RICCI SOLITONS FROM THE PERSPECTIVES OF ENERGY FUNCTION

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ABSTRACT. This article explores to what extent the geometry of gradient Ricci solitons extends to non-gradient Ricci solitons. The primary tool is the energy function E of the soliton. We study consequences of various bounds on E. Under mild assumptions on the scalar curvature, we prove a weighted L^1-Liouville type theorem for both the usual Laplacian and the drifted Laplacian Δ_V associated to soliton vector field V, the former of which implies that Ricci solitons with bounded energy function have at most one nonparabolic end. Finally, we show that the measure $e^{-E}\mathrm{dVol}_g$ is finite for complete shrinking Ricci solitons, partially generalizing a result of Aaron Naber. As a consequence, non-gradient shrinking Ricci solitons also have finite fundamental groups, as in the gradient case.

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Contents

1. Introduction

A Ricci soliton is a quadruple (M^n, g, V, λ), where (M^n, g) is a complete Riemannian manifold, $V \in \chi(M)$, a complete vector field, and $\lambda \in \mathbb{R}$ satisfying

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} = \frac{\lambda}{2} g, \tag{1}$$

where $\mathcal{L}_V$ is the Lie derivative with respect to V and Ric is the Ricci tensor of (M, g). The vector field V is called the soliton vector field. The soliton is said to be shrinking when $\lambda > 0$, steady when $\lambda = 0$, or expanding when $\lambda < 0$. Immediately, one notices that all Einstein manifolds can be considered as Ricci solitons by taking V as zero or a Killing field. Such solitons are said to be trivial. If the soliton vector field is gradient, i.e. $V = \nabla f$, then the soliton is said to be gradient and the function f is called the soliton function. Thus, the soliton function of a gradient Ricci soliton satisfies the equation

$$\mathrm{Hess}\, f + \mathrm{Ric} = \frac{\lambda}{2} g. \tag{2}$$

Following [5], we define the energy function of a Ricci soliton as

$$E = \frac{1}{2} g(V, V). \tag{3}$$

Ricci solitons which are not a priori gradient are called *generic* Ricci solitons (see [3, 10]). This article primarily aims to explore the relationship between the geometry of Ricci solitons and the properties of their associated energy functions. This was also explored by Sharief Deshmukh and Hana Alsodais in [5], primarily relying on the fact that eq. (1) is equivalent to

$$\nabla_X V = \frac{\lambda}{2} X + \phi X - \mathrm{Q}\, X,$$

where Q is the Ricci tensor, and ϕ being the (1,1) tensor given by

$$\frac{1}{2}\mathrm{d}\eta(X, Y) = g(\phi X, Y),$$

with η as the dual 1-form of the potential vector field V, i.e., $\eta(X) = g(X, V)$. Our approach is quite different (independent of η) and the geometric results obtained in the article are also of different flavour.

Throughout this article, we will denote the scalar curvature by S and the $(1,1)$-tensor field dual to Ricci tensor on M by Q. Thus, for any vector fields X, Y on M,

$$g(\mathrm{Q}\,X, Y) = \mathrm{Ric}(X, Y) = g(X, \mathrm{Q}\,Y).$$

The traceless Ricci tensor will be denoted by $\mathring{\mathrm{Q}}$,

$$\mathring{\mathrm{Q}}X = \mathrm{Q}\,X - \frac{\mathrm{S}}{n}X.$$

The norm of any vector field or tensor field T will be denoted by $|T|$.

It is known that for a gradient Ricci soliton with soliton function f (see Lemma 1.1 of [2])

$$\nabla(\mathrm{S} + |\nabla f|^2 - \lambda f) = 0. \tag{4}$$

We establish a weaker form of this for non-gradient Ricci solitons as:

Corollary 1. *For any Ricci soliton (M^n, g, V, λ),*

$$\mathrm{div}(\nabla S + \nabla E + \nabla_V V - \lambda V) = 0. \tag{5}$$

Note that for a gradient Ricci soliton, $V = \nabla f$ and $\nabla E = \nabla_{\nabla f}\nabla f$. Thus, the above equation reduces to

$$\Delta(\mathrm{S} + 2E - \lambda f) = 0.$$

The boundedness of the energy function is a particularly restrictive condition for an expanding soliton. This is captured in the following theorem. Note that a zero energy function implies that a Ricci soliton is trivial.

Theorem 1. *Let (M^n, g, V, λ) be a Ricci soliton with E as its energy function.*

(1) If the soliton is nontrivial and E attains a maximum, then the soliton is shrinking or steady. If further $\mathrm{Ric}(V, V) \leq 0$, then the soliton is steady.
(2) For a nontrivial soliton, if E is bounded from above, then the soliton is steady or shrinking.
(3) If $\mathrm{Ric}(V, V) \leq 0$ and $|\nabla E| \in \mathcal{L}^1(M)$, then the soliton vector field is parallel and the soliton is trivial.

In particular, this theorem partially recovers the well-known result by Perelman (see (2.4) of [14]): There are no nontrivial compact expanding or steady Ricci solitons.

Let Ω be a compact subset of M. With respect to Ω, an end of the manifold is an unbounded connected component of $M \setminus \Omega$. If Ω_i is an exhaustion of M by compact sets, then the number of ends with respect to Ω_i is monotonically nondecreasing sequence. If this sequence is bounded, M is said to have finitely many ends. An end is said to be parabolic if it does not admit a positive harmonic function u that is constant on the its boundary and has a vanishing lim inf towards the infinity of the end. Let $\mathcal{H}_D(M)$ denote the space of harmonic functions u on M with finite dirichlet energy, i.e.,

$$\int |\nabla u|^2 \,\mathrm{dVol}_g < \infty.$$

Li and Tam [9] showed that the number of nonparabolic ends of a manifold is bounded above by the dimension of $\mathcal{H}_D(M)$. In [11], using this estimate, it was proven that a steady gradient Ricci soliton has at most one nonparabolic end. We show that their technique can be modified to solitons with bounded energy function and a scalar curvature lower bound. Note that for any Ricci soliton, the scalar curvature S is always bounded below (see Theorem 2.14, [4]). In view of this, the assumption of lower bound on the scalar curvature is mild one. In particular, the hypothesis of the following theorem is satisfied by steady gradient Ricci solitons.

Theorem 2. *On a noncompact Ricci soliton (M^n, g, V, λ) with $\mathrm{S} \geq \frac{(n-2)}{2}\lambda$, any harmonic function u with*

$$\int |\nabla u|^2 \,\mathrm{dVol}_g < \infty \text{ and } \int |\nabla u|^2 \sqrt{E}\,\mathrm{dVol}_g < \infty \tag{6}$$

is constant.

The theorem implies the following corollary, which can be seen as a generalisation of Corollary 4.2 of [11].

Corollary 2. *Any noncompact Ricci soliton (M^n, g, V, λ) with $\mathrm{S} \geq \frac{(n-2)}{2}\lambda$ and energy function E bounded above has at most one nonparabolic end.*

Proof. We see that for a soliton with bounded energy function E, Theorem 2 implies $\dim \mathcal{H}_D(M) = 1$. It then follows from [9], that M has at most one nonparabolic end. □

The drifted Laplacian, with respect to a vector field V, of a function u is defined as $\Delta_V u = \Delta u - g(V, \nabla u)$. If the vector field V is gradient, say $V = \nabla f$, the operator $\Delta_f u = \Delta u - g(V, \nabla u)$ is called the weighted Laplacian. The weighted Laplacian Δ_f is a self-adjoint operator with respect to the measure $e^{-f}\mathrm{dVol}g$. A function u is said to be V- harmonic if $\Delta_V u = 0$. Similarly, it is said to be f-harmonic if $\Delta_f u = 0$. In [12], Munteanu and Wang studied the dimension of the space of f-harmonic functions with some bounds. Similar to the previous theorem, we have the following result regarding V-harmonic functions on a noncompact Ricci soliton.

