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CURIOUS CROSSING-CRITICAL EDGES

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Abstract. Motivated by Kuratowski's theorem, a Kuratowski subgraph of a graph is a subgraph that is a subdivided K_5 or a subdivided $K_{3,3}$. An edge is crossing-critical if the crossing number decreases after removing the edge. In this note, we present the following examples: a graph with an edge that is crossed in every optimal drawing of the graph, but the edge is not in any Kuratowski subgraph of the graph; a graph with a unique edge that is in every Kuratowski subgraph but is not crossed in any optimal drawing of the graph; and a graph with a crossing-critical edge that is not present in any Kuratowski subgraph and is not crossed in any optimal drawing of the graph. Fáry's theorem implies that the Kuratowski subgraphs are the only obstructions to a graph having a crossing-free drawing with all edges drawn as straight lines. The three example graphs given also hold if we restrict drawings to only have straight line edges, and thus also apply to the rectilinear crossing number.

1. Introduction

In this note, a graph is a finite loopless multigraph, i.e., we allow parallel edges but no loops. Edges are parallel if they have the same endvertices. Given vertices x, y of the graph G, the *multiplicity* of the x-y edge is the number of x-y edges in G. A *simple graph* is a graph G without parallel edges, i.e., a graph where the edge multiplicity is one for each adjacent vertex pair x, y. Given a multigraph G, the *underlying simple graph* is a simple graph H on the same vertex set, such that H has a single edge between vertices x, y precisely when x, y are adjacent in G. If G is a graph, a *representation* of G is the pair (H, w_G), where H is the underlying simple graph, and $w_G : E(H) \to \mathbb{Z}^+$ is defined such that for any x-y edge e of H, $w_G(e)$ is the edge multiplicity of the x-y edge in G. If G is a graph with underlying simple graph H and e is an x-y edge of H, then the x-y-edges of G are *copies* of e.

A *subdivision of a graph H* is a graph G where some of the edges of H are replaced by paths whose internal vertices are not in H and not shared with any other path. The vertices of H are called the *main vertices* of the subdivision, and the internal vertices of the paths are called the *subdividing vertices*.

A *drawing $\mathcal{D}$ of a graph G* is an embedding of the graph in the plane such that the vertices are assigned to points of the plane injectively, the edges are mapped to Jordan arcs connecting to the endvertices of the edge, and if v is not an endvertex of the edge e then the image $\mathcal{D}(v)$ of v is disjoint from the image $\mathcal{D}(e)$ of e.

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A graph is *planar* if it can be drawn in the plane without crossing edges, i.e., such that the images of two edges only have their common endvertices in common; in other words edges do not cross in the drawing. The celebrated Kuratowski theorem [3] states that a graph if planar precisely when it does not contain a subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$. H is called a *Kuratowski subgraph* of the graph G if H is a subgraph of G and H is a subdivision of either $K_{3,3}$ or K_5. Note that Kuratowski subgraphs are simple graphs by definition. In a sense, Kuratowski's theorem states that Kuratowski subgraphs force crossings to appear in any drawing of the graph. Fáry's theorem (proved independently in [2, 7]) states that every planar graph can be drawn without crossings even if we restrict to drawings using straight line edges (the *rectilinear drawings*). Thus, the Kuratowski subgraphs are also the only obstacles to having planar straight line drawings. Moreover, in any drawing of a graph, every Kuratowski subgraph must have a pair of edges that cross.

