Random valuations

Andrii Ilienko[1,2] and Ilya Molchanov[1]

[1]University of Bern
[2]Igor Sikorsky Kyiv Polytechnic Institute

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Abstract

A valuation is a finitely additive function on the family of compact convex sets in $\mathbb{R}^d$. We study non-negative infinitely divisible random valuations, with particular emphasis on monotone, σ-continuous models with independent increments along nested families. After separating the deterministic part, we show that the Lévy measure of such a valuation is generated by pairs (F, r), where F is a non-empty closed convex set and $r > 0$, with each pair contributing $r\mathbf{1}_{F\cap K=\varnothing}$. This yields a Poisson representation and an equivalent formulation through a pure-jump completely random measure on the space of closed convex sets. For stationary valuations, we derive a cylinder–Grassmannian representation of the Lévy measure. In the stationary isotropic case, we obtain a McMullen-type decomposition, at the level of one-dimensional distributions, into independent components stable under dilation of the argument.

Keywords: random valuation, stationarity, set-indexed process, independent increments, Lévy measure, Grassmannian, completely random measure

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1 Introduction

Let $\mathcal{K}^d$ be the family of convex bodies (i.e., compact convex sets) in $\mathbb{R}^d$. While the empty set is typically not considered a convex body, we adopt the convention that it is included in $\mathcal{K}^d$. A *valuation* φ is an additive map from $\mathcal{K}^d$ to an abelian semigroup. Additivity means that

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L) \tag{1.1}$$

for all compact convex sets K and L such that $K \cup L$ is also convex, see [1] and [18, Chapter 6]. Additionally, we will always assume that $\varphi(\varnothing) = 0$. While, in general, the values of valuations may belong to any semigroup, most of the literature on valuations focuses on valuations with values in the set of real or complex numbers or in the family of compact convex sets equipped with Minkowski addition. We consider only real-valued valuations $\varphi : \mathcal{K}^d \to \mathbb{R}$.

While valuations share with measures the finite additivity property, their domain of definition is not an algebra. The additivity property restricted to the case when $K \cup L$ is convex makes the study of valuations complicated and causes a number of problems in their characterisation. For instance, unlike measures, a non-negative valuation is not necessarily monotone and there is no analogue of the Jordan decomposition for signed valuations. Some valuations, such as perimeter or surface area, are given by the Hausdorff measures of the boundaries of sets. However, many valuations are far from being measures, most importantly valuations given by $\varphi(K) = V_d(K + A)$, where V_d is the volume and $K + A = \{x + y : x \in K, y \in A\}$ is the Minkowski sum of K and a fixed compact convex set A.

A common assumption in the theory of valuations is their invariance under a group of transformations of $\mathbb{R}^d$. In most cases, valuations are assumed to be translation invariant, meaning that $\varphi(K + x) = \varphi(K)$ for all $x \in \mathbb{R}^d$, see [1]. While the family of translation-invariant valuations is still not completely described, adding the rotation invariance assumption results in valuations invariant under all rigid motions, which are well understood. By Hadwiger's theorem [18, Theorem 6.4.14], any real-valued valuation on $\mathcal{K}^d$ that is continuous in the Hausdorff metric and invariant under rigid motions can be expressed as a weighted sum of the intrinsic volumes $V_i(K)$, $i = 0, \dots, d$. We refer to [18] for the definition of intrinsic volumes and recall that V_d is the Lebesgue measure, V_{d-1} is one half of the surface area, and $V_0(K)$ is the indicator of $K \neq \varnothing$.

Each measure naturally defines a valuation by restricting its domain to $\mathcal{K}^d$. Random measures are the most important example of random set functions, and the theory of random measures is rich and useful in various applications, see [11].

Random measures are additive processes indexed by (not necessarily convex) sets. An important family of random measures arises in the theory of (weighted) empirical measures constructed as sums of weights attached to locations in the space, see [20] and recently in [8]. Particular problems which arise when the weights are signed have been addressed in [3]. Gaussian measures on the family of convex sets have been considered in [5] and [6].

Dropping the additivity requirement of a set-indexed process results in non-additive measures and capacities, which are also widely used in mathematics, e.g., in large deviations, cooperative games, and mathematical economics. Random non-additive measures appear in relation to the theory of extreme values, see [15, 21]. Set-indexed martingales have been studied in [10], without assuming any additivity.

In this paper we study *random valuations*. In view of what has been mentioned, one can regard them as sandwiched between random measures and general set-indexed processes. From a pure mathematical perspective, random valuations can be considered as valuations taking values in the additive group of random variables. However, some probabilistic assumptions imposed later in this paper become far more transparent if random valuations are treated as stochastic processes on the space of convex bodies.

Imposing invariance assumptions on random valuations *pathwise* makes it possible to utilise available representations of deterministic valuations. In contrast, we study random valuations without imposing any invariance on their paths. Instead, we occasionally impose stationarity and isotropy in distribution. These assumptions do not ensure the invariance of individual realisations and therefore do not permit the direct application of deterministic characterisation results.

The family of valuations not satisfying any pathwise invariance assumptions is very rich and is presently understood only in two cases: valuations on the line and σ-continuous monotone integer-valued valuations on the plane, see [9]. Recall that φ is said to be σ-continuous if $\varphi(K_n) \to \varphi(K)$ whenever $K_n \downarrow K$, see [18, Section 6.2].

We recall a conjecture made in [9] concerning deterministic valuations; it is supported by its version valid for integer-valued σ-continuous monotone valuations on the plane. A measure μ on $\mathcal{C}$ is said to be *locally finite* if μ has finite total variation on the families

$$\mathcal{C}_K = \{F \in \mathcal{C} \colon F \cap K \neq \varnothing\} \tag{1.2}$$

for all compact sets K. The valuation φ is said to be locally finite if $|\varphi(K)|$, $K \subset W$, $K \in \mathcal{K}^d$, is bounded for all compact W.

Conjecture 1.1. *Each locally finite σ-continuous valuation φ is representable as*

$$\varphi(K) = \mu(\mathcal{C}_K), \quad K \in \mathcal{K}^d, \tag{1.3}$$

for a locally finite signed measure μ on $\mathcal{C}$.

It should be noted that such a representation of a monotone φ is not necessarily unique and may involve a signed measure on $\mathcal{C}$, as examples in [9] show.

While a description of general random valuations is surely out of reach, this paper studies infinitely divisible random valuations and, in particular, those which have the property of independent increments. It is important to stress that the independence of increments differs from assuming that the values on disjoint sets are independent — we prove that the latter assumption singles out valuations which are completely random measures. In this connection, note that set-indexed Lévy processes were studied in [3] and further in [7], assuming that the values on disjoint sets are independent.

We show that each non-negative infinitely divisible random valuation with independent increments and no Gaussian component admits a representation in terms of a Poisson process on the product of the family $\mathcal{C}$ of closed convex sets and the positive real half-line. In this way we settle a random variant of our conjecture, providing a complete characterisation of random valuations which admit representation (1.3) with μ being a completely random measure on $\mathcal{C}$. The main reason why the random variant holds is the fact that the sum of two independent random variables is not zero a.s., unless these two random variables are degenerate.

Particular attention in this paper is devoted to *stationary* random valuations which are infinitely divisible, monotone, and have independent increments. While the additivity property is imposed almost surely, stationarity is understood for distributions. It is shown that such stationary random valuations have a representation in terms of measures on the products of Grassmannians of different orders and the real half-line. McMullen's decomposition theorem (see, e.g., Theorem 6.3.5 in [18]) states that every continuous translation-invariant valuation admits a unique decomposition into the sum of valuations homogeneous of degrees $i = 0, \dots, d$. In the random setting, the homogeneity property is replaced by stability. We prove a variant of McMullen's theorem for isotropic random valuations, showing that each stationary isotropic infinitely divisible random valuation without Gaussian component has the same one-dimensional distributions as the sum of stable random valuations.

The paper is organised as follows. Section 2 addresses general properties of additive stochastic processes on $\mathcal{K}^d$, most importantly, their separability. Section 3 introduces the infinite divisibility property of random valuations. The existence of the corresponding Lévy measure is derived by applying results from [16] obtained for stochastic processes indexed by arbitrary sets. The support of the Lévy measure is crucial to construct infinitely divisible valuations. It is shown in Section 4 that assuming independence of increments we can deduce that the Lévy measure is supported by two-valued valuations.

While the family of two-valued valuations is still very large, Section 5 invokes the σ-continuity and monotonicity assumptions to show that the Lévy measure is supported by valuations of the type $\psi(K) = r\mathbf{1}_{F\cap K\neq\varnothing}$ for $r > 0$ and closed convex sets F. This provides a general series representation of σ-continuous monotone infinitely divisible valuations with independent increments.

Section 6 addresses stability properties of random valuations, noting that the scaling can be applied either to the values or to the argument of the valuation. Assuming stationarity, it is possible to further specify the representation of random valuations as done in Section 7. Finally, Section 8 provides a random variant of McMullen's theorem.

2 Basic properties of random valuations

Consider the space $\mathcal{K}^d$ of compact convex sets in $\mathbb{R}^d$ (including the empty set) equipped with the (extended) Hausdorff metric d_{H} (assuming that $\mathsf{d}_{\mathrm{H}}(\varnothing, K) = \infty$ for each non-empty K). Two sets $K, L \in \mathcal{K}^d$ are said to be *admissible* if their union is convex. Let $\mathbb{R}^{\mathcal{K}^d}$ denote the space of all functions $\varphi : \mathcal{K}^d \to \mathbb{R}$, and let $\mathcal{B}^{\mathcal{K}^d}$ be its cylindrical σ-algebra. We fix a complete probability space $(\Omega, \mathfrak{F}, \mathbf{P})$.

Definition 2.1. A *random valuation* Φ is a stochastic process on $\mathcal{K}^d$ such that $\Phi(\varnothing) = 0$ a.s. and, for each fixed admissible $K, L \in \mathcal{K}^d$, we have

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L) \quad \text{a.s.}$$

The distributions of $(\Phi(K_1), \dots, \Phi(K_n))$ for $n \geq 1$, $K_1, \dots, K_n \in \mathcal{K}^d$, are called the finite-dimensional distributions of Φ.

A random valuation Φ is said to be *separable* if there exist a countable family $\mathcal{D} \subset \mathcal{K}^d$ and a set $\Omega_0 \subset \Omega$ of full probability such that, for all $K, L \in \mathcal{K}^d$ with $K \cup L \in \mathcal{K}^d$, there exist sequences (K_n) and (L_n) from $\mathcal{D}$ such that $K_n \cup L_n$ is convex, $K_n \downarrow K$ and $L_n \downarrow L$, and $\Phi(K_n) \to \Phi(K)$, $\Phi(L_n) \to \Phi(L)$, $\Phi(K_n \cup L_n) \to \Phi(K \cup L)$ and $\Phi(K_n \cap L_n) \to \Phi(K \cap L)$ as $n \to \infty$ for all $\omega \in \Omega_0$. The separability property implies *separability in probability*, obtained by replacing the convergence for all $\omega \in \Omega_0$ with convergence in probability.

Lemma 2.2. *If a random valuation Φ is separable, then the realisations of Φ almost surely belong to the family of additive functions on $\mathcal{K}^d$.*

Proof. Note that Φ belongs to the family of additive functions if and only if

$$\sup_{K,L\in\mathcal{K}^d, K\cup L\in\mathcal{K}^d} |\Phi(K) + \Phi(L) - \Phi(K \cap L) - \Phi(K \cup L)| = 0 \quad \text{a.s.}$$

By the separability property, the supremum may be reduced to one over K and L from a countable family $\mathcal{D}$. Then use the fact that the expression under the supremum vanishes a.s. for all admissible K and L from $\mathcal{D}$. $\square$

Recall that there are two basic notions of continuity for a deterministic valuation φ:

(C1) continuity in the Hausdorff metric;

(C2) σ-continuity, meaning that $\varphi(K_n) \to \varphi(K)$ whenever $K_n \downarrow K$.