Theorem 3. *On a noncompact Ricci soliton (M^n, g, V, λ) with $\mathrm{S} > \frac{(n-2)}{2}\lambda$, any V-harmonic function u with*

$$\int |\nabla u|^2 E\,\mathrm{dVol}_g < \infty \text{ and } \int |\nabla u|^2 \,\mathrm{dVol}_g < \infty$$

is constant.

Moreover, we also obtain bounds on scalar curvature on Ricci solitons when scalar curvature is subharmonic with respect to the drifted Laplacian Δ_V (see Proposition 6). We see in [16] that a constant scalar curvature is a very restrictive condition on gradient Ricci solitons. We explore some of these results in Section 4 for the nongradient case. We also give examples of Cigar soliton and Sol_3 solitons supporting some of our results.

Theorem 1 of [13] implies that the measure $e^{-f}\mathrm{dVol}_g$ of a complete shrinking gradient Ricci soliton (M^n, g, f, λ) with bounded Ricci curvature is finite. We extend this result to complete, nongradient shrinking Ricci solitons, by using similar techniques as in this paper. In fact, we show that:

Theorem 4. *Let (M^n, g, V, λ) be a complete shrinking Ricci soliton with bounded Ricci curvature. Then the measure $e^{-E}\mathrm{dVol}_g$ is finite. Consequently, M has finite fundamental group.*

The finiteness of a weighted volume for a generic shrinking Ricci soliton is a new result whereas the finiteness of the fundamental group of a complete shriking Ricci soliton was proven by Wylie in [18] without any bound on Ricci curvature through a different approach. As a particular case of the above theorem, we partially recover the aforementioned result of [13] for gradient Ricci solitons. See Corollary 3 for details.

The paper is divided into 6 sections. The Section 2 is devoted to Preliminaries, Section 3 explores general computations on Ricci solitons, while the Section 4 investigates interplay between the scalar curvature, energy function and soliton vector field while the Section 5 investigates the L^1-Liouville type theorems on Ricci solitons. The final section namely, Section 6 generalizes a particular result of Aaron Naber [13] to nongradient shrinking Ricci solitons.

2. Preliminaries

In this section, we describe some of the basics on any Riemannian manifold (M^n, g) pertaining to Ricci solitons. The concepts and basic formulae can be found in [15] and [19].

In the sequel, the following Bochner's formula will be used.

$$\Delta\left(\frac{1}{2}|X|^2\right) = g(\Delta X, X) + |\nabla X|^2, \tag{7}$$

where Δ on the right hand side indicates the connection Laplacian and is defined as

$$\Delta X = \mathrm{tr}(\nabla^2 X).$$

The gradient version of the Bochner formula is

$$\Delta\left(\frac{1}{2}|\nabla u|^2\right) = \mathrm{Ric}\left(\nabla u, \nabla u\right) + g\left(\nabla u, \nabla\left(\Delta u\right)\right) + |\mathrm{Hess}\, u|^2. \tag{8}$$

The Lie derivative of the connection with respect to a vector field V on M is defined as follows:

$$(\mathcal{L}_V\nabla)(X,Y) = \mathcal{L}_V(\nabla_X Y) - \nabla_{\mathcal{L}_V X}Y - \nabla_X(\mathcal{L}_V Y).$$

We define the curvature tensor as

$$\mathrm{R}(X,Y)Z = \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

and its Lie derivative with respect to V as

$$(\mathcal{L}_V\,\mathrm{R})(X,Y)Z = \mathcal{L}_V(\mathrm{R}(X,Y)Z) - \mathrm{R}(\mathcal{L}_V X, Y)Z - \mathrm{R}(X, \mathcal{L}_V Y)Z - \mathrm{R}(X,Y)\mathcal{L}_V Z.$$

The following formulas can be found in [19], we present their proofs here for the sake of convenience.

Proposition 1 (Yano's formulae)**.** *For any vector fields X, Y, Z on a Riemannian manifold (M^n, g) we have,*

$$\left(\mathcal{L}_V\nabla\right)(X,Y) = \nabla^2_{X,Y}V + \mathrm{R}(V,X)Y, \tag{9}$$

$$\left(\nabla_X\left(\mathcal{L}_V g\right)\right)(Y,Z) = g(\nabla^2_{X,Y}V, Z) + g(Y, \nabla^2_{X,Z}V), \tag{10}$$

$$\left(\nabla_X\left(\mathcal{L}_V g\right)\right)(Y,Z) = g\left(\left(\mathcal{L}_V\nabla\right)(X,Y), Z\right) + g\left(\left(\mathcal{L}_V\nabla\right)(X,Z), Y\right), \tag{11}$$

$$(\mathcal{L}_V\,\mathrm{R})(X,Y)Z = \left(\nabla_X\left(\mathcal{L}_V\nabla\right)\right)(Y,Z) - \left(\nabla_Y\left(\mathcal{L}_V\nabla\right)\right)(X,Z). \tag{12}$$

Proof. The first two equations follow from unwinding the definition.

$$\begin{aligned}(\mathcal{L}_V\nabla)(X,Y) &= \nabla_V\nabla_X Y - \nabla_{\nabla_X Y}V - \nabla_{\nabla_V X-\nabla_X V}Y - \nabla_X(\nabla_V Y - \nabla_Y V),\\ &= \nabla^2_{V,X}Y - \nabla^2_{X,V}Y + \nabla^2_{X,Y}V = \nabla^2_{X,Y}V + \mathrm{R}(V,X)Y.\end{aligned}$$

$$\begin{aligned}(\nabla_X(\mathcal{L}_V g))(Y,Z) &= X(\mathcal{L}_V g(Y,Z)) - (\mathcal{L}_V g)(\nabla_X Y,Z) - (\mathcal{L}_V g)(Y,\nabla_X Z),\\ &= X\left(g(\nabla_Y V,Z) + g(Y,\nabla_Z V)\right)\\ &\qquad - (\mathcal{L}_V g)(\nabla_X Y,Z) - (\mathcal{L}_V g)(Y,\nabla_X Z),\\ &= g(\nabla^2_{X,Y}V,Z) + g(Y,\nabla^2_{X,Z}V).\end{aligned}$$

The third equation follows from the first two and the symmetries of the curvature tensor.

$$\begin{aligned}&g((\mathcal{L}_V\nabla)(X,Y),Z) + g\left((\mathcal{L}_V\nabla)(X,Z),Y\right)\\ &\qquad = g(\nabla^2_{X,Y}V,Z) + g(\nabla^2_{X,Z}V,Y) + \mathrm{R}(V,X,Y,Z) + \mathrm{R}(V,X,Z,Y),\\ &\qquad = (\nabla_X(\mathcal{L}_V g))(Y,Z).\end{aligned}$$

The last equation follows from [19, eqn 5.16] or can be computed directly. □

It is well known that for any function f and vector field X,

$$\operatorname{div}\operatorname{Hess} f(X) = \operatorname{Ric}(\nabla f, X) + X(\Delta f).$$

The following is the nongradient version of the above formula (see also Lemma 2.1 of [16]).

Proposition 2. *Any two vector fields V, X on a Riemannian manifold (M^n, g) satisfy:*

$$[\operatorname{div}(\mathcal{L}_V g)]X = g(\Delta V, X) + \operatorname{Ric}(V,X) + X(\operatorname{div} V). \tag{13}$$

Proof. For a point $p \in M$, let $\{E_i\}$ be an orthonormal frame in a neighbourhood of point p which is parallel at p. At point $p \in M$ we get,

$$X(\operatorname{div} V) = X\left(\sum_{i=1}^n g(E_i, \nabla_{E_i}V)\right),$$

which implies

$$X(\operatorname{div} V) = \sum_{i=1}^n g(E_i, \nabla^2_{X,E_i}V). \tag{14}$$

Now, using eq. (10) at p we obtain,

$$\begin{aligned}[\operatorname{div}(\mathcal{L}_V g)]X &= (\nabla_{E_i}(\mathcal{L}_V g))(E_i,X),\\ &= g(\nabla^2_{E_i,E_i}V,X) + g(E_i,\nabla^2_{E_i,X}V),\\ &= g(\Delta V,X) + g(E_i,\mathrm{R}(E_i,X)V) + g(E_i,\nabla^2_{X,E_i}V),\\ &= g(\Delta V,X) + \operatorname{Ric}(X,V) + X(\operatorname{div} V),\end{aligned}$$

where the last equality uses eq. (14). □

3. Basic Identities on Ricci solitons

This section focuses on extending Proposition 2.1 of [6] to the nongradient case.