The *crossing number* $\mathrm{cr}(\mathcal{D})$ of a drawing $\mathcal{D}$ of G is $\sum\limits_{e,f\in E(G)} |\mathcal{D}(e)^0 \cap \mathcal{D}(f)^0|$, where for an edge e, $\mathcal{D}(e)^0$ is the interior of $\mathcal{D}(e)$, and the summation goes for unordered pairs of distinct edges. The *crossing number* $\mathrm{cr}(G)$ *of a graph* G is the minimum crossing number over all its drawings. A drawing $\mathcal{D}$ of G is called *optimal* if $\mathrm{cr}(\mathcal{D}) = \mathrm{cr}(G)$. An edge $e \in E(G)$ is called *crossing-critical* if $\mathrm{cr}(G - e) < \mathrm{cr}(G)$. The *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ of the graph G is the minimum crossing number of its rectilinear drawings. Clearly $\mathrm{cr}(G) \le \overline{\mathrm{cr}}(G)$, with equality precisely when G has an optimal drawing that is rectilinear. An edge $e \in E(G)$ is called *crossing-critical with respect to rectilinear drawings* if $\overline{\mathrm{cr}}(G - e) < \overline{\mathrm{cr}}(G)$.

We call an edge f of a graph G a *Kuratowski edge* if it belongs to some Kuratowski subgraph of G, and a Kuratowski edge f is further called a *strong Kuratowski edge* if it belongs to every Kuratowski subgraph. The Kuratowski theorem implies that f is a strong Kuratowski edge of the graph G precisely when G is nonplanar but $G - f$ is planar. In particular, strong Kuratowski edges are both crossing-critical and crossing-critical with respect to rectilinear drawings.

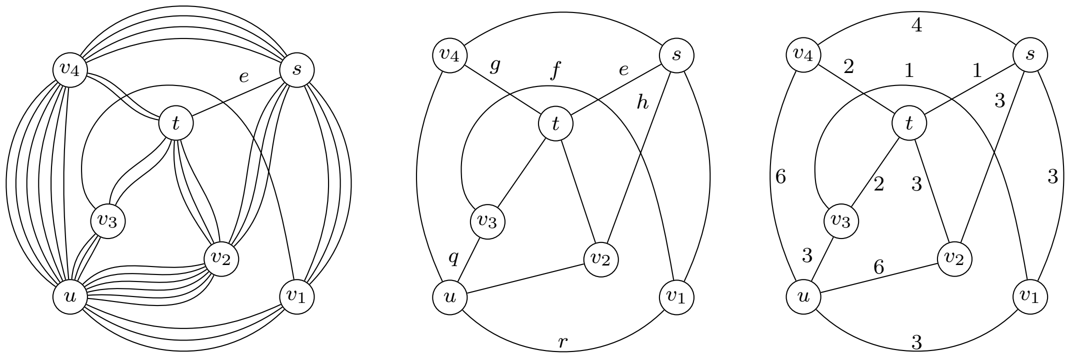
If h is an edge that is crossed in some optimal drawing $\mathcal{D}$ of G, then h is clearly crossing-critical, as the drawing of $G - h$ induced by $\mathcal{D}$ has fewer than $\mathrm{cr}(G)$ crossings; similar statement applies to the rectilinear versions of the concept.

In 1983, Širáň [6] gave an example of a graph G^* that has a crossing-critical edge e such that e is not an edge of any of the Kuratowski subgraphs of G^*. Figure 1 shows an optimal drawing of this graph G^*, in which e is crossed, its underlying simple graph G_0 and a representation of G^*. The edge f on this figure is clearly a strong Kuratowski edge, as its removal renders the graph planar, and it is crossed in the optimal drawing of Figure 1. Figure 2 presents two more optimal drawings of this graph, and illustrates that no edge of G^* is crossed in every optimal drawing.

Based on Širáň's example, we consider the following

Problem 1. *Find the answer to the following questions:*

(1) *If an edge e of a graph G is crossed in every optimal drawing of G, must e be a Kuratowski edge?*



An optimal drawing of the graph G^*.

The underlying simple graph G_0 of G^*.

A representation of G^*.

FIGURE 1. Širáň's example [6]: $\mathrm{cr}(G^* - e) < \mathrm{cr}(G^*) = 6$, e is not in any Kuratowski subgraph of G^*, but is crossed in the optimal drawing shown.

