariant in each of its sections over S.

Let σ be a section of e over S which contains no singular point of the real hypersurface $e = 0$ and no point at which f vanishes identically. We now show that a is sign-invariant in σ. By assumption, e and a are coprime. By Theorem 3, in which we replace n by $n-1$, and put $f = e$ and $g = a$ (hence $r = \mathrm{res}_{x_{n-1}}(e, a)$), the analytic function $a|_{e=0}$ is order invariant in σ, since $r \neq 0$ is order-invariant in S, by hypothesis (and Lemma A.3 of [McC88], if either e or a is reducible, or both are). Hence, a is sign-invariant in σ, by Proposition 3 and the connectedness of σ.

Next we show that f is analytic delineable on σ. By assumption, f and e are well-placed. So, there are two cases to consider. In the first case, d is a product of elements of E. Since e is order-invariant in σ (shown above), it follows by Lemma A.3 of [McC88] that d is order-invariant in σ. Therefore, by Theorem 3.1 of [Bro01], f is degree-invariant in σ. Hence, by Theorem 2 of [McC98], f is analytic delineable on σ. In the second case, e and d are coprime. By Theorem 3, in which we replace n by $n-1$, and put $f = e$ and $g = d$ (hence $r = \mathrm{res}_{x_{n-1}}(e, d)$), the analytic function $d|_{e=0}$ is order invariant in σ, since $r \neq 0$ is order-invariant in S, by hypothesis (and Lemma A.3 of [McC88], if either e or d is reducible, or both are). Therefore, by Theorem 2, f is analytic delineable on σ.

Now let $g \in B \setminus F$, with g assumed to be well-positioned with respect to F and E. There are two cases to consider. For the first case, we assume that $\mathrm{res}_{x_n}(f, g)$ is a product of elements of E. Recall, as was shown above, that e is order-invariant in σ. Therefore, by Lemma A.3 of [McC88], each element of E is order-invariant in σ, and hence $\mathrm{res}_{x_n}(f, g)$ is order-invariant in σ. Therefore, by Theorem 1 (Section 2.1), g is sign-invariant in every section of f over σ.

For the second case, we assume that e and $\mathrm{res}_{x_n}(f, g)$ are coprime. Let f_i be an irreducible factor of f, of positive degree in x_n. Then $r_i := \mathrm{res}_{x_n}(f_i, g)$

and e are coprime, since r_i is a factor of r, and $r_i \in J_{n-1}$, by definition. By Theorem 3, in which we replace n by $n-1$, and put $f = e$ and $g = r_i$ (hence $r = \mathrm{res}_{x_{n-1}}(e, r_i)$), the analytic function $r_i|_{e=0}$ is order invariant in σ, since $r \neq 0$ is order-invariant in S, by hypothesis (and Lemma A.3 of [McC88], if e or r_i is reducible, or both are). Finally, by Theorem 4, g is sign-invariant in every section of f_i over σ, from which it follows that g is sign-invariant in every section of f over σ. $\square$

We conclude this subsection with a few remarks. First, let us carefully consider the two assumptions of Theorem 5 that "S contains no point at which e vanishes identically" and "σ ... contains ... no point at which f vanishes identically". The reader may wonder if these are appropriate, since of the coefficients of f, only the leading one is placed into the first projection set J_{n-1}, and likewise for e in relation to J_{n-2}. So, why is it reasonable to expect that these assumptions will hold in practical applications of Theorem 5? The reason is that our use of the Brown-McCallum projection operator entails the analysis and solution of projection factor coefficient systems prior to the lifting phase of CAD. The solution of such systems is required only when it is determined that there are finitely many such solutions, as the projection operator fails otherwise. Moreover, such solutions – the isolated points at which projection factors vanish identically – must be "added" to the CAD progressively during the stack construction phase [Bro01]. So, in practical applications of Theorem 5, the submanifolds S and σ (which represent arbitrary cells of the CADs of $\mathbf{R}^{n-2}$ and $\mathbf{R}^{n-1}$, respectively) will satisfy the assumptions quoted above.