It is clear that (C1) implies (C2). Furthermore, if φ is monotone, then σ-continuity is equivalent to the upper semicontinuity of φ.

Lemma 2.3. *If Φ is a separable random valuation, then $C_\Phi = \{\Phi$ is continuous$\}$ is an $\mathfrak{F}$-measurable event.*

Proof. Let $\tilde{C}_\Phi$ denote the event that Φ is locally uniformly continuous on $\mathcal{D}$:

$$\tilde{C}_\Phi = \bigcap_{r\geq 1} \bigcap_{m\geq 1} \bigcup_{n\geq 1} \bigcap_{K,L\in\mathcal{D}, K,L\subset B_r, \mathsf{d}_{\mathrm{H}}(K,L)\leq n^{-1}} \big\{|\Phi(L) - \Phi(K)| \leq m^{-1}\big\},$$

where B_r is the Euclidean ball of radius r centred at the origin. Since all intersections and unions are countable, we have $\tilde{C}_\Phi \in \mathfrak{F}$.

To prove the lemma, it suffices to show that $\Omega_0 \cap \tilde{C}_\Phi \subseteq \Omega_0 \cap C_\Phi$. The reverse inclusion follows from the local compactness of $\mathcal{K}^d$ with respect to the Hausdorff metric: any continuous function on $\mathcal{K}^d$ is locally uniformly continuous there, hence, also on $\mathcal{D}$. To prove the direct inclusion, let $K_n \to K$. By the definition of separability, choose $K^i, K_n^i \in \mathcal{D}$ with $K^i \downarrow K$ and $K_n^i \downarrow K_n$ such that $\Phi(K^i) \to \Phi(K)$ and $\Phi(K_n^i) \to \Phi(K_n)$ for all $\omega \in \Omega_0$. Hence,

$$|\Phi(K_n) - \Phi(K)| \leq |\Phi(K_n) - \Phi(K_n^i)| + |\Phi(K_n^i) - \Phi(K^i)| + |\Phi(K^i) - \Phi(K)|,$$

and, therefore,

$$\limsup_{n\to\infty} |\Phi(K_n) - \Phi(K)| = \limsup_{n\to\infty} \limsup_{i\to\infty} |\Phi(K_n) - \Phi(K)| \leq \limsup_{n\to\infty} \limsup_{i\to\infty} |\Phi(K_n^i) - \Phi(K^i)|,$$

which vanishes by the local uniform continuity of Φ on $\mathcal{D}$. Since $\widetilde{C}_\Phi \in \mathfrak{F}$, the probability space is complete, and $\Omega \setminus \Omega_0$ is a null event, it follows that $C_\Phi \in \mathfrak{F}$. $\square$

The following result implies that an a.s. σ-continuous random valuation is separable.

Lemma 2.4. *Each random valuation whose realisations are almost surely σ-continuous is separable.*

Proof. Since the family $\mathcal{D}$ of polytopes (including $\varnothing$) with rational vertices is countable and dense in $\mathcal{K}^d$, we only need to show that $\mathcal{D}$ approximates pairs of convex sets with convex union by sequences which also have convex unions. Let $K, L \in \mathcal{K}^d$ be such that $U = K \cup L \in \mathcal{K}^d$. Choose a sequence $\varepsilon_n \downarrow 0$ and decreasing sequences $U_n, K_n, L_n \in \mathcal{D}$ such that

$$U \subset U_n \subset U + \varepsilon_n B^d, \quad K + \varepsilon_n B^d \subset K_n \subset K + 2\varepsilon_n B^d, \quad L + \varepsilon_n B^d \subset L_n \subset L + 2\varepsilon_n B^d.$$

Let $K_n' = K_n \cap U_n$, $L_n' = L_n \cap U_n$. Then $K_n', L_n' \in \mathcal{D}$ and $K_n' \downarrow K$, $L_n' \downarrow L$, $K_n' \cup L_n' \downarrow K \cup L$.

It remains to check that $K_n' \cup L_n' \in \mathcal{D}$. By the choice of ε_n, we have $U_n \subset K_n \cup L_n$, hence

$$K_n' \cup L_n' = (K_n \cap U_n) \cup (L_n \cap U_n) = (K_n \cup L_n) \cap U_n = U_n \in \mathcal{D}. \qquad \square$$

The family of all valuations is very rich, and some invariance conditions are usually necessary to arrive at meaningful characterisation results. If almost all realisations of Φ are continuous and translation invariant, then, by McMullen's theorem (see, e.g., Theorem 6.3.5 in [18]), each realisation of Φ can be decomposed into the sum of homogeneous valuations of orders $i = 0, \dots, d$. The following result shows that the homogeneous summands can be chosen as random valuations and establishes their measurability.

Proposition 2.5. *Assume that Φ is a random valuation such that almost all of its realisations are continuous translation-invariant valuations. Then*

$$\Phi(K) = \sum_{i=0}^{d} \Phi_i(K), \tag{2.1}$$

where each Φ_i is a random valuation such that almost all its realisations are continuous, translation-invariant valuations which are homogeneous of degree i, that is, $\Phi_i(cK) = c^i \Phi_i(K)$ a.s. for all $K \in \mathcal{K}^d$ and $c > 0$.

Proof. By McMullen's theorem, (2.1) holds for almost all realisations of Φ. Each Φ_i is additive and homogeneous of degree i on $\mathcal{K}^d$. For each $K \in \mathcal{K}^d$, we evaluate Φ at rK for $r > 0$, which shows that

$$\Phi(rK) = \sum_{i=0}^{d} \Phi_i(rK) = \sum_{i=0}^{d} r^i \Phi_i(K).$$

Thus, $\big(\Phi_0(K), \dots, \Phi_d(K)\big)$ is the solution of the above system of linear equations for $d+1$ distinct positive values $r_0, \dots, r_d$ of r. This solution is a linear transform of the values $\Phi(r_0 K), \dots, \Phi(r_d K)$, and so each $\Phi_i(K)$ is a random variable for all $K \in \mathcal{K}^d$. $\square$

A direct consequence of the previous result and Hadwiger's theorem (see [18, Theorem 6.4.14]) is the following characterisation of random valuations almost all of whose realisations are continuous and invariant under rigid motions. In this case the random valuation is the linear combination of intrinsic volumes $V_0, \dots, V_d$ with random coefficients, see [18, Section 4.1] for the definition of intrinsic volumes.

Corollary 2.6. *Assume that Φ is a random valuation almost all of whose realisations are continuous and invariant under rigid motions. Then*

$$\Phi(K) = \sum_{i=0}^{d} \xi_i V_i(K), \tag{2.2}$$

where $(\xi_0, \dots, \xi_d)$ is a random vector, and $V_0, \dots, V_d$ are the intrinsic volumes.

A rich source of random valuations is provided by random (signed) measures on the family $\mathcal{C}$ of closed convex sets in $\mathbb{R}^d$ equipped with the Borel σ-algebra generated by the Fell topology, see [13, Section 1.2] and [14, Appendix C].

Proposition 2.7. *Let Z be a locally finite random signed measure on $\mathcal{C}$ (as defined in (1.2)). Then*

$$\Phi(K) = Z(\{F \in \mathcal{C} \colon K \cap F \neq \varnothing\}), \quad K \in \mathcal{K}^d, \tag{2.3}$$

is a σ-continuous random valuation. Furthermore, Φ is a.s. non-negative if Z is a.s. non-negative. In this case, Φ is necessarily monotone.

Proof. Since Z is a random measure, $\Phi(K)$ is a random variable for every $K \in \mathcal{K}^d$ and thus Φ is a stochastic process on $\mathcal{K}^d$. Assume that $K, L \in \mathcal{K}^d$ are admissible. Since Z is a random measure and its support is contained in the family of closed convex sets,

$$\Phi(K) + \Phi(L) = Z(\mathcal{C}_K) + Z(\mathcal{C}_L) = Z(\mathcal{C}_{K \cup L}) + Z(\mathcal{C}_{K \cap L}) = \Phi(K \cup L) + \Phi(K \cap L) \quad \text{a.s.}$$

Note that convexity of $K \cup L$ is essential above to conclude that $\mathcal{C}_K \cap \mathcal{C}_L = \mathcal{C}_{K \cap L}$. Since $\mathcal{C}_\varnothing = \varnothing$, we have $\Phi(\varnothing) = Z(\varnothing) = 0$ a.s.

Assume that $(K_n)_{n \geq 1}$ is a sequence in $\mathcal{K}^d$ such that $K_n \downarrow K$. Since Z is locally finite, its total variation on $\mathcal{C}_{K_1}$ is finite. Since $\mathcal{C}_{K_n} \downarrow \mathcal{C}_K$ as $n \to \infty$, continuity from above for finite signed measures yields that $\Phi(K_n) = Z(\mathcal{C}_{K_n})$ a.s. converges to $Z(\mathcal{C}_K) = \Phi(K)$ as $n \to \infty$. If Z is non-negative, then Φ is also non-negative and monotone, since $\mathcal{C}_L \subset \mathcal{C}_K$ for $L \subset K$. $\square$

3 Infinitely divisible valuations

A random valuation Φ is said to be *infinitely divisible* if, for each $n \geq 2$, Φ is equal in distribution to the sum $\Phi_{1,n} + \dots + \Phi_{n,n}$, where $\Phi_{1,n}, \dots, \Phi_{n,n}$ are n i.i.d. random valuations. We say that Φ is an *ID valuation* if it is an infinitely divisible separable random valuation. In the following we consider only non-negative Φ, which implies that its distribution does not have a Gaussian component.

Let Φ be a non-negative ID valuation, and let $\mathcal{I} = \{K_1, \dots, K_m\}$ be a finite collection of sets from $\mathcal{K}^d$. Then the random vector $\Phi_{\mathcal{I}} = (\Phi(K_1), \dots, \Phi(K_m))$ is infinitely divisible. By the Lévy–Khinchin representation (see [17]), there exists a measure $\Lambda_{\mathcal{I}}$ on $\mathbb{R}^m_+ \setminus \{0\}$ and $b_{\mathcal{I}} \in \mathbb{R}^m_+$ such that, for each $u \in \mathbb{R}^m_+$,

$$\mathbf{E} \exp\Big\{ - \big\langle u, \Phi_{\mathcal{I}} \big\rangle \Big\} = \exp\Big\{ - \langle u, b_{\mathcal{I}} \rangle - \int_{\mathbb{R}^m_+ \setminus \{0\}} \Big(1 - e^{-\langle u, x\rangle}\Big) \Lambda_{\mathcal{I}}(dx) \Big\}.$$

The uniqueness of $\Lambda_{\mathcal{I}}$ and $b_{\mathcal{I}}$ implies that for $\mathcal{I} \subset \mathcal{I}'$, the vector $b_{\mathcal{I}'}$ restricted to $\mathcal{I}$ equals $b_{\mathcal{I}}$, and $\Lambda_{\mathcal{I}}$ is the projection of $\Lambda_{\mathcal{I}'}$ (on the space excluding the origin).

A measure Λ on $(\mathbb{R}^{\mathcal{K}^d}_+, \mathcal{B}^{\mathcal{K}^d})$ is said to be a *Lévy measure* if the following two conditions hold:

(L1) for all $K \in \mathcal{K}^d$,
$$\int \min(1, \psi(K)) \Lambda(d\psi) < \infty; \tag{3.1}$$

(L2) for all $A \in \mathcal{B}^{\mathcal{K}^d}$, $\Lambda(A) = \Lambda_*(A \setminus \{O_{\mathcal{K}^d}\})$, where Λ_* is the inner measure and $O_{\mathcal{K}^d}$ is the function $\psi \colon \mathcal{K}^d \to \mathbb{R}$ which is identically zero.