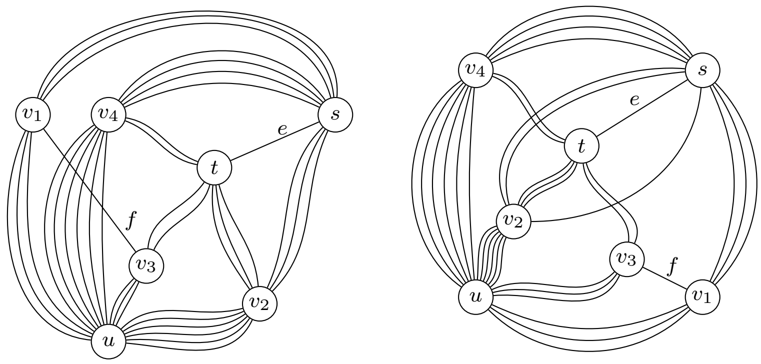


FIGURE 2. Optimal drawings of the graph G^* in Širáň's example [6]. Note that the set of edges crossed in both of these drawings is empty.

(2) *If G has strong Kuratowski edges, must G have an optimal drawing in which some strong Kuratowski edge is crossed?*
(3) *If the edge e of a graph G is not crossed in any optimal drawing of G and is not a Kuratowski edge, is it true that e cannot be crossing-critical?*

We consider the corresponding questions for the rectilinear drawings of simple graphs as well.

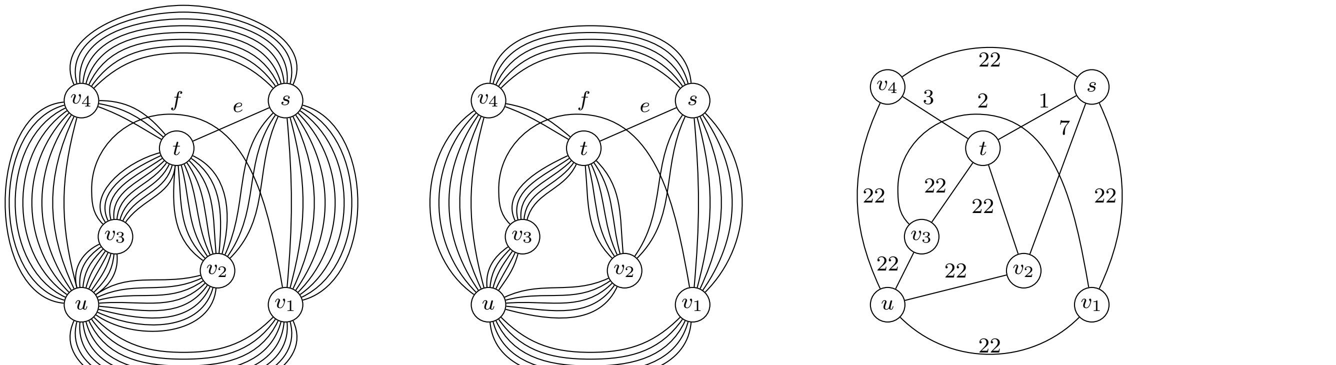
We modify Širáň's example by changing the edge-multiplicities of G^* judiciously to create graphs G_1, G_2, G_3 (see Figure 3) such that each have the same underlying simple graph G_0 that give the following (perhaps surprising) answers to our questions:

Claim 1. *The answer to all questions in* Problem 1 *is in the negative (see Figure 3 for the examples), i.e.,*

(1) *A graph can have an edge that is crossed in every optimal drawing, but is not a Kuratowski sedge. (An example is the graph G_1 and the unique s-t edge e.)*
(2) *A graph with strong Kuratowski edges can be such that no strong Kuratowski edge is crossed in any optimal drawing of the graph. (An example is the graph G_2 and the unique strong Kuratowski edge f.)*
(3) *A graph can have an edge that is crossing-critical, but this edge is not a Kuratowski edge and is not crossed in any optimal drawing of the graph. (An example is the graph G_3 and the unique s-t edge e.)*

Furthermore, all three negative answers can be provided with simple graphs. Moreover, these simple graphs provide the corresponding examples for the rectilinear crossing number as well.