Second, the assumptions of Theorem 5 that "S has codimension at most 1" and "σ ... contains no singular point of $e = 0$" are admittedly restrictive. So, we offer here only a partial validation of the use of the fully reduced projection for the first two steps, given the availability of the equational constraints $f = 0$ and $e = 0$ stipulated. This is enough to justify the use of the fully reduced projection for all $n \leq 4$, provided that the codimension and non-singularity conditions hold in the case $n = 4$. Also, as we show in the next section, there are specific applications where $n > 4$ which could benefit from the application of the theory we have presented in this paper. Moreover, we hope and trust that our results will stimulate further progress in this direction.

Third, we offer a simple way to view the results we have presented. In rough terms, we have shown that, under the conditions specified and assuming that d and e are coprime, we can omit the "troublesome" $\mathrm{discr}_{x_{n-1}}(d)$ from the second projection set J_{n-2}. We can also omit $\mathrm{discr}_{x_{n-1}}(p)$ from J_{n-2}, for each (other) nonconstraint projection polynomial p at level $n-1$. This is a hopeful result, since the inclusion of these polynomials, and others spawned by them during iterated projection, is a significant component of the "doubly-exponential wall" of CAD-based QE in the presence of multiple ECs [DEMU25].

5 Detailed discussion of examples

In this section we revisit two of the examples introduced in Section 1, in the light of the theory presented in Sections 3 and 4.

The experiments with program QEPCADB which we report were conducted on a Unix server with an Intel Xeon Gold CPU (2.30 GHz). For each experiment, 1 megabyte of memory was made available.

5.1 The example of Fortuna, Gianni and Trager

We present details about a system of generic solutions to Example 2 from Section 1, introduced by [FGT01], which we found with the help of algebra and logic software.

Algorithms 1 and 2 (Section 3) have not yet been implemented. However, the existing program QEPCADB can be used to simulate part of the work of Algorithm 1. QEPCADB allows the user to select the projection operation to be used at each level from a short list of available operators. For our first experiment, we selected Hong's projection to be used at all levels, since this is the most efficient operator with no exceptions for its use which is currently available. We used the quantified formula ϕ^* defined in Example 2 (Section 1) as input, with the variable ordering (c, b, a, z, y, x):

$$\phi^* \equiv (\exists z)(\exists y)(\exists x)\phi(c, b, a, z, y, x).$$

The program delivered the expected solution `TRUE` after 14 milliseconds. This solution was found by constructing a suitable partial CAD of $\mathbf{R}^6$, considered to be part of a full CAD $\mathcal{D}$ of $\mathbf{R}^6$, in theory. The CAD $\mathcal{D}_3$ of (c, b, a)-space (that is, the parameter space) induced by $\mathcal{D}$ contains 63 cells. The first cell listed has index $(1, 1, 1)$ and sample point $(-2, -3, -1)$. All the 6 projection factors at levels no greater than 3 have negative sign on this cell. Similar information about each of the other 62 cells could, as for the first cell, be displayed using the command `d-cell`. The program issued a warning about the possible use of McCallum's projection (though this was not selected), since the set A of polynomials occurring in the input is not well-oriented. In fact, it is possible to use McCallum's projection provided that an assumption is used to rule out occurrence of nullifying cells of positive dimension for the first program run. (If this is to be done, then further runs of the program are needed to treat the special cases left out by the first.) Apart from small reductions in computation time and cell count, there was not much benefit to be gained by use of McCallum's projection for this example at this stage.

A succinct and insightful enhanced solution, in the spirit of Sections 1 and 3 above, was found using key information provided by QEPCADB (and alternatively by Gröbner basis computation). The authors of [FGT01] refer to a Gröbner basis for the ideal generated by the polynomials in A. This basis contains the polynomial

$$\rho(c, b, a, z) = z(acz + az + b^2 - bc - b),$$

which is precisely the level 4 polynomial identified by QEPCADB as the unique implicit constraint of ϕ^* at this level. Inspection of ρ suggests that it may be helpful first to consider carefully the assumption $a(c+1) \neq 0$. The reason is that, clearly, for every fixed (c, b, a) satisfying this assumption, there are at most two solutions of the equation $\rho(c, b, a, z) = 0$, hence at most two solutions of the system $\phi(c, b, a, z, y, x)$. Indeed, the solutions of the latter system may be found by explicitly solving $\rho(c, b, a, z) = 0$, obtaining $z = 0$ and $z = b(1-b+c)/(ac+a)$, then back substituting these solutions into the system ϕ. We find $(x, y, z) = (0, 0, 0)$ and

$$(x, y, z) = \left(\frac{b-c-1}{a(c+1)}, \frac{b(1-b+c)}{a(c+1)^2}, \frac{b(1-b+c)}{a(c+1)} \right).$$