Proposition 3. *For a Ricci soliton (M^n, g, V, λ) the following holds:*

$$\operatorname{div} V = \frac{n\lambda}{2} - \mathrm{S}, \tag{15}$$

$$-2\Delta V = 2\,\mathrm{Q}\,V = \lambda V - \nabla E - \nabla_V V, \tag{16}$$

$$|\nabla V|^2 - \Delta E = \mathrm{Ric}(V, V) = \lambda E - V E. \tag{17}$$

Proof. Equation (15) is obtained by contracting the Ricci soliton equation (eq. (1)). Taking divergence of eq. (1), using second contracted Bianchi identity, and comparing with eq. (13), one obtains

$$\Delta V + \mathrm{Q}\,V + \nabla(\operatorname{div} V) = -\nabla\,\mathrm{S}\,.$$

Substituting eq. (15) above gives left half of eq. (16). From eq. (1), we obtain

$$\mathrm{Ric}(X, V) = \frac{\lambda}{2} g(X, V) - \frac{1}{2}(\mathcal{L}_V g)(X, V) = \frac{\lambda}{2} g(X, V) - \frac{1}{2}\Big(XE + g(X, \nabla_V V)\Big)\,.$$

Thus,

$$2\,\mathrm{Q}\,V = \lambda V - \nabla E - \nabla_V V,$$

which proves the right half of eq. (16). Taking inner product of eq. (16) with V and using eq. (7) gives eq. (17). ☐

Lemma 1. *On a Ricci soliton (M^n, g, V, λ), any smooth function u satisfies the following Bochner formula*

$$\Delta_V\left(\frac{1}{2}|\nabla u|^2\right) = \frac{\lambda}{2}|\nabla u|^2 + g\left(\nabla u, \nabla(\Delta_V u)\right) + |\mathrm{Hess}\, u|^2\,. \tag{18}$$

Proof. By definition, $\Delta u = \Delta_V u + g(V, \nabla u)$. Thus,

$$\begin{aligned} g\left(\nabla u, \nabla(\Delta u)\right) &= g\left(\nabla u, \nabla(\Delta_V u)\right) + g\left(\nabla_{\nabla u} V, \nabla u\right) + g(V, \nabla_{\nabla u}\nabla u), \\ &= g\left(\nabla u, \nabla(\Delta_V u)\right) + \frac{1}{2}\mathcal{L}_V g(\nabla u, \nabla u) + \mathrm{Hess}\, u(V, \nabla u). \end{aligned}$$

Substituting in the Bochner formula (eq. (8)) and using the Ricci soliton equation, we obtain

$$\Delta\left(\frac{1}{2}|\nabla u|^2\right) = \lambda\left(\frac{1}{2}|\nabla u|^2\right) + g\left(\nabla u, \nabla(\Delta_V u)\right) + \mathrm{Hess}\, u(V, \nabla u) + |\mathrm{Hess}\, u|^2\,.$$

Laslty, noting

$$V\left(\frac{1}{2}|\nabla u|^2\right) = g\left(\nabla_V \nabla u, \nabla u\right) = \mathrm{Hess}\, u(V, \nabla u)$$

gives the desired equation. ☐

The following is a known result (see Lemma 2.1 of [10]). It allows for an estimate of V-Laplacian of the scalar curvature. We share a different proof based on Yano's formulae derived in Proposition 1

Theorem 5. *For a Ricci soliton (M^n, g, V, λ), the following holds:*

$$\Delta \mathrm{S} - V\,\mathrm{S} - \lambda\,\mathrm{S} = -2\,|\mathrm{Q}|^2\,, \tag{19}$$

$$\Delta \mathrm{S} - V\,\mathrm{S} - \frac{2\,\mathrm{S}}{n}\,(\operatorname{div} V) = -2\left|\mathring{\mathrm{Q}}\right|^2. \tag{20}$$

Proof. The second equation follows from the first by noting $\left|\mathring{\mathrm{Q}}\right|^2 = |\mathrm{Q}|^2 - \frac{\mathrm{S}^2}{n}$ and eq. (15).

Differentiating eq. (1) along an arbitrary vector field, we obtain

$$\nabla_X\left(\mathcal{L}_V g\right) = -2\nabla_X \operatorname{Ric}.$$

Taking cyclic sum of eq. (11) (with subtraction on the last one) and using the above gives

$$\begin{aligned}
&2g\left(\left(\mathcal{L}_V \nabla\right)(X,Y), Z\right)\\
&\quad= \left(\nabla_X\left(\mathcal{L}_V g\right)\right)(Y,Z) + \left(\nabla_Y\left(\mathcal{L}_V g\right)\right)(Z,X) - \left(\nabla_Z\left(\mathcal{L}_V g\right)\right)(X,Y)\\
&\quad= -2\left(\left(\nabla_X \operatorname{Ric}\right)(Y,Z) + \left(\nabla_Y \operatorname{Ric}\right)(Z,X) - \left(\nabla_Z \operatorname{Ric}\right)(X,Y)\right).
\end{aligned}$$

For simplicity of computation we define a $(1,1)$ tensor T by

$$g(T(X,Y),Z) = (\nabla_Z \operatorname{Ric})(X,Y).$$

Also for any $(1,2)$ tensor $\mathcal{U}$, we denote its symmetrization by

$$(\mathcal{U} \circ \mathrm{sym})(X,Y) = \mathcal{U}(X,Y) + \mathcal{U}(Y,X).$$

Then we can write

$$\left(\mathcal{L}_V \nabla\right)(X,Y) = -\left((\nabla_X \mathrm{Q})Y + (\nabla_Y \mathrm{Q})X - T(X,Y)\right).$$

Or simply

$$\mathcal{L}_V \nabla = -\left(\nabla\,\mathrm{Q} \circ \mathrm{sym} - T\right).$$

Therefore,

$$\nabla_X\left(\mathcal{L}_V \nabla\right) = -\left(\nabla_X \nabla\,\mathrm{Q} \circ \mathrm{sym} - \nabla_X T\right).$$

Now using the above in eq. (12),

$$\begin{aligned}
(\mathcal{L}_V \mathrm{R})(X,Y)Z = &-\left(\left(\nabla^2_{X,Y}\mathrm{Q}\right)Z + \left(\nabla^2_{X,Z}\mathrm{Q}\right)Y - \nabla_X T(Y,Z)\right)\\
&+\left(\left(\nabla^2_{Y,X}\mathrm{Q}\right)Z + \left(\nabla^2_{Y,Z}\mathrm{Q}\right)X - \nabla_Y T(X,Z)\right).
\end{aligned}$$

Let $\{E_i\}$ be an orthonormal frame in the neighbourhood of a point point $p \in M$ which is parallel at p. Noting,

$$g((\nabla_X T)(Y,Z), W) = (\nabla^2_{X,W} \operatorname{Ric})(Y,Z),$$

and contracting the above equation w.r.t. X, we obtain

$$\begin{aligned}
(\mathcal{L}_V \operatorname{Ric})(Y,Z) = &\left(-\nabla^2_{E_i,Y}\operatorname{Ric}\right)(Z,E_i) - \left(\nabla^2_{E_i,Z}\operatorname{Ric}\right)(Y,E_i) + \left(\nabla^2_{E_i,E_i}\operatorname{Ric}\right)(Y,Z)\\
&+\left(\nabla^2_{Y,E_i}\operatorname{Ric}\right)(Z,E_i) + \left(\nabla^2_{Y,Z}\operatorname{Ric}\right)(E_i,E_i) - \left(\nabla^2_{Y,E_i}\operatorname{Ric}\right)(E_i,Z).
\end{aligned} \tag{21}$$