We note that the "furthermore" part of the previous claim is obvious: in each of the G_i, subdivide all but one of the parallel copies of each x-y edge to obtain a simple graph G_i^*. The edge e does not have parallel copies, so it is never subdivided, and the edge f does not have parallel copies in G_2. Hence these simple graphs also have the required properties. The "moreover" part follows from the fact that we have straight line drawings of the underlying simple graphs G_i (Figure 4) used in the proof, and the simple graphs G_i^* have corresponding straight line drawing: draw the not subdivided copy of a multi-edge as a straight line, and place the subdivided copies of this edges of the multi-edges close to the original edge.



The graph G_1
$w_1(f) = w_1(e) = 1$,
$w_1(g) = w_1(h) = 3$
$z \notin \{e, f, g, h\}$: $w_1(z) = 8$

The graph G_2
$w_2(f) = w_2(e) = 1$,
$w_2(g) = w_2(h) = 2$
$z \notin \{e, f, g, h\}$: $w_2(z) = 5$

A representation of G_3
$w_3(e) = 1$, $w_3(f) = 2$,
$w_3(g) = 3$, $w_3(h) = 7$
$z \notin \{e, f, g, h\}$: $w_3(z) = 22$

FIGURE 3. The graphs G_1 and G_2 and a representation of G_3. For $i \in \{1, 2, 3\}$ the representation of G_i is (G_0, w_i). e, f, g, h are the edges of the underlying graph G_0 (see Figure 1).

We will show that the graphs G_1, G_2, G_3 and the edges e, f satisfy the properties claimed in Claim 1 by first finding the edges of Kuratowski subgraphs of the graphs in Claim 2, and then (after some preliminary work) verifying the rest of Claim 1 in Claim 6.

2. Proof of our claims

First we determine which edges belong to Kuratowski subgraphs of graphs with underlying simple graph G_0.

Claim 2. *Let G be a graph with representation (G_0, w). The non-Kuratowski edges of G are precisely the copies of e in G. An edge x is a strong Kuratowski edge of G precisely when x is a copy of some $y \in E(G_0) \setminus \{e, q, r\}$ with $w(y) = 1$.*

Proof. Let G be a graph with underlying graph G_0. It is clear that the set of Kuratowski edges of G is the set of copies of the Kuratowski edges of G_0, and a copy of an edge $y \in E(G_0)$ is a strong Kuratowski edge of G precisely when $w(y) = 1$ and y is a strong Kuratowski edge of G_0. Therefore we need to show that the set of Kuratowski edges of G_0 is $E(G_0) \setminus \{e\}$ and the set of strong Kuratowski edges of G_0 is $E(G_0) \setminus \{e, q, r\}$.

G_0 has 3 vertices of degree 4 (u, s and t) and 12 edges.

As any subdivision of K_5 has 5 vertices of degree 4, any Kuratowski subgraph of G_0 is a subdivided $K_{3,3}$.

Let X be a Kuratowski subgraph of G. As X is a subdivided $K_{3,3}$ with $\alpha \geq 0$ subdividing vertices, X has $6 + \alpha$ vertices and $9 + \alpha$ edges. As G_0 has 7 vertices and 12 edges, either $\alpha = 0$ and X is obtained from G_0 by removing a vertex of degree 3, or $\alpha = 1$ and X is obtained from G_0 by removing 2 edges.

If $\alpha = 0$, then for some $i \in \{1, 2, 3, 4\}$ we have $X = G_0 - v_i$. As X is a bipartite 3-regular graph, the neighbors of v_i in G_0 all have degree 4, therefore $i \in \{2, 4\}$. However, if $i \in \{2, 4\}$ then $X = G_0 - v_i$ contains a triangle on u, v_1, v_3, contradicting the fact that X bipartite. Therefore $\alpha = 1$, and X is obtained from G_0 by removing exactly two edges.