Moreover, close inspection of these solution forms suggests that it may be worthwhile to slightly strengthen the assumption: $a(b-c-1)(c+1) \neq 0$. The reason is that, for each fixed (c, b, a) which satisfies the stronger assumption, there are *exactly* two distinct solutions (x, y, z). We see that a system of generic solutions $\mathcal{O}$ for ϕ^* is the collection of 8 connected open slightly skewed "octants" determined (bordered) by the planes $a = 0$, $b-c-1 = 0$ and $c+1 = 0$ in $\mathbf{R}^3$. In each octant, the number ν of distinct solutions (x, y, z) of $\phi(c, b, a, z, y, x)$ is 2, and (x, y, z) is expressed in terms of (a, b, c) by the above formulas. Consideration of each bordering plane in turn reveals that, at every point of the variety defined by $a(b-c-1)(c+1) = 0$, we have $\nu = 1$ or $\nu = \infty$. In light of this, we may conclude that the system of generic solutions $\mathcal{O}$ described above is minimal (in the sense that the cardinality of $\mathcal{O}$ is minimum).

We ran program QEPCADB with input ϕ^*, issuing the command

```
assume [a(b - c - 1)(c + 1) /= 0].
```

and selecting McCallum's projection (which is valid under this assumption). The program found the expected solution formula `TRUE` in 7 milliseconds. The induced CAD of (c, b, a)-space has 24 cells labelled `T` (and 12 cells not labelled, because they do not satisfy the assumption). Presumably, the minimal system described above could in principle be obtained by pasting together adjacent cells for which $\nu = 2$. While the command `d-cell I` gives some information about the cell whose index is `I`, it does not report the number of associated solutions of the system ϕ nor how the solutions depend on the parameters. It would seem to be reasonably straightforward to enhance the capability of QEPCADB to provide such information, as outlined in Section 3.

5.2 More about Solotareff's problem

Recall that Example 3 (Section 1) discussed Solotareff's approximation problem: find the best approximation to a real polynomial of degree n by a real polynomial of degree at most $n-2$. (As in Example 3, n does not denote the number of variables, rather the degree of the given polynomial, for consistency with the existing literature.) Example 3 reviewed a computer-assisted solution to the problem in the case $n = 3$. This subsection carefully considers the case $n = 4$, for which detailed treatments – excepting the complete mathematical justification of the equational projection method used – are found in [Col96, Col98]. Recall that in this case the problem is to find the best approximation to $x^4 + rx^3$ by a polynomial of the form $ax^2 + bx + c$; and we need to consider only the case $r > S_4$, since otherwise theory delivers an explicit solution.

In this case we have $S_4 = 12 - 8\sqrt{2}$, the smaller root of $x^2 - 24x + 16$. Consequently, we can express $r > S_4$ by the formula $r^2 - 24r + 16 < 0 \vee [r > 1 \wedge r^2 - 24r + 16 \geq 0]$. We also note that $c = 1 - a$, from theory. Thus, for $r > S_4$, we can formulate the problem of expressing a in terms of r as the following QE problem ([Col98], Section 8). However the quantified formula which appears in [Col98], Section 8, has some typographical errors which are corrected in [McC98], Section 7, and as follows in our formula ϕ^*:

$$
\begin{aligned}
(\exists b)(\exists u)(\exists v)[[r^2 - 24r + 16 < 0 \vee [r > 1 \wedge r^2 - 24r + 16 \geq 0]]& \\
\wedge\, r - b > 0 \wedge -1 < u \wedge u < v \wedge v < 1& \\
\wedge\, v^4 + rv^3 - av^2 - bv - (1 - a) = r - b& \qquad (3)\\
\wedge\, u^4 + ru^3 - au^2 - bu - (1 - a) = r - b& \qquad (4)\\
\wedge\, 4v^3 + 3rv^2 - 2av - b = 0& \qquad (5)\\
\wedge\, 4u^3 + 3ru^2 - 2au - b = 0].& \qquad (6)
\end{aligned}
$$

[Col96] sets up and solves an equivalent, slight variation of the above.