Condition (L2) is satisfied if there exists a countable set $\mathcal{J} \subset \mathcal{K}^d$ such that

$$\Lambda\big(\{\psi \in \mathbb{R}^{\mathcal{K}^d}_+ \colon \psi(K) = 0 \text{ for all } K \in \mathcal{J}\}\big) = 0.$$

Theorem 3.1. *Let Φ be a non-negative ID valuation. Then there exists a unique (necessarily σ-finite) Lévy measure Λ on $\mathbb{R}^{\mathcal{K}^d}_+$ whose outer measure is concentrated on the family of valuations and a deterministic valuation φ such that, for any $m \geq 1$, finite family $\mathcal{I} = \{K_1, \dots, K_m\} \subset \mathcal{K}^d$, and $u \in \mathbb{R}^m_+$,*

$$\mathbf{E} \exp\Big\{ - \big\langle u, \Phi_{\mathcal{I}} \big\rangle \Big\} = \exp\Big\{ - \langle u, \varphi_{\mathcal{I}} \rangle - \int_{\mathbb{R}^{\mathcal{K}^d}_+ \setminus \{O_{\mathcal{K}^d}\}} \Big(1 - e^{-\langle u, \psi_{\mathcal{I}}\rangle}\Big) \Lambda(d\psi) \Big\}.$$

Furthermore, Φ is monotone if and only if φ is monotone and the Λ-outer measure is supported by monotone valuations.

Proof. We let $\varphi(K) = b_{\mathcal{I}}$ with $\mathcal{I} = \{K\}$ for $K \in \mathcal{K}^d$. Since Φ is additive on admissible sets, the uniqueness of $b_{\mathcal{I}}$ implies that φ is additive and so is a valuation. The additivity follows from the Lévy–Khinchin formula applied to the vector $(\Phi(K), \Phi(L), \Phi(K \cup L), \Phi(K \cap L))$, which takes values from a linear subspace of $\mathbb{R}^4$.

By adapting the proof of Theorem 2.8 from [16] to the family of all non-negative additive functions on $\mathcal{K}^d$, it is possible to patch together the measures $\Lambda_{\mathcal{I}}$ to come up with a single measure on $\mathcal{B}^{\mathcal{K}^d}$ which admits them as projections. The same works for the family of all monotone functions on $\mathcal{K}^d$. Indeed, all these properties of stochastic processes on $\mathcal{K}^d$ are determined by their finite-dimensional distributions. The σ-finiteness of Λ follows from separability of Φ as in Corollary 2.18 from [16] for general stochastic processes. $\square$

Let N be a Poisson random measure on $(\mathbb{R}_+^{\mathcal{K}^d}, \mathcal{B}^{\mathcal{K}^d})$ with intensity Λ, see [12, Definition 15.2]. Note that $N\big(\{\psi\colon \psi(K) > \varepsilon\}\big)$ has Poisson distribution with parameter $\Lambda\big(\{\psi\colon \psi(K) > \varepsilon\}\big) < \infty$, and so is almost surely finite for each given $K \in \mathcal{K}^d$. Since the space of valuations with the pointwise convergence is not a standard Borel space, it is not immediately possible to interpret the Poisson random measure N as a Poisson point process on the family of valuations, see [12, Section 2.1]. Proposition 2.10 from [16] implies the following.

Proposition 3.2. *A non-negative ID valuation Φ with vanishing deterministic part has the same distribution as*

$$\tilde{\Phi}(K) = \int \psi(K) N(d\psi), \quad K \in \mathcal{K}^d, \tag{3.2}$$

where N is the Poisson random measure on $(\mathbb{R}_+^{\mathcal{K}^d}, \mathcal{B}^{\mathcal{K}^d})$ with intensity Λ; the outer measure associated with Λ is concentrated on valuations.

The deterministic part $\varphi(K)$ can be recovered as the essential infimum $\operatorname{ess\,inf} \Phi(K)$ for all $K \in \mathcal{K}^d$. Unlike monotonicity and non-negativity, σ-continuity is not a finite-dimensional constraint and therefore does not pass immediately from Φ to its Lévy measure. The following result addresses this issue.

Lemma 3.3. *If Φ is a monotone ID valuation whose realisations are almost surely σ-continuous, then its Lévy measure is supported by σ-continuous valuations, meaning that the Λ-outer measure of the family of valuations that are not σ-continuous is zero.*

Proof. By Theorem 3.1, the deterministic part φ of Φ is monotone and the Λ-outer measure is concentrated on monotone valuations. Fix $K_0 \in \mathcal{K}^d$ and $\varepsilon > 0$, and let $A_{K_0,\varepsilon}$ be the family of all $\psi \in \mathbb{R}_+^{\mathcal{K}^d}$ such that $\psi(K_0) \geq \varepsilon$. By (3.1), $\Lambda(A_{K_0,\varepsilon}) < \infty$. Use the Poisson representation of Φ and split its Poisson random measure into its restrictions to $A_{K_0,\varepsilon}$ and its complement. This gives, on one probability space, $\widetilde{\Phi} = \varphi + X + Y$, where

$$X(K) = \int_{A_{K_0,\varepsilon}} \psi(K) N(d\psi)$$

is a compound Poisson valuation and Y is the remaining independent Poisson integral. The valuation $\widetilde{\Phi}$ has the same distribution as Φ. Hence its outer distribution is concentrated on σ-continuous valuations.

The three terms φ, X, and Y are monotone. Consider a realisation for which $\widetilde{\Phi}$ is σ-continuous, and let $K_n \downarrow K$. By monotonicity, the limits $\varphi(K_n) \downarrow \varphi_*$, $X(K_n) \downarrow X_*$, and $Y(K_n) \downarrow Y_*$ exist and satisfy $\varphi_* \geq \varphi(K)$, $X_* \geq X(K)$ and $Y_* \geq Y(K)$. On the other hand, the σ-continuity of $\widetilde{\Phi}$ yields $\varphi_* + X_* + Y_* = \varphi(K) + X(K) + Y(K)$. Since all three defects are non-negative, each defect vanishes. Thus X is σ-continuous.

Let $\lambda = \Lambda(A_{K_0,\varepsilon})$. If $\lambda = 0$, there is nothing to prove. If $\lambda > 0$, condition on the event that the compound Poisson random measure defining X has exactly one atom. On this event, $X = \psi$, where ψ has distribution $\lambda^{-1}\Lambda|_{A_{K_0,\varepsilon}}$. Since X is almost surely σ-continuous, it follows that the outer measure associated with $\Lambda|_{A_{K_0,\varepsilon}}$ is concentrated on σ-continuous valuations.

Finally, let $\mathcal{D}$ be the countable cofinal family of rational polytopes. Then the family of non-zero monotone valuations is a subset of the union of $A_{K_0,\varepsilon}$ over all $K_0 \in \mathcal{D}$ and rational $\varepsilon > 0$. Thus, the Λ-outer measure of the family of valuations that are not σ-continuous is zero. □

Example 3.4. Let $\{F_i, i \geq 1\}$ be a point process on the family $\mathcal{C}$ of closed convex sets in $\mathbb{R}^d$. If this process is locally finite, that is, only finitely many of the sets F_i intersect any given compact set, then

$$\Phi(K) = \sum_i \mathbf{1}_{K \cap F_i \neq \varnothing} \tag{3.3}$$

is an integer-valued monotone valuation. It is ID if the point process $\{F_i, i \geq 1\}$ is infinitely divisible, meaning that, for any $n \geq 2$, it can be represented as the superposition of n independent copies of another point process on $\mathcal{C}$. It follows from [9] that on the plane all monotone σ-continuous integer-valued valuations are given by (3.3) with coefficients $r_i \in \{-1, 1\}$ in front of the indicators.

Example 3.5. If almost all realisations of Φ are continuous and invariant under rigid motions, then Corollary 2.6 implies that it is infinitely divisible if and only if the vector of coefficients $(\xi_0, \dots, \xi_d)$ in (2.2) is infinitely divisible.

4 Independence of increments

As in the theory of random measures (see [11]) and set-indexed processes (see [2]), it seems to be natural to consider valuations with completely independent values, meaning that the values of Φ on disjoint sets are independent. However, such random valuations are not very interesting as Theorem 4.5 below shows. In view of this, we impose the following condition of independent increments.

A random valuation Φ is said to have *independent increments* if, for each $n \geq 2$ and each nested sequence $L_1 \supset \cdots \supset L_n$ of compact convex sets and $L_{n+1} = \varnothing$, the random variables $\Phi(L_i) - \Phi(L_{i+1})$, $i = 1, \dots, n$, are jointly independent.

If the paths of a non-trivial continuous random valuation Φ are almost surely invariant under rigid motions, then Corollary 2.6 implies that Φ does not have independent increments unless $\Phi(K) = \xi \mathbf{1}_{K \neq \varnothing} + \varphi(K)$ for a random variable ξ and a deterministic valuation φ which is given by the weighted sum of the intrinsic volumes. The same is the case for random valuations whose realisations are continuous and translation invariant, as the following result shows.

Proposition 4.1. *Assume that Φ is a continuous random valuation whose realisations are a.s. translation invariant. Then Φ has independent increments if and only if $\Phi(K) = \xi \mathbf{1}_{K \neq \varnothing} + \varphi(K)$ where ξ is a random variable and φ is a deterministic continuous translation-invariant valuation.*

Proof. By Proposition 2.5, Φ admits the representation (2.1). Since Φ_0 in (2.1) is homogeneous of degree zero, we have $\Phi_0(K) = \Phi_0(cK)$ for all $K \in \mathcal{K}^d$ and $c > 0$. Letting $c \downarrow 0$ and using continuity of Φ_0, we obtain that $\Phi_0(K) = \Phi_0(\{0\}) = \xi$ if $K \neq \varnothing$. Furthermore, $\Phi_i(\{0\}) = 0$ for $i = 1, \dots, d$.

Fix $K \in \mathcal{K}^d$ and, by translation invariance, assume that $0 \in K$. For $t > 0$ amd $m \in \{1, \dots, d\}$, set

$$X_t = \frac{1}{t^m}\Big(\Phi(tK) - \Phi(\{0\}) - \sum_{i=1}^{m-1} t^i \Phi_i(K)\Big),$$

$$Y_t = \frac{1}{t^m}\Big(\Phi(2tK) - \Phi(tK) - \sum_{i=1}^{m-1} (2^i - 1) t^i \Phi_i(K)\Big).$$

Let $m = 1$. Since $\{0\} \subset tK \subset 2tK$, the two corresponding increments of Φ are independent if $m = 1$. Hence, X_t and Y_t are independent. By homogeneity, $X_t \to \Phi_1(K)$ and $Y_t \to \Phi_1(K)$ a.s. as $t \downarrow 0$, and so $\Phi_1(K)$ is deterministic. This holds for all K from a countable dense family, hence, for all $K \in \mathcal{K}^d$ by continuity. For $m \geq 2$, we apply the induction argument.

Conversely, if $\Phi(K) = \xi \mathbf{1}_{K \neq \varnothing} + \varphi(K)$ with deterministic φ, then along any decreasing chain all increments of φ are deterministic, while at most one increment of the first term is random. Hence the increments are jointly independent. □

The following result shows that for infinitely divisible random valuations it suffices to check the independence of two successive increments associated with every nested family of four sets.