The contracted Bianchi identity implies,

$$(\nabla_{E_i}\operatorname{Ric})(E_i,Z)=\frac{1}{2}\mathrm{d}\,\mathrm{S}(Z),$$

$$(\nabla_Y\nabla_{E_i}\operatorname{Ric})(E_i,Z)+(\nabla_{E_i}\operatorname{Ric})(E_i,\nabla_Y Z)=\frac{1}{2}(\nabla_Y\mathrm{d}\,\mathrm{S})(Z)+\frac{1}{2}\mathrm{d}\,\mathrm{S}(\nabla_Y Z),$$

$$(\nabla^2_{Y,E_i}\operatorname{Ric})(E_i,Z)=\frac{1}{2}\operatorname{Hess}\mathrm{S}(Y,Z).$$

Furthermore,

$$\begin{aligned}(\nabla^2_{E_i,Y}\operatorname{Ric})(E_i,Z)&=(\mathrm{R}(E_i,Y)\operatorname{Ric})(E_i,Z)+(\nabla^2_{Y,E_i}\operatorname{Ric})(E_i,Z),\\&=(\mathrm{R}(E_i,Y)\operatorname{Ric})(E_i,Z)+\frac{1}{2}\operatorname{Hess}\mathrm{S}(Y,Z).\end{aligned}$$

Substituting in eq. (21),

$$\begin{aligned}(\mathcal{L}_V\operatorname{Ric})(Y,Z)=&-(\mathrm{R}(E_i,Y)\operatorname{Ric})(E_i,Z)-\frac{1}{2}\operatorname{Hess}\mathrm{S}(Y,Z)\\&-(\mathrm{R}(E_i,Z)\operatorname{Ric})(E_i,Y)-\frac{1}{2}\operatorname{Hess}\mathrm{S}(Z,Y)\\&+(\Delta\operatorname{Ric})(Y,Z)+\frac{1}{2}\operatorname{Hess}\mathrm{S}(Y,Z)+\operatorname{Hess}\mathrm{S}(Y,Z)-\frac{1}{2}\operatorname{Hess}\mathrm{S}(Y,Z).\end{aligned}\tag{22}$$

Noting

$$(\mathrm{R}(E_i,Y)\operatorname{Ric})(E_i,Z)=-\operatorname{Ric}(\mathrm{R}(E_i,Y)E_i,Z)-\operatorname{Ric}(E_i,\mathrm{R}(E_i,Y)Z),\tag{23}$$

$$=g(\mathrm{Q}\,Y,\mathrm{Q}\,Z)-\mathrm{R}(E_i,Y,Z,\mathrm{Q}\,E_i).\tag{24}$$

Using eq.(24) in eq. (22) and simplifying we obtain,

$$\begin{aligned}(\mathcal{L}_V\operatorname{Ric})(Y,Z)=&-g(\mathrm{Q}\,Y,\mathrm{Q}\,Z)+\mathrm{R}(E_i,Y,Z,\mathrm{Q}\,E_i)\\&-g(\mathrm{Q}\,Z,\mathrm{Q}\,Y)+\mathrm{R}(E_i,Z,Y,\mathrm{Q}\,E_i)+(\Delta\operatorname{Ric})(Y,Z).\end{aligned}\tag{25}$$

Tracing the above equation,

$$\mathcal{L}_V g(\mathrm{Q}\,E_i,E_i)+2V\,\mathrm{S}=\Delta\,\mathrm{S}\,.$$

Now eq. (1) gives,

$$2(\mathcal{L}_V g)(E_i,\mathrm{Q}\,E_i)=2\lambda\,\mathrm{S}-4\,|\mathrm{Q}|^2\,.$$

Therefore,

$$\lambda\,\mathrm{S}-2\,|\mathrm{Q}|^2+V\,\mathrm{S}=\Delta\,\mathrm{S}\,.$$

□

The rest of this section is dedicated to proving Corollary 1. It will involve various auxiliary identities involving the scalar curvature and the energy function. We begin by establishing the following lemma.

Lemma 2. *Let (M^n,g) be a Riemannian manifold. Then for any symmetric $(1,1)$-tensor T on M, and any vector field X on M,*

$$\operatorname{div}(TX)=(\operatorname{div}T)X+\frac{1}{2}g\left(\mathring{T},\mathcal{L}_X g\right)+\frac{\operatorname{tr}T}{n}\operatorname{div}X.$$

Proof. Let (E_i) be an orthonormal frame in some neighbourhood of a point p. Then

$$\begin{aligned} g(T, \mathcal{L}_X g) &= g(T(E_i), E_j)\ (\mathcal{L}_X g)(E_i, E_j) \\ &= T_i^j \left(g(\nabla_{E_i} X, E_j) + g(E_i, \nabla_{E_j} X) \right) \\ &= 2g(TE_i, \nabla_{E_i} X) = 2g(E_i, T(\nabla_{E_i} X)). \end{aligned}$$

Now for any vector field Y,

$$\nabla_Y(TX) = (\nabla_Y T)X + T(\nabla_Y X).$$

Contracting the above equation over Y,

$$g(\nabla_{E_i}(TX), E_i) = g\left((\nabla_{E_i} T)X, E_i) + g(T(\nabla_{E_i} X), E_i)\right).$$

Hence,

$$\operatorname{div}(TX) = (\operatorname{div} T)X + \frac{1}{2} g(T, \mathcal{L}_X g).$$

Substituting $\mathring{T} = T - \frac{\operatorname{tr} T}{n} g$ produces the result. □

Remark 1. For a closed manifold, integrating the above equation implies Theorem 4.2 of [8].

Proposition 4. *For any Ricci soliton* (M^n, g, V, λ),

$$\frac{1}{2}\left(V\,\mathrm{S} + \operatorname{div}(\nabla E) + \operatorname{div}(\nabla_V V) \right) = \frac{1}{n}(\operatorname{div} V)^2 + \left| \mathring{Q} \right|^2. \tag{26}$$

Proof. Taking $T = \mathrm{Q}$ and $X = V$ in Lemma 2, we have

$$\operatorname{div}(\mathrm{Q}\, V) = (\operatorname{div} \mathrm{Q})V + \frac{1}{2} g\left(\mathring{\mathrm{Q}}, \mathcal{L}_V g\right) + \frac{\operatorname{tr} T}{n} \operatorname{div} V.$$

Now availing eq. (1) and second contracted Bianchi identity viz., $2 \operatorname{div} \mathrm{Ric} = d\,\mathrm{S}$, we obtain

$$\operatorname{div}(\mathrm{Q}\, V) = \frac{1}{2} V\,\mathrm{S} - \left| \mathring{Q} \right|^2 + \frac{\mathrm{S}}{n} \operatorname{div} V.$$

Comparing the above with eq. (16),

$$\frac{1}{2} V\,\mathrm{S} - \left| \mathring{Q} \right|^2 + \frac{\mathrm{S}}{n} \operatorname{div} V = \operatorname{div}(\mathrm{Q}\, V) = \frac{1}{2}\left(\lambda \operatorname{div} V - \Delta E - \operatorname{div}(\nabla_V V) \right).$$

Or simply

$$\frac{1}{2}\left(V\,\mathrm{S} + \operatorname{div}(\nabla E) + \operatorname{div}(\nabla_V V) \right) = \left| \mathring{Q} \right|^2 - \frac{\mathrm{S}}{n}(\operatorname{div} V) + \frac{\lambda}{2} \operatorname{div} V. \tag{27}$$

Lastly eq. (15) implies

$$\frac{1}{2}\left(V\,\mathrm{S} + \operatorname{div}(\nabla E) + \operatorname{div}(\nabla_V V) \right) = \frac{1}{n}(\operatorname{div} V)^2 + \left| \mathring{Q} \right|^2.$$

□

Remark 2. For any noncompact Ricci soliton with constant scalar curvature, the above corollary implies the existence of a vector field $W = \nabla E + \nabla_V V$ which has a nonnegative divergence, and W is divergence-free iff the soliton is trivial (Einstein).