We aim here to justify the use of the fully reduced equational projections in the solutions provided by [Col96, Col98]. The variable ordering chosen was (r, a, b, u, v). Inspecting the above formulation, we see that there are four explicit equational constraints, two in the variable v and two in u. Let us denote by $f_1(r, a, b, u, v)$ the polynomial obtained by shifting all terms to the left hand side of the equation on line (3) in the above formula; notice that the equation on line (5) is then $f_1' = 0$, where f_1' denotes the derivative of f_1 with respect to v. Thus, the two constraints in v are $f_1 = 0$ and $f_1' = 0$. Similarly, denote by $f_2(r, a, b, u)$ the polynomial obtained by shifting all terms to the left hand side of the equation on line (4) above; then the equation on line (6) is $f_2' = 0$, where this time f_2' denotes the derivative with respect to u. The two constraints in u are $f_2 = 0$ and $f_2' = 0$.

For the first projection, f_1 is chosen as the pivot and hence $\mathrm{discr}_v(f_1) = \mathrm{res}_v(f_1, f_1')$ occurs "twice" (in computation only once, of course) in the first projection set J_4, amongst other polynomials. Let $e_1(r, a, b)$ denote the greatest squarefree divisor of $\mathrm{discr}_v(f_1)$. Then e_1 is the product of two irreducible factors, say p_1 and q_1; and $e_1 = 0$ is an *implicit* constraint at level 3. For the second projection, f_2 is chosen as the pivot; hence $\mathrm{discr}_u(f_2) = \mathrm{res}_u(f_2, f_2')$ occurs in the second projection set J_3, amongst other polynomials. Let $e_2(r, a, b)$ denote the greatest squarefree divisor of $\mathrm{discr}_u(f_2)$. Then e_2 is the product of two irreducible factors, say p_2 and q_2; and $e_2 = 0$ is another implicit constraint at level 3. For the third projection, e_2 is selected as the pivot constraint. The third projection set J_2 hence contains (factors of) $\delta(r, a) := \mathrm{discr}_b(e_2)$ and

$\rho(r,a) := \mathrm{res}_b(e_2, e_1)$, amongst other polynomials. Collins observed that the latter polynomial $\rho(r,a)$ – the propagated bivariate constraint – has seven irreducible factors; and he identified one of these factors, which we shall denote by $\rho^*(r,a)$, as crucial to his solution:

$$\begin{aligned}&324a^4 + 324r^2a^3 - 2016a^3 + 108r^4a^2 - 1128r^2a^2 + 4576a^2 \\ &\quad + 12r^6a - 224r^4a + 1392r^2a - 4480a - 15r^6 + 112r^4 - 608r^2 + 1600.\end{aligned}$$

We shall state the solution shortly; but for now we continue our step-by-step summary of its working out. For the fourth and last projection, $\rho(r,a)$ is used as the pivot, with the univariate projection set $J_1 = P_{\{\rho\}}(J_2)$ obtained accordingly. QEPCAD then obtains an irreducible basis for J_1, and constructs a CAD $\mathcal{D}_1$ of $\mathbf{R}^1$ for which each element of this basis, and hence of J_1, is order-invariant in every cell of $\mathcal{D}_1$. Proposition 1 may now be applied, showing that $\rho(r,a)$ is analytic delineable on each 1-cell c of $\mathcal{D}_1$, and that each element of J_2 is order-invariant in every such section. (Strictly speaking, Proposition 1 makes equivalent conclusions about irreducible bases for the sets $\{\rho\}$ and J_2, respectively.) Finally, Collins used the interactive stepwise stack construction capability of QEPCAD to construct a claimed partial CAD $\mathcal{D}$ of $\mathbf{R}^5$ sufficient to infer the solution involving ρ^* to the given QE problem. Careful reading of Collins' work reveals that his solution actually comprises two separate, related parts: a theorem T about ρ^* which he discovered, and a formula ϕ' equivalent to the given ϕ^*, whose definition relies upon the theorem T. The theorem T states that for every $r > S_4$, $\rho^*(r,a)$ (as a polynomial in a) has exactly two distinct real roots. The solution formula ϕ' may then be expressed as follows: "$r > S_4$ and a is the larger real root of $\rho^*(r,a)$". An astute reader will notice that the formal expressions of both theorem T and formula ϕ' in the language of Tarski algebra may require quantifers, so T and ϕ' are not, strictly speaking, quantifier-free.