Proposition 4.2. *An infinitely divisible random valuation has independent increments if and only if $\Phi(K_2) - \Phi(K_1)$ and $\Phi(K_4) - \Phi(K_3)$ are independent for all compact convex sets $K_1 \subset K_2 \subset K_3 \subset K_4$.*

Proof. The result follows from the fact that for infinitely divisible random vectors joint independence is equivalent to pairwise independence, see Exercise 12.10(ii) from [17]. □

Lemma 4.3. *Let Φ be an ID valuation with the Lévy measure Λ, and let $a_1, \dots, a_n$ and $b_1, \dots, b_m$ be real numbers and $K_1, \dots, K_n$ and $L_1, \dots, L_m$ be compact convex sets. Then the finite sums $\sum a_i \Phi(K_i)$ and $\sum b_j \Phi(L_j)$ are independent if and only if the Λ-outer measure vanishes on the complement of the set of valuations ψ such that $\sum a_i \psi(K_i) = 0$ or $\sum b_j \psi(L_j) = 0$.*

Proof. The statement follows from the already mentioned Exercise 12.10 from [17]. □

The following result shows that it is possible to weaken the independence of increments property if Φ is assumed to be monotone.

Proposition 4.4. *Let Φ be a monotone ID valuation. Then Φ has independent increments if and only if $\Phi(K) - \Phi(L)$ and $\Phi(L)$ are independent for all $K, L \in \mathcal{K}^d$ such that $L \subset K$.*

Proof. We need to prove only sufficiency. Let $K_1 \subset K_2 \subset K_3 \subset K_4$. By assumption, $\Phi(K_4) - \Phi(K_2)$ and $\Phi(K_2)$ are independent. By Theorem 3.1 and Lemma 4.3, Λ is concentrated on monotone valuations ψ such that $\psi(K_2) = 0$ or $\psi(K_2) = \psi(K_4)$. In the first case, monotonicity yields $\psi(K_2) - \psi(K_1) = 0$, while in the second case $\psi(K_4) - \psi(K_3) = 0$. Lemma 4.3 therefore implies that $\Phi(K_2) - \Phi(K_1)$ and $\Phi(K_4) - \Phi(K_3)$ are independent. The claim follows from Proposition 4.2. $\square$

Next, we characterise ID valuations with independent values. Recall that a random measure is said to be *completely random* if its values on disjoint sets are independent, see [12, Section 15.3].

Theorem 4.5. *Assume that Φ is a σ-continuous monotone ID random valuation with vanishing deterministic part and such that $\Phi(K)$ and $\Phi(L)$ are independent for disjoint $K, L \in \mathcal{K}^d$. Then Φ has the same finite-dimensional distributions as the restriction to $\mathcal{K}^d$ of a completely random measure η on $\mathbb{R}^d$.*

Proof. By Lemma 3.3, the Lévy measure Λ of Φ is supported by σ-continuous valuations. By Lemma 4.3, outside of a set of zero Λ-outer measure, ψ is monotone and σ-continuous and $\psi(K) = 0$ or $\psi(L) = 0$ for every disjoint pair from the countable family $\mathcal{D}$ of rational polytopes. This extends to arbitrary disjoint $K, L \in \mathcal{K}^d$ by taking their outer approximations.

For each ψ, there exists $K \in \mathcal{K}^d$ such that $c = \psi(K) > 0$. Without loss of generality, assume that K is not a singleton. By splitting K into three parts L_1, L_2, L_3 using two parallel hyperplanes, we see that L_1 and L_3 are disjoint. If $\psi(L_1) = 0$, then the additivity and monotonicity imply $\psi(L_2 \cup L_3) = \psi(K)$. If $\psi(L_1) > 0$, then $\psi(L_3) = 0$, consequently $\psi(L_1 \cup L_2) = \psi(K)$. Continuing this procedure, we obtain a decreasing family of sets which shrinks to a point $x \in K$ with $\psi(\{x\}) = \psi(K) = c$, also taking into account the σ-continuity of ψ. Then $\psi(L) = 0$ for every $L \in \mathcal{K}^d$ with $x \notin L$.

Let $L \in \mathcal{K}^d$ contain x. By monotonicity, $\psi(L) \geq c$. Successively cut off convex pieces not containing x, retaining at each step a convex set $L_n \ni x$ with $\psi(L_n) = \psi(L)$. By σ-continuity, $\psi(L) = c$. Consequently, $\psi(L) = c\mathbf{1}_{x \in L}$ for all $L \in \mathcal{K}^d$. Thus, the Λ-outer measure is supported by valuations ψ with this property.

Each ψ of this form can be represented as a pair $(x, c) \in \mathbb{R}^d \times (0, \infty)$ and the map $(x, r) \mapsto r\mathbf{1}_{x \in K}$ and its inverse are measurable with respect to the cylindrical σ-algebra. Thus, the Lévy measure of Φ is the image of a measure on $\mathbb{R}^d \times (0, \infty)$ under this map. Therefore, the Poisson random measure N in (3.2) can be realised as the pushforward of a Poisson point process $\{(x_i, r_i) : i \geq 1\}$ on $\mathbb{R}^d \times (0, \infty)$, and thus Φ has the same distribution as

$$\eta(K) = \sum_i r_i \mathbf{1}_{x_i \in K}, \quad K \in \mathcal{K}^d, \tag{4.1}$$

which is a completely random measure restricted to $\mathcal{K}^d$, see [12, Section 15.3]. $\square$

While the complete independence assumption reduces σ-continuous monotone ID valuations to completely random measures, the independence of increments property identifies a richer family of random valuations, as the following section shows.

5 Monotone σ-continuous ID valuations with independent increments

If Φ is monotone, then $\Phi(K) \geq \Phi(\varnothing) = 0$ for all $K \in \mathcal{K}^d$ and so Φ is non-negative. For deterministic valuations, the non-negativity property does not imply monotonicity, e.g., for compact convex sets $M \subset L$, the valuation $\varphi(K) = \mathbf{1}_{K \cap L \neq \varnothing} - \mathbf{1}_{K \cap M \neq \varnothing}$ is non-negative, but not monotone. However, such a situation is impossible for ID valuations with independent increments.

Proposition 5.1. *If an ID valuation Φ with independent increments is non-negative and has vanishing deterministic part, then Φ is necessarily monotone.*

Proof. For $L \subset K$ put $X = \Phi(K) - \Phi(L)$ and $Y = \Phi(L)$. By independent increments, X and Y are independent. Since the deterministic part vanishes, the essential infimum of Y is zero. Moreover, $Y \geq 0$ and $X + Y \geq 0$. If $X < -\varepsilon$ with a positive probability, then $\mathbf{P}\{Y < \varepsilon\} > 0$ and independence would give that $\mathbf{P}\{X + Y < 0\} > 0$, a contradiction. Hence $X \geq 0$ a.s. Separability then gives pathwise monotonicity on the common full-probability event. $\square$

The following result provides a complete description of σ-continuous monotone ID valuations with independent increments. Recall that $\mathcal{C}$ denotes the family of all closed convex sets in $\mathbb{R}^d$.

Theorem 5.2. *A monotone σ-continuous ID valuation Φ has independent increments if and only if its Lévy measure Λ is the pushforward of a Borel measure ν on $\mathcal{C} \times (0, \infty)$ by the map*

$$(F, r) \mapsto \psi(K) = r\mathbf{1}_{K \cap F \neq \varnothing}, \quad K \in \mathcal{K}^d. \tag{5.1}$$

Furthermore, Φ has the same distribution as

$$\tilde{\Phi}(K) = \varphi(K) + \sum_i r_i \mathbf{1}_{K \cap F_i \neq \varnothing}, \quad K \in \mathcal{K}^d, \tag{5.2}$$

where $\{(F_i, r_i), i \geq 1\}$ is the Poisson point process on $\mathcal{C} \times (0, \infty)$ with intensity ν and φ is a deterministic monotone σ-continuous valuation. The measure ν necessarily satisfies

$$\int_{\mathcal{C} \times (0,\infty)} \min(1, r)\mathbf{1}_{F \cap K \neq \varnothing}\, \nu(d(F, r)) < \infty, \quad K \in \mathcal{K}^d. \tag{5.3}$$

Proof. Since the deterministic part $\varphi(K) = \operatorname{ess\,inf} \Phi(K)$ is a deterministic valuation, we replace Φ by $\Psi = \Phi - \varphi$ and assume that Φ does not have a deterministic part. If $K_n \downarrow K$, then

$$[\varphi(K_n) - \varphi(K)] + [\Psi(K_n) - \Psi(K)] = \Phi(K_n) - \Phi(K) \to 0,$$

so each of the non-negative terms on the left-hand side converges to zero, meaning that φ and Ψ are σ-continuous.

By Proposition 4.4 and Lemma 4.3, Φ has independent increments if and only if Λ-outer measure is supported by ψ such that

$$\psi(K) = \psi(L) \quad \text{or} \quad \psi(L) = 0 \tag{5.4}$$

for all $K, L \in \mathcal{K}^d$ with $L \subset K$. Let $K \in \mathcal{K}^d$ with $c = \psi(K) \neq 0$. If $K \subset W$ for $W \in \mathcal{K}^d$, then $\psi(W) = c$. For each $L \in \mathcal{K}^d$, take $W \in \mathcal{K}^d$ such that $(K \cup L) \subset W$. Then $\psi(L)$ is either zero or $\psi(L) = \psi(W) = c$. By applying this to a countable dense collection $K, L \in \mathcal{D}$ and using σ-continuity of ψ, we obtain that the Λ-outer measure is supported by non-decreasing valuations ψ with two values $\{0, c\}$ with $c > 0$ depending on ψ. By Lemma 3.3, all such ψ are σ-continuous.

It follows from [9] that a non-negative σ-continuous valuation ψ with two values is uniquely determined by its support, which is the set F of all points x such that $\psi(\{x\}) > 0$. Hence, $\psi(K) = c\mathbf{1}_{F \cap K \neq \varnothing}$ for some $c > 0$. This set F is necessarily nonempty, since otherwise ψ would have empty support and so would be zero. By σ-continuity of ψ we immediately obtain that F is closed. Indeed, if $x_n \in F$ and $x_n \to x$, we let K_n be the convex hull of $\{x\}$ and $\{x_i, i \geq n\}$, so that $c = \psi(K_n) \to \psi(\{x\})$ and so $x \in F$.

Let $L = \mathrm{conv}(\{x, y\})$ for any $x, y \in F$, and choose $z \in L$. Since ψ is non-decreasing and takes only two values 0 and c, we have $\psi(L) = c$. Similarly, $\psi(L_1) = \psi(L_2) = c$ for the segments with end-points x, z and y, z, respectively. Additivity therefore yields that $\psi(\{z\}) = c$, so $z \in F$. Thus, F is a convex set. The map between $\psi(K) = r\mathbf{1}_{F \cap K \neq \varnothing}$, $K \in \mathcal{K}^d$, with its cylindrical σ-algebra and the pair $(F, r) \in (\mathcal{C}, (0, \infty))$ with the product σ-algebra is bimeasurable. Hence, the Lévy measure of Φ corresponds to a measure on $\mathcal{C} \times (0, \infty)$.

Property (5.3) follows from (3.1). The space $\mathcal{C} \times (0, \infty)$ is a standard Borel space and so a Poisson random measure with intensity ν can be realised as a Poisson point process η on this space, see [12]. Thus, the integral in (3.2) can be represented as the sum (5.2).

In the reverse direction, the increments of $\tilde{\Phi}$ are determined by the Poisson process on $\mathcal{C} \times (0, \infty)$ over disjoint sets and so are independent. $\square$

The Poisson process $\{(F_i, r_i), i \geq 1\}$ on $\mathcal{C} \times (0, \infty)$ from Theorem 5.2 defines a pure-jump random measure Z on the family $\mathcal{C}$ of closed convex sets in $\mathbb{R}^d$ by letting

$$Z(\mathcal{M}) = \sum_i r_i \mathbf{1}_{F_i \in \mathcal{M}}$$

for all measurable $\mathcal{M} \subset \mathcal{C}$. This random measure on $\mathcal{C}$ is completely random, meaning that its values on disjoint subsets are independent, see [12, Section 15.3]. In contrast to Theorem 4.5, which relates monotone ID valuations with independent values to a completely random measure on $\mathbb{R}^d$, the random measure Z is defined on the family $\mathcal{C}$ of closed convex sets in $\mathbb{R}^d$. Then, the representation in Theorem 5.2 can be formulated as follows.