Corollary 1. *For any Ricci soliton* (M^n, g, V, λ),

$$\operatorname{div}(\nabla S + \nabla E + \nabla_V V - \lambda V) = 0. \tag{5}$$

Proof. From eq. (20) and eq. (27)

$$\Delta \mathrm{S} = V\,\mathrm{S} - 2\left|\mathring{\mathrm{Q}}\right|^2 + \frac{2\,\mathrm{S}}{n}(\operatorname{div} V) = \lambda \operatorname{div} V - \operatorname{div}(\nabla E + \nabla_V V).$$

□

Thus we affirm Corollary 1 as stated in the introduction.

4. Interplay between the scalar curvature, energy function and soliton vector field

In this section, we establish Theorem 1 as stated in the introduction. To prove Theorem 1, we will use the Omori-Yau maximum principle, Theorem 7, and Lemma 3. In the later part of the section, motivated by the results of [16], we explore Ricci solitons of constant scalar curvature. In particular, we obtain bounds on scalar curvature and the energy function.

First we wish to showcase a result from [17] which shows the importance of the energy function. Rewriting eq. (4) as

$$\lambda \nabla f = \nabla(\mathrm{S} + 2E),$$

we see that for a given nonsteady Ricci soliton (M^n, g, V, λ), up to a constant, the best candidate for a soliton function is $\frac{1}{\lambda}(\mathrm{S} + 2E)$. Sharma et. al. showed that this is indeed the case when ∇V is symmetric.

Theorem 6. *For a non-steady Ricci soliton (M^n, g, V, λ), if ∇V is symmetric, then the soliton is gradient.*

Proof. When ∇V is symmetric, we have

$$\mathcal{L}_V g(X,Y) = 2g(\nabla_X V, Y) = 2g(X, \nabla_Y V) \text{ and}$$
$$(\nabla_Z(\mathcal{L}_V g))(X,Y) = 2g(\nabla^2_{Z,X} V, Y) = 2g(X, \nabla^2_{Z,Y} V).$$

It then follows that (also see proof of Proposition 2),

$$(\operatorname{div}(\mathcal{L}_V g))Y = 2g(\Delta V, Y) = 2\operatorname{Ric}(V,Y) + 2Y(\operatorname{div} V).$$

Therefore, using eq. (1) and the second contracted Bianchi identity, one obtains

$$-\nabla \mathrm{S} = -2\operatorname{div} \mathrm{Q} = \operatorname{div}(\mathcal{L}_V g) = 2\Delta V = 2(\mathrm{Q}\,V + \nabla(\operatorname{div} V)).$$

Using eq. (15) above,

$$\mathrm{Q}\,V = \frac{1}{2}\nabla \mathrm{S} = -\Delta V. \tag{28}$$

Now, the definition of E and the symmetry of ∇V implies

$$XE = g(\nabla_X V, V) = \frac{1}{2}\mathcal{L}_V g(X,V) = \frac{\lambda}{2} g(X,V) - \operatorname{Ric}(X,V),$$

and thus,

$$\nabla E = \frac{\lambda}{2} V - \mathrm{Q}\,V.$$

Comparing with eq. (28),

$$\lambda V = \nabla\left(2E + S\right).$$

□

Now we give the different and shorter proof of the following lemma than [20, p. 660]. In what follows we will work with the cut off function ϕ on M which is supported on $B(p, 2R)$ and satisfying $|\nabla\phi| \leq \frac{C}{R}$ on $B(p, 2R)$.

Lemma 3. *For a subharmonic function f on a noncompact manifold (M^n, g), if $|\nabla f| \in \mathcal{L}^1(M)$, then $\Delta f = 0$.*

Proof. Fix point p, and let $A_R = B(p, 2R)\setminus B(p, R)$. We consider ϕ to be the cutoff function supported in $B(p, 2R)$, such that $|\nabla\phi| \leq \frac{C}{R}$. Then,

$$\begin{aligned}\int_M \phi\Delta f \,\mathrm{dVol}_g = -\int_{A_R} g\left(\nabla\phi, \nabla f\right) \mathrm{dVol}_g &\leq \int_{A_R} |\nabla\phi| \, |\nabla f| \; \mathrm{dVol}_g \\ &\leq \frac{C}{R}\int_{A_R} |\nabla f| \; \mathrm{dVol}_g \to 0 \text{ as } R \to \infty.\end{aligned}$$

Since the integrand on the LHS is positive, the result follows from the monotone convergence theorem. □

In [3], it was proven that for any (generic) Ricci soliton with soliton vector field V, the Omori-Yau maximum principle holds for the operator Δ_V. Effectively, we have

Theorem 7 ([3])**.** *Let (M^n, g, V, λ) be a Ricci soliton. Then for any $u \in C^2(M)$ with $u^* = \sup_M u < \infty$, there exists a sequence of points $x_k \in M$ such that*

$$\lim_{k\to\infty} u(x_k) > u^* - \frac{1}{k}, \quad \left|\nabla u\right|(x_k) < \frac{1}{k}, \quad \Delta_V u(x_k) < \frac{1}{k}.$$

Theorem 1. *Let (M^n, g, V, λ) be a Ricci soliton with E as its energy function.*

1. *If the soliton is nontrivial and E attains a maximum, then the soliton is shrinking or steady. If further* $\mathrm{Ric}(V, V) \leq 0$, *then the soliton is steady.*
2. *For a nontrivial soliton, if E is bounded from above, then the soliton is steady or shrinking.*
3. *If* $\mathrm{Ric}(V, V) \leq 0$ *and $|\nabla E| \in \mathcal{L}^1(M)$, then the soliton vector field is parallel and the soliton is trivial.*

Proof. For a nontrivial Ricci soliton, the energy function E is not identically zero.

(1) Equation (17) implies

$$\Delta E + \lambda E = \left|\nabla V\right|^2 + g(V, \nabla E). \tag{29}$$

If E attains a maximum at p, then $\nabla E|_p = 0$ and $\Delta E|_p \leq 0$. Thus, $\lambda \geq 0$.
If further, $\mathrm{Ric}(V, V) \leq 0$, the other part of eq. (17) implies

$$\Delta E = \left|\nabla V\right|^2 - \mathrm{Ric}(V, V) \geq 0.$$

Thus E is subharmonic and if it attains a maximum, it must be constant. Then $|\nabla V| = 0$ and $\lambda E = 0$, which gives the result.

(2) The Omori-Yau maximum principle (Theorem 7) implies the existence of a sequence (x_k) such that

$$\lim_{k\to\infty} E(x_k) = E^*, \; \left|\nabla E\right|(x_k) < \frac{1}{k}, \; \Delta_V E(x_k) < \frac{1}{k}.$$

Noting $g(V, \nabla E) \geq -|V|\,|\nabla E| = -\sqrt{2E}\,|\nabla E|$ and evaluating eq. (29) at x_k

$$\begin{aligned}\lambda E(x_k) &= |\nabla V|^2(x_k) - \Delta_V E(x_k)\\ &> |\nabla V|^2(x_k) - \frac{1}{k}\end{aligned}$$

As $k \to \infty$, we see that $\lambda E^* \geq 0$, which implies $E = 0$ or $\lambda \geq 0$.