To fulfil our aim of justifying Collins' solution and methodology we shall use Theorem 5 to verify the claimed partial CAD $\mathcal{D}$. By the known existence and uniqueness of a solution to Solotareff's problem [Ach56], it will suffice to verify that, for every 1-dimensional section $S \subset \mathbf{R}^2$ of ρ^*, for which $r > S_4$ and a is the larger real root of $\rho^*(r,a)$, the pivot constraints used at levels 3, 4 and 5 determine a tree $\mathcal{T}_S$ of cells such that the root node is the cell S, and the children of each node n are the cells in a stack over n. Additionally, $\mathcal{T}_S$ should have the properties that every leaf is at the same level, and the quantifier-free part ϕ of the given ϕ^* is true throughout at least one leaf of $\mathcal{D}_S$.

Let $S \subset \mathbf{R}^2$ be an arbitrary such 1-dimensional section of ρ^*. By Proposition 1, as noted above, each element of J_2 is order-invariant in S. Firstly, we shall use Theorem 5 to verify the delineability of the pivot constraint polynomials e_2 and f_2 used at levels 3 and 4, respectively. We will see that e_2 is delineable over S, and f_2 is delineable over every section of e_2 over S. Indeed, in the notation of Theorem 5, let us put $f = f_2$, $d = \mathrm{discr}_u(f_2)$, $a = \mathrm{ldcf}_u(f_2) = 1$, and $e = e_2$ $(= d)$. Noticing that f and e are well-placed (because $d = e$), e and a are coprime (since $a = 1$), and S contains no point at which e vanishes identically (since $\mathrm{ldcf}_b(e) = 27$), we see that all the hypotheses of Theorem 5 hold. Therefore, by Theorem 5, e is analytic delineable on S; hence the sections of e over S determine a stack over S. Let σ be a section of e over S. By inspection of the detailed output of QEPCAD, we see that ρ^* and $\mathrm{discr}_b(e)$ are coprime. Hence, by the order-invariance of $\mathrm{discr}_b(e)$ (a product of elements of J_2) in S, we must have $\mathrm{discr}_b(e) \neq 0$ throughout S. It follows that σ contains no singular point of the real algebraic surface $e = 0$. Again, by Theorem 5, f is analytic delineable on σ; hence the sections of f over σ determine a stack over σ. With B denoting the set of level 4 irreducible projection factors and F the set of irreducible factors of f, we may observe that every $g \in B \setminus F$ is well-positioned with respect to F and E. Once again, by Theorem 5, each such g is sign-invariant in every section of f over σ.

Secondly, we shall employ a different application of Theorem 5 to verify the delineability of the level 5 constraint polynomial f_1 – as a polynomial in the 4 variables r, a, b, v – over the arbitrary section $\sigma \subset \mathbf{R}^3$ of e_2 over S introduced in the previous paragraph. Indeed, in the notation of Theorem 5, we now put $f = f_1$, $d = \mathrm{discr}_v(f_1)$, $a = \mathrm{ldcf}_v(f_1) = 1$ and $e = e_2$. Noticing that f and e are well-placed (because now e and d are coprime), and e and a are coprime (since $a = 1$), we see again that all the hypotheses of Theorem 5 hold. Therefore, by Theorem 5, f is analytic delineable on σ, and consequently f is analytic delineable on every section $\hat{\sigma}$ of f_2 over σ. Hence the sections of f over $\hat{\sigma}$ determine a stack over $\hat{\sigma}$. With B now denoting the set of all level 5 irreducible projection factors and F now denoting the set of irreducible factors of f, we may observe that every $g \in B \setminus F$ whose variables lie in $\{r, a, b, v\}$ is well-positioned with respect to F and E. By Theorem 5, each such g is sign-invariant in every section of f over $\hat{\sigma}$. Since $v - u$, the only exception, is linear in v, it is not difficult to see that it too is sign-invariant in every section of f over $\hat{\sigma}$. This concludes our validation of the existence of the claimed tree $\mathcal{T}_S$.