Corollary 5.3. *A non-negative σ-continuous random valuation Φ with* $\operatorname{ess\,inf} \Phi(K) = 0$ *for all $K \in \mathcal{K}^d$ is ID with independent increments if and only if Φ has the same distribution as $Z(\mathcal{C}_K)$ for a non-negative pure-jump completely random measure Z on $\mathcal{C}$ with intensity ν satisfying* (5.3).

Corollary 5.3 settles a random variant of Conjecture 1.1 for non-negative ID valuations with independent increments. If Φ is integrable, it implies that $\mathbf{E}\Phi(K) = \mu(\mathcal{C}_K)$ for a measure μ on $\mathcal{C}$. Given that, in general, the measure μ representing the deterministic valuation $\mathbf{E}\Phi(K)$ may be signed (see Examples 3.3 and 3.4 in [9]), we conclude that not every deterministic monotone σ-continuous valuation can be obtained as the expectation of an ID valuation with independent increments.

Remark 5.4 (Special cases of ν)**.** If $\nu(\mathcal{C}_K \times (0,\infty)) < \infty$, then the sum in (5.2) involves a finite number of terms for any given K. If $\nu(\mathcal{C} \times (0,\infty)) < \infty$, then the sum in (5.2) involves only a finite number of terms simultaneously for all K.

The ID valuation Φ with independent increments and vanishing deterministic part is a restriction to $\mathcal{K}^d$ of a random measure on $\mathbb{R}^d$ if and only if ν is supported by singletons, that is, by the family $\{(\{x\}, r)\colon r \in (0,\infty), x \in \mathbb{R}^d\}$. In this case, the values of Φ on disjoint sets are independent.

Remark 5.5 (Uniqueness of ν)**.** In general, a measure ν on $\mathcal{C} \times (0,\infty)$ is not uniquely determined by its values on $\mathcal{C}_K \times A$ for all $K \in \mathcal{K}^d$ and Borel $A \subset (0,\infty)$, that is, by the valuation it generates. For example, on the line, let ν_1 be the sum of Dirac measures at $([0,3],1)$ and $([1,2],1)$, and ν_2 at $([0,2],1)$ and $([1,3],1)$. Then $\nu_1(\mathcal{C}_K \times \cdot) = \nu_2(\mathcal{C}_K \times \cdot)$ for all K.

However, in case of ID valuations, the measure ν is determined uniquely by the distribution of the random valuation if one takes into account the finite-dimensional distributions of Φ. First, for any $K \in \mathcal{K}^d$, the measure $\nu(\mathcal{C}_K \times \cdot)$ is the Lévy measure M_K of the infinitely divisible random variable $\Phi(K)$. Second, by comparing the Lévy measure $M_{K,L}$ of the random vector $(\Phi(K), \Phi(L))$ with the Lévy measure of $(\sum_i r_i \mathbf{1}_{K\cap F_i \neq \varnothing}, \sum_i r_i \mathbf{1}_{L\cap F_i \neq \varnothing})$, we see that the restriction of $M_{K,L}$ on the x-axis gives us $\nu((\mathcal{C}_K \setminus \mathcal{C}_L) \times \cdot)$, on the y-axis $\nu((\mathcal{C}_L \setminus \mathcal{C}_K) \times \cdot)$, and on the diagonal $\{(t,t) : t > 0\}$ (equivalently, to $(0,\infty)^2$) we obtain $\nu((\mathcal{C}_K \cap \mathcal{C}_L) \times \cdot)$. In this way, finite-dimensional distributions of Φ yield the values of the Lévy measure ν on $(\mathcal{C}_{K_1} \cap \cdots \cap \mathcal{C}_{K_m}) \times A$ for all $K_1, \dots, K_m \in \mathcal{K}^d$, $m \geq 1$, and Borel $A \subset (0,\infty)$. Since this family of sets is closed under finite intersections, generates the Borel σ-algebra, and ν is finite on it outside the origin, the values of ν on this family determine uniquely ν on the whole Borel σ-algebra on the product space $\mathcal{C} \times (0,\infty)$.

Example 5.6. Let ν be the pushforward of the product of the Lebesgue measure V_d and a Lévy measure θ on $(0,\infty)$ under the map $(x,r) \mapsto (x+M, r)$, where M is a deterministic compact convex set. Furthermore, let $\varphi(K) = 0$ in (5.2). Then

$$\Phi(K) = \sum_i r_i \mathbf{1}_{K\cap(x_i+M)\neq\varnothing},$$

where $\{(x_i, r_i), i \geq 1\}$ is the Poisson point process on $\mathbb{R}^d \times (0,\infty)$ of intensity $V_d \otimes \theta$. Since $K \cap M + x \neq \varnothing$ if and only if $x \in K + (-M)$, where $(-M) = \{-x : x \in M\}$, we have

$$\mathbf{E}e^{-u\Phi(K)} = \exp\Big\{ -V_d(K + (-M)) \int_0^\infty \big(1 - e^{-ur}\big)\theta(dr)\Big\}.$$

If $\theta = \delta_1$ is the Dirac measure at 1, then $r_i = 1$ for all i and $\Phi(K) = \eta(K + (-M))$, where η is a homogeneous Poisson random measure on $\mathbb{R}^d$. In analogy with the deterministic setting, where valuations $V_d(K + M)$, $M \in \mathcal{K}^d$, span a dense subset of the family of all continuous translation-invariant valuations (see [4, Corollary 3.3]), it is tempting to conjecture that linear combinations of random valuations $\eta(K + M)$, $M \in \mathcal{K}^d$, are dense (in the topology of pointwise a.s. convergence) in the family of all stationary σ-continuous monotone ID valuations with independent increments.

If the independence of increments is relaxed as described in Appendix A, then the Lévy measure is supported by valuations $\psi(K)$ given by suprema of upper semicontinuous quasi-concave functions over K. In the square-integrable case, the absence of correlation between increments is equivalent to the fact that the variance of Φ is a valuation itself, see Appendix B.

6 Homogeneity and stability properties

Below we consider only σ-continuous monotone valuations. The scaling of a valuation φ by a positive number t can be defined either by multiplying its values by t, which yields the valuation $t\varphi$, or by dilating the argument K of φ. The latter scaling transforms the valuation φ into the valuation $T_t\varphi$ given by

$$(T_t\varphi)(K) = \varphi(t^{-1}K), \quad K \in \mathcal{K}^d.$$

A random valuation Φ is said to be *β-homogeneous (in distribution)* with some $\beta \geq 0$ if $t^{-\beta}\Phi$ has the same distribution as $T_t\Phi$ for all $t > 0$. Note that this concept of homogeneity is different from the pathwise homogeneity, which appears in McMullen's representation from Proposition 2.5.

A random valuation Φ is said to be *α-stable (in argument)* with some $\alpha \neq 0$ if, for all $n \geq 2$, the valuation $T_{n^{-1/\alpha}}\Phi$ has the same distribution as the sum of n independent copies of Φ. This immediately implies that Φ is ID. This α-stability property is different from the fact that the values $\Phi(K)$ are α-stable random variables for all $K \in \mathcal{K}^d$; the valuation Φ is said to be *α-stable (in value)* if the sum of its n independent copies has the same distribution as $n^{1/\alpha}\Phi$.

Proposition 6.1. *Let Φ be a non-trivial σ-continuous non-negative ID valuation with independent increments and vanishing deterministic part. Assume that the distribution of Φ is determined by the measure ν on $\mathcal{C} \times (0, \infty)$, see Theorem 5.2.*

(i) Φ is α-stable in argument if and only if

$$\nu(t^{1/\alpha}\mathcal{M} \times A) = t\nu(\mathcal{M} \times A) \tag{6.1}$$

for all rational $t > 0$, measurable $\mathcal{M} \subset \mathcal{C}$ and Borel $A \subset (0, \infty)$, and then $\alpha \in (0, \infty)$. Here the scaling of $\mathcal{M}$ is applied to all its elements.

(ii) Φ *is α-stable in value if and only if*

$$\nu(\mathcal{M} \times tA) = t^{-\alpha}\nu(\mathcal{M} \times A) \tag{6.2}$$

for all $t > 0$, measurable $\mathcal{M} \subset \mathcal{C}$ and Borel $A \subset (0, \infty)$, and then $\alpha \in (0, 1)$.

(iii) Φ *is β-homogeneous if and only if $\nu(t\mathcal{M} \times A) = \nu(\mathcal{M} \times (t^{-\beta}A))$ for all $t > 0$, measurable $\mathcal{M} \subset \mathcal{C}$ and Borel $A \subset (0, \infty)$.*

In each assertion, it suffices to verify the corresponding identity for $\mathcal{M} = \mathcal{C}_L$, where L ranges over finite unions of compact convex sets and $A \subset [\varepsilon, \infty)$ for any $\varepsilon > 0$.

Proof. All mentioned properties of Φ can be equivalently formulated for ν. If

$$\Phi(K) = \sum_i r_i \mathbf{1}_{K \cap F_i \neq \varnothing}, \quad K \in \mathcal{K}^d,$$

then

$$T_t\Phi(K) = \sum_i r_i \mathbf{1}_{K \cap tF_i \neq \varnothing} \quad \text{and} \quad t\Phi(K) = \sum_i tr_i \mathbf{1}_{K \cap F_i \neq \varnothing}.$$

The intensity measure of the Poisson point process $\{(tF_i, r_i), i \geq 1\}$ is given by $\nu((t^{-1}\mathcal{M}) \times A)$ and of the Poisson point process $\{(F_i, tr_i), i \geq 1\}$ by $\nu(\mathcal{M} \times t^{-1}A)$. Furthermore, the Lévy measure of the sum of n i.i.d. copies of Φ is $n\nu$.

(i) *Sufficiency* is obvious, since (6.1) says that the Lévy measure of $T_{n^{-1/\alpha}}\Phi$ is exactly the Lévy measure of Φ multiplied by n. Hence, $\Phi(n^{1/\alpha}K)$ has the same distribution as the sum of n independent copies of $\Phi(K)$, and the same holds for finite-dimensional distributions. Applying this for K which contains the origin and using the monotonicity of Φ show that $\alpha > 0$.

Necessity. The stability property implies that the Lévy measure M_K of the random variable $\Phi(K)$ is homogeneous under scaling of K, namely, $M_{n^{1/\alpha}K} = nM_K$. Thus, $\nu(\mathcal{C}_{n^{1/\alpha}K} \times A) = n\nu(\mathcal{C}_K \times A)$ for all $K \in \mathcal{K}^d$ and Borel $A \subset (0, \infty)$. Note that

$$\mathcal{C}_{n^{1/\alpha}K} = \{F \in \mathcal{C} : F \cap n^{1/\alpha}K \neq \varnothing\} = \{n^{1/\alpha}F : F \cap K \neq \varnothing\} = n^{1/\alpha}\mathcal{C}_K.$$

Furthermore, $\nu((\mathcal{C}_{K_1} \cap \cdots \cap \mathcal{C}_{K_m}) \times A)$ is recovered from the restriction of the Lévy measure $M_{K_1,\dots,K_m}$ of $(\Phi(K_1), \dots, \Phi(K_m))$ onto $(0, \infty)^m$ and so inherits the homogeneity property of $M_{K_1,\dots,K_m}$ and satisfies

$$\nu((\mathcal{C}_{n^{1/\alpha}K_1} \cap \cdots \cap \mathcal{C}_{n^{1/\alpha}K_m}) \times A) = n\nu((\mathcal{C}_{K_1} \cap \cdots \cap \mathcal{C}_{K_m}) \times A).$$

Since finite intersections of $\mathcal{C}_{K_i}$ generate the σ-algebra on $\mathcal{C}$, the measure ν is uniquely determined and satisfies $\nu(n^{1/\alpha}\mathcal{M} \times A) = n\nu(\mathcal{M} \times A)$ for all $n \geq 2$ on the whole σ-algebra on $\mathcal{C}$. By a standard argument, (6.1) holds for all rational t.