(3) From eq. (17) and the hypothesis, we have $\Delta E = |\nabla V|^2 - \mathrm{Ric}(V, V) \geq 0$. Consequently, by the hypothesis that $|\nabla E| \in L^1(M)$ and from Lemma 3, we obtain that $\Delta E = 0$ and hence $|\nabla V|^2 = \mathrm{Ric}(V, V) = 0$.

□

Example 1. Consider $(\mathbb{R}^2, g, V, \lambda)$ with

$$g = \frac{2}{1+y^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right),$$

$V = -x\partial_x - y\partial_y$ and $\lambda = -1$. Its scalar curvature $\mathrm{S} = \frac{1-y^2}{1+y^2}$. Its energy function

$$E = \frac{x^2+y^2}{1+y^2}$$

is not bounded above and also the soliton is expanding.

Example 2. The Sol3 soliton $(\mathbb{R}^3, g, V, \lambda)$ is noncompact, nongradient expanding Ricci soliton with

$$g = e^{2t}\mathrm{d}x^2 + e^{-2t}\mathrm{d}y^2 + \mathrm{d}t^2,$$

$V = -2x\partial_x + 2y\partial_y$ and $\lambda = -4$. It has constant scalar curvature $\mathrm{S} = -2$. See [1] for a detailed exposition. Its energy function

$$E = 2\left(e^{2t}x^2 + e^{-2t}y^2\right)$$

is not bounded above.

5. L^1-Liouville type theorem with respect to energy function

In this section, we prove Theorems 2 and 3. To handle the equality case of the lower bound on the scalar curvature in Theorem 2, we require a technical lemma. We take a brief look at Ricci solitons of constant scalar curvature before proving Lemma 4.

In [7] gradient Ricci solitons of constant scalar curvature were investigated. It is shown that for such solitons the possible values of scalar curvature is of the form $\frac{k\lambda}{2}$ for $0 \leq k \leq n$. The main idea of their proof depends on the theory of isoparametric functions, and analysis of eq. (4). However, an analogue of eq. (4) is absent in the nongradient case. Nevertheless, we have the following results. Note that by eq. (15) constant S implies that $\operatorname{div} V$ is constant.

Proposition 5. *A compact Ricci soliton (M^n, g, V, λ) with constant scalar curvature is Einstein.*

Proof. Follows directly from eq. (26) using Stokes' theorem. □

Proposition 6. *Let (M^n, g, V, λ) be a Ricci soliton with $\Delta_V \mathrm{S} \geq 0$, then*

$$0 \leq \mathrm{S} \leq \frac{n\lambda}{2} \quad OR \quad \frac{n\lambda}{2} \leq \mathrm{S} \leq 0. \tag{30}$$

Further,

(1) $\lambda = 0$ or $\mathrm{S} = 0$ implies M is Ricci flat.
(2) $\mathrm{S} = \frac{n\lambda}{2}$ implies M is Einstein.

In particular, the conclusion holds for solitons of constant scalar curvature.

Proof. The proof follows simply by employing $\Delta_V \mathrm{S} \geq 0$ in eq. (19) and eq. (20). In fact, respectively we obtain,

$$\lambda \mathrm{S} = 2 \left|\mathrm{Q}\right|^2 \geq 0,$$
$$\frac{2\,\mathrm{S}}{n}\left(\frac{n\lambda}{2} - \mathrm{S}\right) = 2\left|\mathring{\mathrm{Q}}\right|^2 \geq 0.$$

□

Gaffney's Stokes theorem states that for a complete (orientable) Riemannian manifold, if $|V|$ and $\operatorname{div} V$ are integrable, then $\int(\operatorname{div} V)\,\mathrm{d}\,\mathrm{Vol}_g = 0$. This implies the following

Lemma 4. *Let (M^n, g, V, λ) be an (orientable) Ricci soliton with constant scalar curvature $\mathrm{S} \neq \frac{n\lambda}{2}$ of finite volume, then $\sqrt{E} \notin \mathcal{L}^1(M)$.*

Proof. As M has finite volume, constant $\operatorname{div} V$ implies $\operatorname{div} V$ is integrable. If $\sqrt{E}$ is integrable, so is $|V|$, and then Gaffney's Stokes theorem implies $\mathrm{S} = \frac{n\lambda}{2}$, which is contradictory to the given hypothesis. □

We are now ready to prove Theorems 2 and 3. Both proofs rely on the divergence theorem to transfer the derivative from u to a cut-off function and then applying the respective Bochner formulas.

Theorem 2. *On a noncompact Ricci soliton (M^n, g, V, λ) with $\mathrm{S} \geq \frac{(n-2)}{2}\lambda$, any harmonic function u with*

$$\int |\nabla u|^2 \,\mathrm{dVol}_g < \infty \ \text{ and } \int |\nabla u|^2 \sqrt{E}\,\mathrm{dVol}_g < \infty \tag{6}$$

is constant.

Proof. For any point p, let ϕ be the standard cut-off function on $\mathrm{B}_p(2r)$. Let $A_r = \mathrm{B}_p(2r) \setminus \mathrm{B}_p(r)$. Let u be a harmonic function satisfying eq. (6).

From eq. (1),

$$\int_M \phi^2 \operatorname{Ric}(\nabla u, \nabla u)\,\mathrm{dVol} = \frac{\lambda}{2}\int_M \phi^2 \left|\nabla u\right|^2 \mathrm{dVol} - \int_M \phi^2 g(\nabla_{\nabla u} V, \nabla u)\,\mathrm{dVol} \tag{31}$$

Since ϕ has compact support, noting $\Delta u = 0$ and applying the divergence theorem to the vector fields $(Vu)\phi^2\nabla u$ and $\phi^2 \left|\nabla u\right|^2 V$ yields

$$-\int_M \phi^2 g(\nabla_{\nabla u} V, \nabla u)\,\mathrm{dVol} = \int_M \phi^2 \operatorname{Hess} u(V, \nabla u)\,\mathrm{dVol} + \int_M (Vu) g(\nabla u, \nabla\phi^2)\,\mathrm{dVol} \text{ and}$$

$$2\int_M \phi^2 \operatorname{Hess} u(V, \nabla u)\,\mathrm{dVol} = -\int_M \phi^2 \left|\nabla u\right|^2 (\operatorname{div} V)\,\mathrm{dVol} - \int_M \left|\nabla u\right|^2 g(V, \nabla\phi^2)\,\mathrm{dVol}$$

Combined, the above equations give

$$
\begin{aligned}
(32)\qquad -\int_M \phi^2 g(\nabla_{\nabla u} V, \nabla u)\,\mathrm{dVol} &= \int_M (Vu) g(\nabla u, \nabla \phi^2)\,\mathrm{dVol} \\
&\quad - \frac{1}{2}\int_M \phi^2 \left|\nabla u\right|^2 (\operatorname{div} V)\,\mathrm{dVol} - \frac{1}{2}\int_M \left|\nabla u\right|^2 g(V, \nabla\phi^2)\,\mathrm{dVol}
\end{aligned}
$$

Now

$$
\begin{aligned}
\int_M \phi^2 \Delta\left(\frac{1}{2}\left|\nabla u\right|^2\right)\,\mathrm{dVol} &= -\frac{1}{2}\int_M g(\nabla\phi^2, \nabla(\left|\nabla u\right|^2))\,\mathrm{dVol} \\
&\leq \frac{1}{2}\int_M 2\phi\left|\nabla\phi\right| 2\left|\nabla u\right| \left|\nabla\left|\nabla u\right|\right|\,\mathrm{dVol} \\
&\leq \frac{1}{2}\int_M \left(\phi^2\left|\nabla\left|\nabla u\right|\right|^2 + 4\left|\nabla\phi\right|^2\left|\nabla u\right|^2\right)\mathrm{dVol}
\end{aligned}
$$