As the removal of these two edges results in a subdivision of a 3-regular graph, we have that the 4 endvertices of these 2 edges must be s, t, u, v_i for some $i \in \{1, 2, 3, 4\}$. Therefore, one of the removed edges must have both endpoints in $\{s, t, u\}$. The only such edge is e, which is not in any Kuratowski subgraph of G_0. So, $X = G - \{e, z\}$ for some edge z incident upon u. Since X must be triangle-free, z is an edge of the triangle on u, v_1, v_3, so $x \in \{q, r\}$.

Therefore we have that if X is a Kuratowski subgraph of G_0, then $X \in \{G_0 - \{e, q\}, G_0 - \{e, r\}\}$, and it is easy to check that both of these choices result in a Kuratowski subgraph. This finishes our proof. $\square$

Next we consider the drawings. It is well-known [5] that any optimal drawing of a graph must be *good*, i.e., it must be such that any two arcs representing edges in the drawing do not touch each other, they cross at most once, and arcs representing edges that share an endvertex do not cross.

If H is an edge-weighted simple graph with edge-weights $w : E(G) \to \mathbb{R}^+$, *the weighted crossing number* $\mathrm{cr}_w(\mathcal{D})$ *of a drawing* $\mathcal{D}$ is

$$\mathrm{cr}_w(\mathcal{D}) = \sum_{e,f \in E(G)} w(e) \cdot w(f) \cdot |\mathcal{D}(e)^0 \cap \mathcal{D}(f)^0|.$$

and $\mathrm{cr}_w(H)$ is the minimum of the weighted crossing numbers of good drawings of H.

We will make use of the following easy claim:

Claim 3. *Let G be any graph with representation (H, w), $\mathcal{D}$ be any optimal drawing of G. The following statements are true:*

(i) *For any good drawing $\mathcal{D}_H$ of H, there is a drawing $\mathcal{D}_{H,G}$ of G in which two edges of G cross precisely when they are copies of two edges that cross in $\mathcal{D}_H$. In particular* $\mathrm{cr}(G) \le \mathrm{cr}_w(\mathcal{D}_H)$.
(ii) *For any edge e of H, its parallel copies in G have the same number of crossings in $\mathcal{D}$.*
(iii) *If z is an edge of H such that $w(z) > \mathrm{cr}(G)$ then no copy of z has a crossing in $\mathcal{D}$.*
(iv) *Let $e \in E(H)$, $Y \subseteq E(H)$ such that G has a copy e' of e such that in $\mathcal{D}$ the edge e' crosses at least one copy of every edge in Y. Then $w(e)\sum_{y\in Y} w(y) \le \mathrm{cr}(G)$.*
(v) $\mathrm{cr}(G) = \mathrm{cr}_w(H)$.

Proof. Note that $\mathcal{D}$ is a good drawing.

Proof of (i): Create a drawing $\mathcal{D}_{H,G}$ from $\mathcal{D}_H$ as follows: for any edge e of H draw the copies of e in G close enough to the image of e in $\mathcal{D}_H$ so that copies of two edges that are uncrossed in $\mathcal{D}_H$ do not cross in the drawing created.

Proof of (ii): Assume to the contrary that e_1, e_2 are parallel copies of the edge e of H that do not cross the same number of edges in $\mathcal{D}$, without loss of generality e_1 has fewer crossings. Redraw e_2 close to e_1 so e_2 crosses the exact same edges as e_1. The new drawing of G has fewer crossings that $\mathcal{D}$, a contradiction,

Proof of (iii) follows from (ii).