Let $\{x_i, i \geq 1\}$ be a Poisson point process on $\mathbb{R}^d$ with intensity measure having density $\|x\|^{\alpha-d}$ for any $\alpha \in (0, \infty)$. The corresponding intensity measure satisfies $\mu(tL) = t^\alpha\mu(L)$

for each compact set L. Therefore, the valuation given by (4.1) is a random measure which is α-stable in argument, so that all values $\alpha \in (0, \infty)$ are possible.

Next, $\alpha < 0$ is not possible if Φ is σ-continuous and monotone, since then $(T_{n^{-1/\alpha}}\Phi)(K) = \Phi(n^{1/\alpha}K) \to \Phi(\{0\})$ a.s. as $n \to \infty$ if $0 \in K$. On the other hand, stability gives that $\Phi(n^{1/\alpha}K)$ coincides in distribution with $\Phi_1(K) + \cdots + \Phi_n(K)$ for independent copies of Φ. If $\mathbf{P}\{\Phi(K) > 0\} > 0$, then the sum on the right-hand side tends to ∞ in probability, which is a contradiction. If $0 \notin K$, replace K with a sufficiently large ball and use monotonicity.

(ii) *Sufficiency* is obvious, since (6.2) means that the Lévy measure of $n^{1/\alpha}\Phi$ is exactly the Lévy measure of Φ multiplied by n.

Necessity. The stability property implies that the Lévy measure M_K of the random variable $\Phi(K)$ is homogeneous. Furthermore, $\nu((\mathcal{C}_{K_1} \cap \cdots \cap \mathcal{C}_{K_m}) \times A)$ is recovered from the restriction of $M_{K_1,\dots,K_m}$ onto $(0,\infty)^m$ and so inherits the homogeneity property of $M_{K_1,\dots,K_m}$. Since finite intersections of $\mathcal{C}_{K_i}$ generate the σ-algebra on $\mathcal{C}$, the measure ν is uniquely determined and is homogeneous on the whole σ-algebra.

Since Φ is non-trivial, there exists K for which $\Phi(K)$ is a non-degenerate non-negative strictly α-stable random variable, and hence necessarily $\alpha \in (0, 1)$. Existence for every $\alpha \in (0, 1)$ follows by taking Φ to be the restriction to $\mathcal{K}^d$ of a non-negative independently scattered α-stable random measure with a locally finite control measure.

(iii) The valuation Φ is β-homogeneous if and only if the distribution of the Poisson point process $\{(tF_i, r_i), i \geq 1\}$ coincides with the distribution of $\{(F_i, t^{-\beta} r_i), i \geq 1\}$ for all $t > 0$. This is exactly the formulated homogeneity property of ν. □

7 Stationary random valuations

A random valuation Φ is said to be *stationary* if all its finite-dimensional distributions are translation invariant, that is, the joint distribution of $(\Phi(K_1), \dots, \Phi(K_m))$ is the same as the joint distribution of $(\Phi(K_1 + x), \dots, \Phi(K_m + x))$ for all $x \in \mathbb{R}^d$, all $K_1, \dots, K_m \in \mathcal{K}^d$, and all $m \geq 1$. Similarly, Φ is *isotropic* if all its finite-dimensional distributions are invariant under rotations applied to the argument. We assume that random valuations do not have a deterministic part.

Example 7.1 (Stationary pure-jump completely random measure)**.** Let $\{(x_i, r_i), i \geq 1\}$ be the Poisson point process on $\mathbb{R}^d \times (0, \infty)$ with intensity measure equal to the product of the Lebesgue measure on $\mathbb{R}^d$ and a measure μ on $(0, \infty)$ such that $\int \min(1, r)\mu(dr) < \infty$. Then η given by (4.1) is a stationary pure-jump completely random measure. If restricted to compact convex sets, it defines a stationary σ-continuous ID valuation with independent increments and independent values, see Theorem 4.5.

If Φ is an ID valuation, then the uniqueness of the Lévy measure implies that Φ is stationary (isotropic) if and only if its Lévy measure Λ is invariant under translations (rotations). The stationarity property of a σ-continuous monotone ID valuation with independent increments is equivalent to translation invariance of the measure ν on $\mathcal{C} \times (0, \infty)$ (see Theorem 5.2) with respect to translations of the first component, that is,

$$\nu((\mathcal{C}_{K_1} \cap \cdots \cap \mathcal{C}_{K_m}) \times A) = \nu((\mathcal{C}_{K_1+x} \cap \cdots \cap \mathcal{C}_{K_m+x}) \times A)$$

for all $x \in \mathbb{R}^d$, $K_1, \dots, K_m \in \mathcal{K}^d$, $m \geq 1$, and Borel $A \subset (0, \infty)$.

The family of translation-invariant locally finite measures on $\mathcal{C}$ has been characterised in Theorem 5.4.1 from [13]. Below we present its variant for measures on $\mathcal{C} \times (0, \infty)$. Denote by $G(d, k)$ the Grassmannian in $\mathbb{R}^d$ that consists of linear subspaces of dimension k. For $E \in G(d, k)$, denote by $\Pi_{E^\perp} L$ the projection of a compact set L onto the orthogonal complement $E^\perp$ of E.

The following result characterises translation invariant Lévy measures and so describes monotone stationary σ-continuous ID random valuations which have independent increments and vanishing deterministic part.

Theorem 7.2. *Let ν be a measure on $\mathcal{C} \times (0, \infty)$ which satisfies* (5.3) *and such that $\nu(\{\varnothing\} \times (0, \infty)) = 0$. Then ν is invariant under translations on $\mathcal{C}$ if and only if, for each compact set $L \subset \mathbb{R}^d$ and Borel $A \subset (0, \infty)$, we have*

$$\nu(\mathcal{C}_L \times A) = \sum_{k=0}^{d} \int_A \int_{G(d,k)} \mathbf{E}(V_{d-k}(Y_{E,r} + \Pi_{E^\perp} L)) G_{k,r}(dE) \mu(dr), \tag{7.1}$$

where, for each $k = 0, \dots, d$, $r \mapsto G_{k,r}$ is a measurable map from $(0, \infty)$ to the space of finite measures on $G(d, k)$ with the weak topology, and μ is a σ-finite measure on $(0, \infty)$. Furthermore, for each $E \in G(d, k)$ and $r > 0$, $Y_{E,r}$ is a non-empty random compact convex set in $E^\perp$ such that

$$\int_{(0,\infty)} \min(1, r) \int_{G(d,k)} \mathbf{E}(V_{d-k}(Y_{E,r} + \Pi_{E^\perp} B)) G_{k,r}(dE) \mu(dr) < \infty \tag{7.2}$$

for all $k = 0, \dots, d$, where B is the unit Euclidean ball in $\mathbb{R}^d$, and, for every compact set L in $\mathbb{R}^d$, the map

$$(E, r) \mapsto \mathbf{E}(V_{d-k}(Y_{E,r} + \Pi_{E^\perp} L)) \tag{7.3}$$

is measurable.

Proof. For $k = 0, \dots, d$, let $\mathcal{Z}_k$ denote the family of closed convex cylinders with k-dimensional direction space, namely, each $F \in \mathcal{Z}_k$ can be written in the form $F = E + C$, where $E \in G(d, k)$ and C is a non-empty compact convex subset of $E^\perp$. As in the proof of [13, Theorem 5.4.1], the translation-invariant measure ν is concentrated on $\bigcup_{k=0}^d \mathcal{Z}_k \times (0, \infty)$. This follows from the mentioned theorem for $\nu(\cdot \times (1/n, \infty))$ and then taking the union over n.

Denote by $s(C)$ the Steiner point of C, see [18, Eq. (1.31)], and put $Y = C - s(C)$ and $x = s(C)$. Then every $F \in \mathcal{Z}_k$ has a unique representation

$$F = E + Y + x, \tag{7.4}$$

where $E \in G(d,k)$, $Y \subset E^\perp$, $s(Y) = 0$, and $x \in E^\perp$. Let

$$\mathcal{Y}_k = \{(E, Y) : E \in G(d,k),\ Y \subset E^\perp \text{ is non-empty compact convex},\ s(Y) = 0\}.$$

Equipped with the Borel σ-algebra inherited from $G(d,k) \times \mathcal{K}^d$, the space $\mathcal{Y}_k$ is a standard Borel space. Indeed, the map $C \mapsto s(C)$ is continuous in the Hausdorff metric, and the conditions $Y \subset E^\perp$ and $s(Y) = 0$ are Borel. The correspondence in (7.4) and its inverse are Borel measurable.

Restrict ν to $\mathcal{Z}_k \times (0,\infty)$ and transport this restriction under the map $(F, r) \mapsto (E, Y, x, r)$ determined by (7.4). Denote the resulting measure by $\widetilde{\nu}_k$. It is a measure on

$$\mathcal{X}_k = \{(E, Y, x, r) : (E, Y) \in \mathcal{Y}_k,\ x \in E^\perp,\ r > 0\}.$$

The transported measure $\widetilde{\nu}_k$ is invariant under $(E, Y, x, r) \mapsto (E, Y, x + \Pi_{E^\perp} z, r)$. Let $B_{E^\perp}$ denote the unit Euclidean ball in $E^\perp$ and, for Borel $H \subset \mathcal{Y}_k \times (0,\infty)$, define

$$\Theta_k(H) = \frac{1}{\kappa_{d-k}} \widetilde{\nu}_k\big(\{(E, Y, x, r) : (E, Y, r) \in H,\ x \in B_{E^\perp}\}\big),$$

where κ_{d-k} is the volume of the unit Euclidean ball in $\mathbb{R}^{d-k}$. For $k = d$, we let $\kappa_0 = 1$. By translation invariance and the uniqueness of Haar measure on each fibre $E^\perp$, a monotone-class argument yields

$$\int_{\mathcal{X}_k} h(E, Y, x, r)\, \widetilde{\nu}_k(d(E, Y, x, r)) = \int_{\mathcal{Y}_k \times (0,\infty)} \int_{E^\perp} h(E, Y, x, r)\, dx\ \Theta_k(d(E, Y, r)) \tag{7.5}$$

for every non-negative Borel function h.