Using eq. (8) for harmonic u and Kato's inequality,

$$
\Delta\left(\frac{1}{2}\left|\nabla u\right|^2\right) = \operatorname{Ric}(\nabla u, \nabla u) + \left|\operatorname{Hess} u\right|^2 \geq \operatorname{Ric}(\nabla u, \nabla u) + \left|\nabla\left|\nabla u\right|\right|^2.
$$

Therefore,

$$
(33)\quad \int_M \phi^2 \operatorname{Ric}(\nabla u, \nabla u)\,\mathrm{dVol} \leq -\frac{1}{2}\int_M \phi^2\left|\nabla\left|\nabla u\right|\right|^2\,\mathrm{dVol} + 2\int_M \left|\nabla\phi\right|^2\left|\nabla u\right|^2\,\mathrm{dVol}
$$

Combining eqs. (31) to (33)

$$
\begin{aligned}
(34)\qquad &\int_M \frac{1}{2}(\lambda - \operatorname{div} V)\phi^2\left|\nabla u\right|^2\,\mathrm{dVol} + \frac{1}{2}\int_M \phi^2\left|\nabla\left|\nabla u\right|\right|^2\,\mathrm{dVol} \\
&\leq \frac{1}{2}\int_M g(\left|\nabla u\right|^2 V - 2(Vu)\nabla u, \nabla\phi^2)\,\mathrm{dVol} + 2\int_M \left|\nabla\phi\right|^2\left|\nabla u\right|^2\,\mathrm{dVol}
\end{aligned}
$$

Lastly, noting eq. (15) and

$$
\left|\left|\nabla u\right|^2 V - 2(Vu)\nabla u\right|^2 = 2E\left|\nabla u\right|^4,
$$

we obtain

$$
\begin{aligned}
(35)\qquad &\int_M \left(\mathrm{S} - (n-2)\frac{\lambda}{2}\right)\phi^2\left|\nabla u\right|^2\,\mathrm{dVol} + \frac{1}{2}\int_M \phi^2\left|\nabla\left|\nabla u\right|\right|^2\,\mathrm{dVol} \\
&\leq \frac{1}{2}\int_M \sqrt{2E}\left|\nabla u\right|^2\left|\nabla\phi^2\right|\,\mathrm{dVol} + 2\int_M \left|\nabla\phi\right|^2\left|\nabla u\right|^2\,\mathrm{dVol}
\end{aligned}
$$

Since $\nabla\phi$ is nonzero only on A_r, the integral on the right hand side is over A_r. Equation (6) implies that the right hand side vanishes as $r \to \infty$.

Since $\mathrm{S} \geq \frac{(n-2)}{2}\lambda$, we have that $|\nabla u|$ is constant and at least one of the following is true: u is constant or $\mathrm{S} = \frac{(n-2)}{2}\lambda$. Suppose u is not constant and subsequently $\mathrm{S} = \frac{(n-2)}{2}\lambda$. Constant $|\nabla u|$ and the hypothesis eq. (6) imply that the manifold has finite volume and $\sqrt{E} \in \mathcal{L}^1(M)$. This is in contradiction to Lemma 4. ☐

Theorem 3. *On a noncompact Ricci soliton (M^n, g, V, λ) with $\mathrm{S} > \frac{(n-2)}{2}\lambda$, any V-harmonic function u with*

$$
\int \left|\nabla u\right|^2 E\,\mathrm{dVol}_g < \infty \text{ and } \int \left|\nabla u\right|^2\,\mathrm{dVol}_g < \infty
$$

is constant.

Proof. For any point p, let ϕ be the standard cut-off function on $\mathrm{B}_p(2r)$. Let $A_r = \mathrm{B}_p(2r) \setminus \mathrm{B}_p(r)$. Throughout this proof, we will denote $\mathcal{U} = \frac{1}{2}|\nabla u|^2$.

$$\begin{aligned}
-\int_M (V\mathcal{U})\,\phi^2\,\mathrm{dVol}_g &= \int_M \mathcal{U}\left(\phi^2 \operatorname{div} V + 2\phi V\phi\right)\,\mathrm{dVol}_g \\
&\le \int_M \mathcal{U}(\phi^2 \operatorname{div} V)\,\mathrm{dVol}_g + \int_{A_r} \mathcal{U}(2\phi\,|V|\,|\nabla\phi|)\,\mathrm{dVol}_g \\
&\le \int_M \mathcal{U}(\phi^2 \operatorname{div} V)\,\mathrm{dVol}_g + \int_{A_r} \mathcal{U}\left(\phi^2\,|V|^2 + |\nabla\phi|^2\right)\,\mathrm{dVol}_g \\
&\le \int_M \mathcal{U}(\phi^2 \operatorname{div} V)\,\mathrm{dVol}_g + \int_{A_r} \mathcal{U}\left(2\phi^2 E + |\nabla\phi|^2\right)\,\mathrm{dVol}_g.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_M (\Delta\mathcal{U})\,\phi^2\,\mathrm{dVol}_g &= -\int_M 2\phi g\left(\nabla\mathcal{U}, \nabla\phi\right)\,\mathrm{dVol}_g \\
&= -\int_{A_r} 2\phi\,\mathrm{Hess}\,u(\nabla u, \nabla\phi)\,\mathrm{dVol}_g \\
&\le \int_{A_r} 2\phi\,|\mathrm{Hess}\,u|\,|\nabla u|\,|\nabla\phi|\,\mathrm{dVol}_g \\
&\le \int_{A_r} \left(\phi^2\,|\mathrm{Hess}\,u|^2 + |\nabla u|^2\,|\nabla\phi|^2\right)\,\mathrm{dVol}_g \\
&\le \int_{A_r} \left(\phi^2\,|\mathrm{Hess}\,u|^2 + 2\mathcal{U}\,|\nabla\phi|^2\right)\,\mathrm{dVol}_g.
\end{aligned}$$

Using the above inequalities,

$$\begin{aligned}
\int_M (\Delta_V\mathcal{U})\,\phi^2\,\mathrm{dVol}_g \le \int_M \mathcal{U}\left(\phi^2 \operatorname{div} V\right)\,\mathrm{dVol}_g &+ \int_{A_r}\left(\phi^2\,|\mathrm{Hess}\,u|^2\right)\,\mathrm{dVol}_g \\
&+ \int_{A_r}\left(3\mathcal{U}\,|\nabla\phi|^2\right)\,\mathrm{dVol}_g + \int_{A_r}(2\phi^2\mathcal{U}E)\,\mathrm{dVol}_g.
\end{aligned}$$

Comparing this with eq. (18) for a V-harmonic function u and noting,

$$\int_M |\mathrm{Hess}\,u|^2\,\phi^2\,\mathrm{dVol}_g \ge \int_{A_r} |\mathrm{Hess}\,u|^2\,\phi^2\,\mathrm{dVol}_g,$$

we obtain

$$\int_M (\lambda - \operatorname{div} V)\,\mathcal{U}\phi^2\,\mathrm{dVol}_g \le \int_{A_r}\left(3\mathcal{U}\,|\nabla\phi|^2\right)\,\mathrm{dVol}_g + \int_{A_r}(2\phi^2\mathcal{U}E)\,\mathrm{dVol}_g.$$

Lastly, taking $r \to \infty$, and using eq. (15),

$$0 \le \int_M \left(\mathrm{S} - \frac{n-2}{2}\lambda\right)\left(\frac{1}{2}\,|\nabla u|^2\right)\,\mathrm{dVol}_g \le \lim_{r\to\infty}\int_{A_r}\left(\frac{1}{2}\,|\nabla u|^2\right)(2E)\,\mathrm{dVol}_g = 0.$$

By hypothesis on scalar curvature, the above inequality implies that u is constant. □

Example 3 (p. 79, [4]). Hamilton's Cigar soliton, $(\mathbb{R}^2, g, \nabla f, 0)$ is a noncompact steady gradient Ricci soliton where

$$g = \frac{4\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)}{1 + x^2 + y^2},$$

and $f = -\ln\left(1 + x^2 + y^2\right)$. Its scalar curvature $\mathrm{S} = \frac{1}{1+x^2+y^2} > 0$. The gradient of f is

$$\nabla f = -\frac{1}{2}(x\partial_x + y\partial_y).$$

Since, the energy function

$$E = \frac{1}{2}\frac{x^2 + y^2}{1 + x^2 + y^2} \leq \frac{1}{2}$$

is bounded above, Theorems 2 and 3 imply that there is no f-harmonic or harmonic function u on $(\mathbb{R}^2, g)$ such that $|\nabla u|^2 \in L^1(M)$. Thus, the manifold has at most one nonparabolic end.