Proof of (iv): Let $y_1, y_2, \dots, y_k$ be an enumeration of the edges of Y, and let y_i' be a copy of y_i such that e' and y_i' crosses in $\mathcal{D}$. Create the sequence of drawings $\mathcal{D}^0, \mathcal{D}^1, \dots \mathcal{D}^k$ satisfying $\mathrm{cr}(\mathcal{D}^i) = \mathrm{cr}(G)$ as follows:

For $\mathcal{D}^0$ redraw every parallel copy of e different from e' close to e' so they cross the same edges as e'. By (ii) $\mathrm{cr}(\mathcal{D}^0) = \mathrm{cr}(\mathcal{D}) = \mathrm{cr}(G)$.

If for some $i \in [k]$ we have $\mathcal{D}^{i-1}$, create $\mathcal{D}^i$ by redrawing every copy of y_i except y_i' close to y_i'. By (ii) $\mathrm{cr}(\mathcal{D}^i) = \mathrm{cr}(\mathcal{D}^{i-1}) = \mathrm{cr}(G)$.

In $\mathcal{D}^k$ every copy of e crosses every copy of every edge in Y, consequently $w(e)\sum_{y\in Y} w(y) \le \mathrm{cr}(\mathcal{D}^k) = \mathrm{cr}(G)$.

To show (v), we apply the same procedure in (iv): Let $h_1, h_2, \dots, h_m$ be an enumeration of the edges of H, and $h_1', \dots, h_k'$ be edges of G such that h_i' is a fixed copy of h_i. Define the sequence of drawings $\mathcal{D}^0 = \mathcal{D}, \mathcal{D}^1, \mathcal{D}^2, \dots, \mathcal{D}^m$ such that for all $i \in [m]$, $\mathcal{D}^i$ is obtained from $\mathcal{D}^{i-1}$ by redrawing the edges parallel to h_i'. As before $\mathrm{cr}(\mathcal{D}^i) = \mathrm{cr}(G)$. Let $\mathcal{D}_H$ be the subdrawing of H generated by the edges $h_1', \dots, h_m'$ in $\mathcal{D}^m$. Then $\mathrm{cr}_w(H) \le \mathrm{cr}_W(\mathcal{D}_H) = \mathrm{cr}(\mathcal{D}^m) = \mathrm{cr}(G)$. Using (i), the statement follows. $\square$

We detour a bit by defining when two drawings of a graph are equivalent (see standard graph theory books, e.g., [1]). Using $\mathcal{D}(G)$ for the set of points that are in the drawing, we can define a face of the drawing as a maximal connected component of $\mathbb{R}^2 - \mathcal{D}(G)$ (note that this does not assume that the drawing is planar).

Let S^2 be the two-dimensional sphere, with north pole $(0,0,1)$, and $\pi : S^2 - \{(0,0,1)\} \to \mathbb{R}^2$ is the stereographic projection of the sphere to the plane. Let G be a graph and $\mathcal{D}_1, \mathcal{D}_2$ be two drawings of G. Drawings $\mathcal{D}_1, \mathcal{D}_2$ are called *equivalent* if there is a homeomorphism $\phi : S^2 \to S^2$ such that $\pi \circ \phi \circ \pi^{-1}$ maps $\mathcal{D}_1$ to $\mathcal{D}_2$. Note that the use of π, π^{-1} allows us to switch the outer face of the drawing without changing any salient features of the drawing.

Recall that for $k \in \mathbb{N}$, a *simple* graph is k-connected if it has more than k vertices and the removal of fewer than k vertices does not disconnect the graph. Whitney [8] showed that any two crossing-free drawings of a 3-connected *simple* planar graph are equivalent. The Jordan-Schönflies theorem [4] states that if γ, δ are both closed Jordan curves in $\mathbb{R}^2$, and f is a homeomorphism between them, then f may be extended to a homeomorphism of the entire plane to itself. The analogous statement with $\mathbb{S}^2$ in the place of $\mathbb{R}^2$ also holds. These imply that any two crossing-free drawings of a subdivision of 3-connected simple planar graph are equivalent.

Claim 3 allows us to reduce our questions about optimal drawing of a graph into questions about drawings of the underlying simple graph. So, we will first consider drawings of the simple graph G_0.