Let $L \subset \mathbb{R}^d$ be compact. For a cylinder $F = E + Y + x$, we have $F \cap L \neq \varnothing$ if and only if $(Y + x) \cap \Pi_{E^\perp} L \neq \varnothing$. Hence

$$\int_{E^\perp} \mathbf{1}_{(E+Y+x)\cap L \neq \varnothing}\, dx = V_{d-k}\big(\Pi_{E^\perp} L + (-Y)\big).$$

Applying the measurable involution $Y \mapsto -Y$ to Θ_k, and denoting the resulting measure again by Θ_k, we obtain from (7.5)

$$\nu(\mathcal{C}_L \times A) = \sum_{k=0}^{d} \int_{\mathcal{Y}_k \times A} V_{d-k}(Y + \Pi_{E^\perp} L)\, \Theta_k(d(E, Y, r)). \tag{7.6}$$

It remains to disintegrate the measures Θ_k. Let $\theta_k(A) = \Theta_k(\mathcal{Y}_k \times A)$ for Borel $A \subset (0,\infty)$. Each θ_k is σ-finite. Indeed, since Y is non-empty,

$$V_{d-k}(Y + \Pi_{E^\perp} B) \geq \kappa_{d-k},$$

and therefore, for every $n \geq 1$,

$$\kappa_{d-k} \theta_k([n^{-1}, \infty)) \leq \nu(\mathcal{C}_B \times [n^{-1}, \infty)) < \infty$$

by (5.3). Set $\mu = \theta_0 + \cdots + \theta_d$, which is a σ-finite measure on $(0,\infty)$. Since all spaces involved are standard Borel, the disintegration theorem yields finite measurable kernels $H_{k,r}$ on $\mathcal{Y}_k$ such that

$$\Theta_k(d(E,Y,r)) = H_{k,r}(d(E,Y))\,\mu(dr).$$

Let $G_{k,r}$ be the E-marginal of $H_{k,r}$, that is, $G_{k,r}(D) = H_{k,r}\big(\{(E,Y) \in \mathcal{Y}_k : E \in D\}\big)$ for Borel $D \subset G(d,k)$. Then $r \mapsto G_{k,r}$ is a measurable kernel and $G_{k,r}$ is a finite measure for μ-almost every r. Define the joint measure

$$\bar{H}_k(dr, dE, dY) = H_{k,r}(d(E,Y))\mu(dr)$$

and its (r,E)-marginal

$$\bar{G}_k(dr, dE) = G_{k,r}(dE)\mu(dr).$$

Redefining $G_{k,r} = 0$ on a μ-null set if necessary, we may assume that $G_{k,r}$ is finite for every $r > 0$. Since these are measures on standard Borel spaces, disintegration gives a jointly measurable probability kernel $Q_k(r,E,dY)$ such that

$$\bar{H}_k(dr, dE, dY) = Q_k(r,E,dY)G_{k,r}(dE)\mu(dr).$$

On the set where $G_{k,r}$ vanishes, the kernel Q_k may be chosen arbitrarily, so that the assertion holds for every r.

Let $Y_{E,r}$ denote a random compact convex set with distribution $Q_k(r,E,\cdot)$. Then (7.6) becomes

$$\nu(\mathcal{C}_L \times A) = \sum_{k=0}^{d} \int_A \int_{G(d,k)} \mathbf{E} V_{d-k}(Y_{E,r} + \Pi_{E^\perp} L) G_{k,r}(dE)\mu(dr),$$

which is (7.1). Notice that the construction also proves the measurability of (7.3), since $(E,Y) \mapsto V_{d-k}(Y + \Pi_{E^\perp} L)$ is Borel and Q_k is a measurable probability kernel. Finally, taking $L = B$ in (7.1) and using (5.3) gives

$$\begin{aligned}\infty &> \int_{(0,\infty)} \min(1,r)\,\nu(\mathcal{C}_B \times dr)\\ &= \sum_{k=0}^{d} \int_{(0,\infty)} \min(1,r) \int_{G(d,k)} \mathbf{E} V_{d-k}(Y_{E,r} + \Pi_{E^\perp} B) G_{k,r}(dE)\mu(dr).\end{aligned}$$

Since all summands are non-negative, each of them is finite, which yields (7.2).

Conversely, a measure satisfying (7.1) is translation invariant. Indeed, for $x \in \mathbb{R}^d$,

$$\Pi_{E^\perp}(L + x) = \Pi_{E^\perp} L + \Pi_{E^\perp} x,$$

and translation invariance of Lebesgue measure gives

$$V_{d-k}(Y_{E,r} + \Pi_{E^\perp}(L + x)) = V_{d-k}(Y_{E,r} + \Pi_{E^\perp} L).$$

Thus $\nu(\mathcal{C}_{L+x} \times A) = \nu(\mathcal{C}_L \times A)$ for all compact L, all $x \in \mathbb{R}^d$, and all Borel $A \subset [n^{-1}, \infty)$ with any $n \geq 1$. Letting n grow and using the fact that compact hitting sets form a measure-determining class on $\mathcal{C}$, this implies translation invariance of ν. $\square$

It is important to stress that (7.1) provides a representation of the values of ν on $\mathcal{C}_L \times A$ for all (not necessarily convex) compact sets L and Borel $A \subset [n^{-1}, \infty)$ with any sufficiently large n. In particular, (7.1) yields the values of ν on $(\mathcal{C}_{K_1} \cup \cdots \cup \mathcal{C}_{K_m}) \times A = \mathcal{C}_{K_1 \cup \cdots \cup K_m} \times A$ for all collections $K_1, \dots, K_m \in \mathcal{K}^d$ and then by the inclusion-exclusion formula for all $(\mathcal{C}_{K_1} \cap \cdots \cap \mathcal{C}_{K_m}) \times A$, hence for all measurable $\mathcal{M} \subset \mathcal{C}$.

It is instructive to separate out in (7.1) the terms corresponding to the trivial Grassmannians (for $k = 0$ and $k = d$), and then

$$\begin{aligned}\nu(\mathcal{C}_L \times A) = &\int_A \mathbf{E} V_d(Y_{0,r} + L) g_0(r) \mu(dr) + \mathbf{1}_{L \neq \varnothing} \int_A g_d(r) \mu(dr) \\ &+ \sum_{k=1}^{d-1} \int_A \int_{G(d,k)} \mathbf{E}(V_{d-k}(Y_{E,r} + \Pi_{E^\perp} L)) G_{k,r}(dE) \mu(dr),\end{aligned}$$

where g_0 and g_d are two non-negative measurable functions. This representation shows that the Poisson point process $\{(F_i, r_i), i \geq 1\}$ with the intensity measure ν can be obtained as the superposition of the stationary Poisson point process of compact convex sets with marks in $(0, \infty)$ for $k = 0$, the Poisson process $\{(\mathbb{R}^d, r_i), i \geq 1\}$ for $k = d$, and the Poisson processes of random cylinders with k-dimensional direction spaces and compact convex bases in $E^\perp$, $k = 1, \dots, d-1$.

Example 7.3. The term with $k = d$ in (7.1) corresponds to $\nu(\mathcal{C}_L \times A) = \mathbf{1}_{L \neq \varnothing} \theta(A)$ for a Lévy measure θ on $(0, \infty)$. Then $\Phi(K) = \xi$ for all non-empty $K \in \mathcal{K}^d$, where ξ is the infinitely divisible random variable with the Lévy measure θ.

If $\mu = \delta_1$, then the term corresponding to $k = 0$ results in the Lévy measure

$$\nu(\mathcal{C}_L \times A) = c \mathbf{E} V_d(Y_0 + L) \delta_1(A)$$

for $c = G_{0,1}(\{0\})$. The corresponding ID valuation Φ is given by

$$\Phi(K) = \sum_i \mathbf{1}_{(Y_i + x_i) \cap K \neq \varnothing},$$

where $\{(Y_i, x_i), i \geq 1\}$ is the Poisson point process on $\mathcal{C} \times \mathbb{R}^d$ with the intensity being c times the product of the distribution of a random compact convex set Y_0 (called the typical grain) and the Lebesgue measure. In other words, $\Phi(K)$ is the number of sets from the germ-grain model which intersect K, see [19, Section 4.3]. The case of all $Y_i = M$ with $s(M) = 0$ appears in Example 5.6.

In the stationary isotropic case we get the following result. Denote

$$c_i^k = \frac{\Gamma((k+1)/2)}{\Gamma((i+1)/2)} \quad \text{and } c_{i,j}^{k,m} = c_i^k c_j^m.$$

Theorem 7.4. *If Φ is a σ-continuous non-negative stationary isotropic ID valuation with independent increments, then the corresponding Lévy measure on $\mathcal{C} \times (0, \infty)$ is given by*

$$\nu(\mathcal{C}_L \times A) = \sum_{k=0}^{d} \int_A \int_{G(d,k)} \mathbf{E}(V_{d-k}(Y_{E,r} + \Pi_{E^\perp} L))\gamma_k(dE) a_{k,r} \mu(dr) \tag{7.7}$$

for each compact set L in $\mathbb{R}^d$ and Borel $A \subset (0, \infty)$, where γ_k is the normalised Haar measure on $G(d, k)$, $Y_{E,r}$ is a random compact convex set in $E^\perp$ such that $Y_{RE,r}$ has the same distribution as $RY_{E,r}$ for each deterministic rotation R, and $a_{k,r}$ is a measurable non-negative function of $r > 0$. If $K \in \mathcal{K}^d$, then

$$\nu(\mathcal{C}_K \times A) = \sum_{k=0}^{d} \sum_{j=0}^{d-k} c_{0,d}^{d-k-j,k+j} V_{d-k-j}(K) \int_A \mathbf{E} V_j(Y_{E,r}) a_{k,r} \mu(dr). \tag{7.8}$$

Proof. The measure ν is invariant under rotations and translations. Referring to [13, Section 5.4] or examining the proof of Theorem 5.4.1 ibid. shows that $G_{k,r} = a_{k,r}\gamma_k$, where $a_{k,r} \geq 0$ and γ_k is the Haar measure on $G(d, k)$, and that the random compact convex set $Y_{E,r}$ for $E \in G(d, k)$ is such that $Y_{RE,r}$ has the same distribution as $RY_{E,r}$ for each deterministic rotation R. For this, we can redefine the kernels on a μ-null set. This yields (7.7).

If $L = K$ is convex, averaging the right-hand side of (7.1) with the normalised Haar measure on the family of all rotations applied to K and referring to Theorem 6.1.1 from [19] yield that

$$\nu(\mathcal{C}_K \times A) = \sum_{k=0}^{d} \int_A \int_{G(d,k)} \sum_{j=0}^{d-k} c_{0,d-k}^{j,d-k-j} \mathbf{E} V_j(Y_{E,r}) V_{d-k-j}(\Pi_{E^\perp} K)\gamma_k(dE) a_{k,r} \mu(dr).$$

By [19, Theorem 6.2.2],

$$\int_{G(d,k)} V_{d-k-j}(\Pi_{E^\perp} K)\gamma_k(dE) = c_{d,j}^{d-k,k+j} V_{d-k-j}(K). \qquad \square$$

Proposition 7.5. *Assume that Φ is a stationary σ-continuous monotone ID valuation with independent increments and vanishing deterministic part which is also almost surely simple, that is, $\Phi(K) = 0$ a.s. for all $K \in \mathcal{K}^d$ of dimension at most $d - 1$. Then Φ is a stationary pure-jump completely random measure.*

Proof. Let $L_j = B \cap e_j^\perp$, $j = 1, \dots, d$, where B is the unit Euclidean ball and $e_1, \dots, e_d$ are the coordinate vectors. Each L_j has dimension $d-1$, and hence the simplicity assumption yields $\Phi(L_j) = 0$ a.s. Consequently, the Lévy measure of $\Phi(L_j)$ vanishes, and so $\nu(\mathcal{C}_{L_j} \times A) = 0$ for every Borel $A \subset (0, \infty)$. Applying representation (7.1) to L_j which yields zero value and noticing that all summands are non-negative show that

$$\int_A \int_{G(d,k)} \mathbf{E} V_{d-k}(Y_{E,r} + \Pi_{E^\perp} L_j) G_{k,r}(dE) \mu(dr) = 0 \tag{7.9}$$

for every $j = 1, \dots, d$ and $k = 0, \dots, d$ and for all Borel $A \subset (0, \infty)$.