Remark 3. Note that this can be generalised to the broader Bryant soliton case which is also a steady Ricci soliton. Since the scalar curvature of Bryant soliton is positive, and by our previous discussion $\mathrm{S} + 2E$ is constant (see eq. (4)). Consequently, E is bounded above. Hence, the same conclusion as above holds.

6. Finiteness of E-Volume of Shrinking Ricci Solitons

We now turn our attention to proving Theorem 4. We first begin with a lower estimate on the energy function of a Ricci soliton with bounded Ricci curvature.

Lemma 5. *Let (M^n, g, V, λ) be a noncompact shrinking Ricci soliton with the bounded Ricci curvature. Then there exists an L_0 such that for every unit speed geodesic $\gamma : [0, L] \to M$ with $L \geq L_0$ and $\gamma(0) = p \in M$,*

$$E(L) \geq \frac{\lambda^2}{8}L^2 + \frac{\lambda}{2}aL + \frac{a^2}{2},$$

where a is a constant that depends only on the geometry of the soliton on the unit ball $B(p, 1)$.

Proof. We first assume that $L \geq 2$. Let $E^i(p)$ denote an orthonormal basis along γ with $E^n(t) = \gamma'(t)$ and $E^i(t)$ denote the parallel transport of E^i over $\gamma(t)$. Write $V = V^i E_i$. Then for any Lipschitz function $\phi : [0, L] \to [0, 1]$ with $\phi(0) = \phi(L) = 0$, the second variation formula implies

$$(n-1)\int_0^L (\phi')^2 \,\mathrm{d}t \geq \int_0^L \phi^2 \operatorname{Ric}(\gamma', \gamma') \,\mathrm{d}t.$$

Using the Ricci soliton equation and $\frac{1}{2}\mathcal{L}_V g(\gamma', \gamma') = \dot{V^n}$, in the above equation, we get

$$(n-1)\int_0^L (\phi')^2 \,\mathrm{d}t \geq \frac{\lambda}{2}\int_0^L \phi^2 \,\mathrm{d}t - \int_0^L \phi^2 \dot{V^n} \,\mathrm{d}t.$$

Subsequently,

$$(n-1)\int_0^L (\phi')^2 \,\mathrm{d}t \geq \frac{\lambda}{2}\int_0^L \phi^2 \,\mathrm{d}t - \int_0^L \dot{V^n} \,\mathrm{d}t + \int_0^L (1 - \phi^2)\dot{V^n} \,\mathrm{d}t.$$

Using the Ricci soliton equation again and noting the bound on Ricci curvature,

$$\begin{aligned}\int_0^L \left(1-\phi^2\right) \dot{V^n}\,\mathrm{d}t &= \int_0^L \left(1-\phi^2\right)\left(\frac{\lambda}{2} - \mathrm{Ric}\left(\gamma', \gamma'\right)\right)\mathrm{d}t \\ &\geq \int_0^L \left(1-\phi^2\right)\left(\frac{\lambda}{2} - C\right)\mathrm{d}t.\end{aligned}$$

Therefore,

$$(n-1)\int_0^L (\phi')^2\,\mathrm{d}t \geq -V^n(L) + V^n(0) + \int_0^L \left(\frac{\lambda}{2} - C\right)\mathrm{d}t + C\int_0^L \phi^2\,\mathrm{d}t. \tag{36}$$

Now we choose,

$$\phi = \begin{cases} t, & 0 \leq t \leq 1; \\ 1, & 1 \leq t \leq L-1; \\ L-t, & L-1 \leq t \leq L. \end{cases}$$

Substituting in eq. (36),

$$V^n(L) \geq V^n(0) - 2(n-1) + \frac{\lambda}{2}L - \frac{4}{3}C,$$

or

$$\sqrt{2E}(L) \geq V^n(L) \geq \frac{\lambda}{2}L + a,$$

where a is a constant that depends only on the geometry of the soliton on the unit ball $B(p,1)$. Since $\lambda > 0$, we can choose L_0 large enough so that the right hand side of the above equation is positive. Then we have for $L \geq L_0$,

$$E(L) \geq \frac{\lambda^2}{8}L^2 + \frac{\lambda}{2}aL + \frac{a^2}{2}.$$

□

The proof of Theorem 4 employs the same technique as in [13]. The only major point of difference is the usage of estimates for E in place of f.

Theorem 4. *Let (M^n, g, V, λ) be a complete shrinking Ricci soliton with bounded Ricci curvature. Then the measure $e^{-E}\mathrm{dVol}_g$ is finite. Consequently, M has finite fundamental group.*

Proof. Using exponential coordinates at p we have, since $\mathrm{Ric} \geq -C$, by the standard comparison that $\sqrt{\det g} \leq (\sinh\sqrt{C}r)^{n-1} \lesssim e^{(n-1)\sqrt{C}r}$. Here notation, $s \lesssim t$ means $s \leq At$ for some constant A.

$$\begin{aligned}\mathrm{Vol}_E(M) &= \int_{S^{n-1}}\int_0^\infty e^{-E}\sqrt{\det g}\,\mathrm{d}r\,\mathrm{d}s^{n-1} \\ &\lesssim \int_0^{L_0} e^{-E}e^{(n-1)\sqrt{C}r}\mathrm{d}r + \int_{L_0}^\infty e^{-\frac{\lambda^2}{8}r^2 + \left((n-1)\sqrt{C} - \frac{\lambda}{2}a\right)r - \frac{a^2}{2}}\,\mathrm{d}r < \infty.\end{aligned}$$

It should be observed that $\sqrt{\det g}(x) = 0$ for x outside the segment domain of p. Hence, all of the above integrals are meaningful.

Now we lift to the universal cover $(\tilde{M}, \tilde{g})$ of (M, g) and note that $(\tilde{M}, \tilde{g}, \tilde{V}, \lambda)$ is a shrinking Ricci soliton, where $\tilde{V}$ is the pullback of V. Furthermore, the corresponding energy function $\tilde{E}$ is also the pullback of the energy function E of M.

Since both the integrals in

$$\int_{\tilde{M}} e^{-\tilde{E}}\, \mathrm{dVol}_{\tilde{g}} = |\pi_1(M)| \int_M e^{-E}\, \mathrm{dVol}_g$$

are finite, it follows that $\pi_1(M)$ is finite. □

We end this article by recovering the result of [13] for gradient shrinking Ricci solitons using the Theorem 4.

Corollary 3. *A complete gradient shrinking Ricci soliton (M^n, g, f, λ) with bounded Ricci curvature has finite f-volume.*

Proof. Without loss of generality, we can assume that $\lambda = 2$ and that eq. (4) gives

$$\mathrm{S} + |\nabla f|^2 - 2f = 0.$$

Also, it is known for shrinking gradient Ricci solitons that $\mathrm{S} \geq 0$ (see Theorem 2.14, [4]). Combining these imply

$$E = \frac{1}{2} |\nabla f|^2 \leq f.$$

Theorem 4 then implies

$$\int_M e^{-f}\, \mathrm{dVol}_g \leq \int_M e^{-E}\, \mathrm{dVol}_g < \infty.$$

□

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