Claim 4. *Assume that $\mathcal{D}$ is a drawing of G_0 such that the induced subdrawing of $G_0 - f$ is planar, and the set of edges that f crosses in $\mathcal{D}$ is a subset of $\{e, g, h\}$. Then f crosses each of e,g and h in $\mathcal{D}$.*

Proof. Recall the drawing $\mathcal{D}_0$ of G_0 (Figure 4), and set $H_0 = G_0 - f$. H_0 is a subdivision of a K_5 minus an edge, and K_5 minus an edge is a planar 3-connected simple graph. Therefore the subdrawings $\mathcal{D}_0|_{H_0}, \mathcal{D}|_{H_0}$ of H_0 induced by the drawings $\mathcal{D}_0, \mathcal{D}$ are equivalent. The edges $E(G_0) - \{e, f, g, h\}$ form the closed walk $v_3tv_2uv_1sv_4uv_3$. The subdrawings of this closed walk in $\mathcal{D}$ (and $\mathcal{D}_0$) divide the plane into the same 3 regions, one of which contains the edges e, g, h (Figure 4, left picture). As G is non-planar, f must cross some edges in $\mathcal{D}$ (and $\mathcal{D}_0$). Since f can only cross the edges of e, g, h, edge f must also lie in the same region as e, g, h. As the two endvertices of f lie on the u-t and u-s edges, f crosses all of e, g, h. □

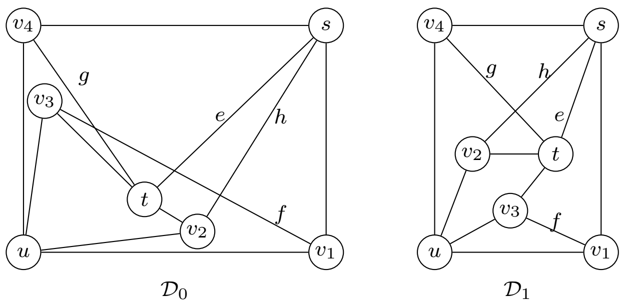


FIGURE 4. The (rectilinear) drawings $\mathcal{D}_0$ and $\mathcal{D}_1$ of G_0. In $\mathcal{D}_0$, the edge f crosses the edges e, g, h. In $\mathcal{D}_1$, the edges g and h cross.

Claim 5. *Let G be a graph with representation (G_0, w) and set*

$$\kappa = \min\left(w(g)w(h),\ w(f)(w(e)+w(g)+w(h))\right).$$

If for all $z \in E(G_0) \setminus \{e, f, g, h\}$ we have $w(z) > \kappa$, then $\mathrm{cr}(G) = \kappa$. Moreover

(i) *If $w(g)w(h) < w(f)(w(e)+w(g)+w(h))$, then in any optimal drawing of G only copies of g and h cross. In particular, e and f are never crossed in an optimal drawings.*

(ii) *If $w(g)w(h) > w(f)(w(e)+w(g)+w(h))$,, then in any optimal drawings of G, all copies of f cross all copies of e, g and h. In particular, e is crossed in any optimal drawings.*

Proof. Claim 3 (i) and the drawings $\mathcal{D}_0, \mathcal{D}_1$ of G_0 on Figure 4 imply that $\mathrm{cr}(G) \le \kappa$. Since for all $z \in E(G_0) \setminus \{e, f, g, h\}$ we have $w(z) > \kappa$, Claim 3 (iii) gives that in any optimal drawing of G, only copies of e, f, g, h may cross. Since in optimal drawings incident edges do not cross, if a crossing does not involve a copy of f, it must happen between copies of g and h.

Let $\mathcal{D}$ be an optimal drawing of G, and let g', h' be a copy of g and h respectively. We have one of following:

(a) If g' and h' cross in $\mathcal{D}$, then by Claim 3 (iv), $\kappa \le w(g)w(h) \le \mathrm{cr}(G) \le \kappa$.