We make two remarks in conclusion of this subsection.

Remark 8. *Collins' solution method for this problem instance is essentially a variant of Algorithm 1 (Section 3). For this variant, the number p of parameters is 1, and the number u of unknowns is also 1: indeed, the parameter is r and the unknown is a. A slight change in terminology is required: for $\alpha \in \mathbf{R}$, the value $\beta \in \mathbf{R}$ is a solution associated with α if $\phi^*(r, a)$ is true. In this variant of Algorithm 1, using the notation of that algorithm, pivots and projection sets are computed down to e_2 and J_1, respectively, in Steps 1, 2, and 3. In Step 5, we compute the open intervals of a J_1-invariant CAD $\mathcal{D}_1$ of parameter space $\mathbf{R}^1$, together with solution cardinality information. From theory, $\nu = 1$ holds for each open interval c with $r > S_4$ in this problem instance. Finally, in Step 6, for each open interval c with $r > S_4$, we construct the defining formula for a in terms of $r \in c$. Perhaps most important of all to note is that Collins [Col96, Col98] used the fully reduced projection operation $P_{E_j}(J_j)$ at each projection step, although this has been rigorously validated in general only for the cases $j = n$ [McC99] and $j = 2$ (Proposition 1 of the present paper). Yet, we have outlined above a new justification for the use of the fully reduced projection operation for $2 < j < n$*

also, for this problem instance.

Remark 9. *Daniel Lazard [Laz06] described an adaptation of the concepts and methods of [LR07] which is effective in solving Solotareff's problem up to $n = 9$ in a certain sense. No explicit solution formulas for the desired coefficients of the best approximation polynomial in terms of the parameter r were provided, though these could presumably be obtained in principle. In any case, our aim here in this paper is to enhance CAD-based QE as a general purpose tool when multiple equations are present. We simply illustrated the potential benefit of our enhancements using Solotareff's problem, rather than making any claim about superior performance relative to other methods for this particular problem.*

6 Conclusion and planned further work

We summarize the work reported in this paper. Sections 3 and 4 are the technical heart of the new contributions that we present. Section 3 contains a formal algorithm which, given a parametric system in the form of an existentially quantified formula $\phi^* \equiv (\exists x_{n-k+1}) \cdots (\exists x_n)\phi(x_1, \ldots, x_n)$, where $s = n - k$ is the number of parameters, finds a description of a system of generic solutions for ϕ^*. The details concerning the structure of the expected input ϕ^* and the constitution of the generic solution description in the output are provided in the specification of Algorithm 1. A supporting proposition and worked examples are included. Elaboration of the general definition of the candidate constraint set at each level is needed.

Section 4 introduces the concept of the order of a real analytic function $g(x_1, \ldots, x_n)$ relative to the real variety V_f of another real analytic function $f(x_1, \ldots, x_n)$ at a non-singular point p of V_f: in symbolic notation, this is denoted by $\mathrm{ord}_p g|_{f=0}$. A trilogy of new delineability related theorems using this concept is presented. The purpose of this theory is to validate the use of the fully reduced equational projection operation for the second projection step of Algorithm 1 and its variants under suitable assumptions. This is a relatively small, but useful, step toward "pushing back the doubly exponential wall" of CAD when many equations are present in ϕ.

It would be desirable to improve Theorem 3 by removing the hypothesis about the codimension of S. Further to that, a simplified and extended version of the theory presented in Section 4 would be a worthwhile aim in the future. Perhaps this could be achieved using the concept of the multiplicity of a variety at a point or a closely related notion. It would be useful to implement the algorithms of Section 3 and evaluate them experimentally. Investigation of the idea of replacing the single projection step paradigm of CAD using a "multi-step" projection operation, when many equational constraints are available, would also seem to be a promising direction.

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