(b) If g' and h' do not cross in $\mathcal{D}$, then select a subdrawing of G_0 by selecting one copy z' of each edge $z \in E(G_0)$ such that g', h' are selected. As in this drawing g', h' do not cross, the only crossings involve f' with e', g', h'. By Claim 4 f' crosses all of e', g', h' and Lemma 3 (iv) gives that $\kappa \le w(f)(w(e)+w(g)+w(h)) \le \mathrm{cr}(G) \le \kappa$.

This gives $\mathrm{cr}(G) = \kappa$.

If $w(g)w(h) < w(f)(w(e)+w(g)+w(h))$, then by (b) no copy of f may cross any copy of g, h, e in $\mathcal{D}$. Part (i) follows.

If $w(g)w(h) > w(f)(w(e)+w(g)+w(h))$, then by (a) any copies of g and h do not cross in $\mathcal{D}$. Part (ii) follows. □

Now we are ready to prove our final claim:

Claim 6. *The following statements are true:*

(i) $\mathrm{cr}(G_1) = 7$*, and e is crossed in every optimal drawing of G_1.*

(ii) $\mathrm{cr}(G_2) = 1$*, and f is not crossed in any optimal drawing of G_2.*

(iii) $\mathrm{cr}(G_3) = 21$*, and e is not crossed in any optimal drawing of G_3, but* $\mathrm{cr}(G - e) < \mathrm{cr}(G)$.

Proof. Recall that for $i \in \{1, 2, 3\}$, the representation of G_i is (G_0, w_i).

$w_1(h)w_1(g) = 9$, $w_1(f)(w_1(e) + w_1(h) + w_1(h)) = 7$ and $w_1(z) = 8$ for $z \notin \{e, f, g, h\}$. Part (i) follows from Claim 5.

$w_2(h)w_2(g) = 4$, $w_2(f)(w_2(e) + w_2(h) + w_2(h)) = 5$ and $w_2(z) = 5$ for $z \notin \{e, f, g, h\}$. Part (ii) follows from Claim 5.

$w_3(h)w_3(g) = 21$, $w_3(f)(w_3(e)+w_3(h)+w_3(h)) = 22$ and $w_3(z) = 22$ for $z \notin \{e, f, g, h\}$. Claim 5 implies that $\mathrm{cr}(G_3) = 21$, and e is not crossed in any optimal drawing of G_3. Using the subdrawing $\mathcal{D}'$ of $G_0 - e$ induced by the drawing $\mathcal{D}_0$ (see Figure 4), Claim 3 (i) gives that $\mathrm{cr}(G-e) \le \mathrm{cr}_{w_3}(\mathcal{D}') = 20 < \mathrm{cr}(G)$ □

It is clear that Claim 2 and Claim 6 implies Claim 1.

References

[1] R. Diestel, Graph Theory, Sixth edition 2025, Springer-Verlag, Heidelberg Graduate Texts in Mathematics, Volume 173.
[2] I. Fáry, On straight-line representation of planar graphs, Acta Sci. Math. (Szeged, 1948), 11: 229–233
[3] K. Kuratowski, Sur le problème des courbes gauches en topologie, Fund. Math. (in French) 15 (1930), 271–283.
[4] B. Mohar, C. Thomassen, Graphs on Surfaces, 2001, Johns Hopkins University Press
[5] M. Schaefer, Crossing numbers of graphs, CRC Press, Taylor and Francis group, Discrete Mathematics and its Applications, 2018.
[6] J. Širáň, Crossing-critical edges and Kuratowski subgraphs of a graph, J. Comb. Theory B 35(2) (1983), 83–92. https://doi.org/10.1016/0095-8956(83)90064-3
[7] K. Wagner Bemerkungen zum Vierfarbenproblem Jahresbericht der Deutschen Mathematiker-Vereinigung (in German, 1936) 46: 26–32.
[8] H. Whitney, Congruent graphs and the connectivity of graphs, Am. J. Math. 54 (1932) 150–168.

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