We first consider $1 \le k \le d-1$. For every $E \in G(d,k)$, there exists $j \in \{1, \dots, d\}$ such that $e_j \notin E^\perp$, since $E^\perp$ is a proper linear subspace of $\mathbb{R}^d$. For such j, the orthogonal projection of $e_j^\perp$ onto $E^\perp$ is surjective. Hence, the dimension of $\Pi_{E^\perp} L_j$ is $d-k$, and $\Pi_{E^\perp} L_j$ has non-empty interior in $E^\perp$. Since $Y_{E,r}$ is non-empty,

$$V_{d-k}(Y_{E,r} + \Pi_{E^\perp} L_j) > 0.$$

It follows that, for every $E \in G(d,k)$,

$$\sum_{j=1}^{d} \mathbf{E} V_{d-k}(Y_{E,r} + \Pi_{E^\perp} L_j) > 0.$$

Summing (7.9) over $j = 1, \dots, d$ therefore shows that $G_{k,r} = 0$ as a measure on $G(d,k)$ for μ-almost every r. Thus, all components with $k = 1, \dots, d-1$ vanish.

If $k = d$, then $E^\perp = \{0\}$ and, since $Y_{E,r}$ is non-empty, $V_0(Y_{E,r} + \Pi_{E^\perp} L_j) = 1$. Hence (7.9) implies that $G_{d,r} = 0$ for μ-almost every r, consequently, the component with $k = d$ also vanishes.

It remains to consider $k = 0$. In this case $E = \{0\}$ and $E^\perp = \mathbb{R}^d$. Since $G(d,0) = \{\{0\}\}$, write $g_0(r) = G_{0,r}(\{\{0\}\})$. We claim that $Y_{0,r}$ is a singleton almost surely for almost every r with respect to the corresponding intensity measure. Suppose otherwise. Then, with positive probability, $Y_{0,r}$ contains a non-degenerate segment with direction $u \neq 0$. There exists $j \in \{1, \dots, d\}$ such that $\langle u, e_j \rangle \neq 0$. Since L_j has non-empty relative interior in the hyperplane $e_j^\perp$, the Minkowski sum of this segment and L_j has non-empty interior in $\mathbb{R}^d$. Therefore, $V_d(Y_{0,r} + L_j) > 0$. Consequently, the sum of $V_d(Y_{0,r} + L_j)$ over j is strictly positive whenever $Y_{0,r}$ is non-singleton. Summing (7.9) over j for $k = 0$ shows that this event has zero probability for almost every relevant r. Hence $Y_{0,r}$ is a singleton a.s. for $g_0(r)\mu(dr)$-almost every r. In the centred parametrisation used in the proof of Theorem 7.2, we have $s(Y_{0,r}) = 0$. A singleton whose Steiner point is the origin is necessarily $\{0\}$. Thus $Y_{0,r} = \{0\}$ almost surely for almost every relevant r.

Hence only the $k = 0$ component remains, and its cylinders are precisely the translated singletons $\{x\}$, $x \in \mathbb{R}^d$. Therefore, the Lévy measure ν is concentrated on $\{(\{x\}, r) : x \in \mathbb{R}^d, r > 0\}$. Translation invariance implies that the image of ν under $(\{x\}, r) \mapsto (x, r)$ is the product of the Lebesgue measure ℓ on $\mathbb{R}^d$ and a measure θ on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} \min(1, r)\theta(dr) < \infty. \tag{7.10}$$

Consequently, Φ has the same distribution as the restriction to $K \in \mathcal{K}^d$ of a stationary pure-jump completely random measure η given by (4.1) with a Poisson point process $\{(x_i, r_i), i \geq 1\}$ on $\mathbb{R}^d \times (0, \infty)$ of intensity $\ell \otimes \theta$. $\square$

8 Stationary stable ID valuations and McMullen's decomposition

Below we specialise the representation of stationary ID valuations to those which are α-stable in argument and show that the stability exponent α necessarily belongs to $\{1,\dots,d\}$. The crucial fact is that the Lévy measure ν is supported by an affine Grassmannian $A(d,k)$ for $k=d-\alpha$. Recall that $A(d,k)$ is the family of all k-dimensional affine subspaces of $\mathbb{R}^d$.

Theorem 8.1. *Let Φ be a non-trivial stationary σ-continuous non-negative ID valuation with independent increments and vanishing deterministic part. If Φ is α-stable in argument, then $\alpha \in \{1,\dots,d\}$, and its Lévy measure has representation*

$$\nu(\mathcal{C}_L \times A) = \int_A \int_{G(d,d-\alpha)} V_\alpha(\Pi_{E^\perp} L) G_{d-\alpha,r}(dE)\mu(dr) \tag{8.1}$$

for all compact sets L in $\mathbb{R}^d$ and Borel $A \subset (0,\infty)$. Furthermore, in the corresponding Poisson point process $\{(F_i,r_i), i \geq 1\}$ all sets F_i are affine subspaces of dimension $d-\alpha$. Conversely, for each $\alpha \in \{1,\dots,d\}$, representation (8.1) *implies that Φ is α-stable in argument.*

Proof. By Proposition 6.1, Φ is α-stable in argument if and only if $\nu(\mathcal{C}_{t^{1/\alpha}L} \times A) = t\nu(\mathcal{C}_L \times A)$ for all compact L, rational $t>0$, and Borel $A \subset (0,\infty)$. Letting A be a subset of $[\varepsilon,\infty)$ with $\varepsilon>0$, L be the unit Euclidean ball B, and using the Steiner formula, (7.1) yields that

$$\nu(\mathcal{C}_{t^{1/\alpha}B} \times A) = \sum_{j=0}^{d} \kappa_j t^{j/\alpha} \sum_{k=0}^{d-j} \int_A \int_{G(d,k)} \mathbf{E} V_{d-j-k}(Y_{E,r}) G_{k,r}(dE)\mu(dr), \tag{8.2}$$

where κ_j is the volume of the unit ball in $\mathbb{R}^j$. Again, the Steiner formula yields that

$$t\nu(\mathcal{C}_B \times A) = t \sum_{j=0}^{d} \kappa_j \sum_{k=0}^{d-j} \int_A \int_{G(d,k)} \mathbf{E} V_{d-j-k}(Y_{E,r}) G_{k,r}(dE)\mu(dr).$$

Thus, only the term with $j=\alpha$ appears on the right-hand side of (8.2) and thus $\alpha \in \{1,\dots,d\}$. This holds for all Borel $A \subset (0,\infty)$ by monotone exhaustion. Therefore,

$$\sum_{k=0}^{d-j} \int_A \int_{G(d,k)} \mathbf{E} V_{d-j-k}(Y_{E,r}) G_{k,r}(dE)\mu(dr) = 0, \quad j \neq \alpha. \tag{8.3}$$

All terms of (8.3) with $j>\alpha$ (if $\alpha<d$) vanish, and so $G_{k,r}=0$ as a measure on $G(d,k)$ for μ-almost every r and $k \leq d-\alpha-1$. Looking at the terms with $j<\alpha$ and $k=d-j$ shows that $G_{k,r}$ vanishes for μ-almost all r and $k \geq d-\alpha+1$. Leaving in (8.3) only the terms with $k=d-\alpha$ yields that, for $j=\alpha-1$,

$$\int_A \int_{G(d,d-\alpha)} \mathbf{E} V_1(Y_{E,r}) G_{d-\alpha,r}(dE)\mu(dr) = 0,$$

so that $Y_{E,r}$ is a singleton for $G_{d-\alpha,r}(dE)\mu(dr)$-almost every (E,r). Then (8.1) holds and all sets F_i are affine subspaces of dimension $d-\alpha$. In particular, if $\alpha = d$, then it is possible to combine $G_{0,r}(\{\{0\}\})$ with $\mu(dr)$ to obtain new measure $\tilde{\mu}$, so that $\nu(\mathcal{C}_L \times A) = \tilde{\mu}(A)V_d(L)$ for all compact L.

The converse easily follows from (8.1) by writing it for $t^{1/\alpha}L$. $\square$

Below we address situations in which Φ is decomposable into a sum of valuations which are stable in argument. The following result is a random analogue of McMullen's decomposition theorem for translation-invariant valuations, see [1, Theorem 2.0.2] and [4, Theorem 3.1].

Two random valuations Φ and Ψ are said to be *weakly equivalent* if the distribution of $\Phi(K)$ coincides with that of $\Psi(K)$ for all $K \in \mathcal{K}^d$, that is, the one-dimensional distributions of Φ and Ψ are identical. If Φ and Ψ are square-integrable and have independent increments, then their weak equivalence implies that their covariances agree on pairs of admissible sets.

Theorem 8.2. *Let Φ be a σ-continuous non-negative stationary isotropic ID valuation with independent increments and vanishing deterministic part. Then there exist a non-negative infinitely divisible random variable ξ and independent non-negative stationary isotropic ID valuations Ψ_m, $m = 1, \dots, d$, with independent increments, such that*

$$\Psi(K) = \xi \mathbf{1}_{K \neq \varnothing} + \sum_{m=1}^{d} \Psi_m(K), \qquad K \in \mathcal{K}^d,$$

is weakly equivalent to Φ. Moreover, Ψ_m is m-stable in argument for every $m = 1, \dots, d$.

More precisely, there exist non-negative measures $b_0, \dots, b_d$ on $(0,\infty)$ satisfying (7.10) such that

$$\nu(\mathcal{C}_K \times A) = \sum_{m=0}^{d} V_m(K) b_m(A), \qquad K \in \mathcal{K}^d, \tag{8.4}$$

for every Borel $A \subset (0,\infty)$, where $V_0(K) = \mathbf{1}_{K \neq \varnothing}$.

Proof. Recall the representation (7.8) for the Lévy measure obtained in Theorem 7.4, where, by isotropy, $\mathbf{E}V_\ell(Y_{E,r})$ does not depend on the particular $E \in G(d,k)$. Put $m = d-k-j$. Then (7.8) can be written as

$$\nu(\mathcal{C}_K \times A) = \sum_{m=0}^{d} V_m(K) b_m(A),$$

where

$$b_m(A) = c_{0,d}^{m,d-m} \sum_{k=0}^{d-m} \int_A \mathbf{E}V_{d-k-m}(Y_{E,r}) a_{k,r} \mu(dr), \quad m = 0, \dots, d, \tag{8.5}$$

are non-negative measures on $(0,\infty)$. Taking K to be the unit Euclidean ball, and using (5.3) yields (7.10).

We now construct independent random valuations corresponding to these measures. Let ξ be a non-negative infinitely divisible random variable with vanishing deterministic part and Lévy measure b_0. Then $K \mapsto \xi V_0(K) = \xi \mathbf{1}_{K \neq \varnothing}$ gives the degree-zero component.

For $m = 1, \dots, d$, let ρ_m be the motion-invariant measure on the affine Grassmannian $A(d, d-m)$, normalised so that the Crofton formula takes the form

$$\rho_m(\{F \in A(d, d-m) : F \cap K \neq \varnothing\}) = V_m(K), \qquad K \in \mathcal{K}^d. \tag{8.6}$$

Let N_m, $m = 1, \dots, d$, be independent Poisson point processes on $A(d, d-m) \times (0, \infty)$ with intensity measures $\rho_m(dF) b_m(dr)$, and assume that they are also independent of ξ. Define

$$\Psi_m(K) = \int r \mathbf{1}_{F \cap K \neq \varnothing} \, N_m(d(F, r)). \tag{8.7}$$

The integral is well defined, since

$$\int_{A(d,d-m)\times(0,\infty)} \min(1, r) \mathbf{1}_{F \cap K \neq \varnothing} \rho_m(dF) b_m(dr) = V_m(K) \int_{(0,\infty)} \min(1, r) b_m(dr) < \infty.$$

By construction, Ψ_m is non-negative, σ-continuous, stationary and isotropic, and it has independent increments.

We next verify stability in argument. The motion-invariant measure ρ_m is homogeneous of degree m under dilations of affine subspaces. Indeed, writing an affine $(d-m)$-flat as $E+x$, with $E \in G(d, d-m)$ and $x \in E^\perp$, its translation component is m-dimensional Lebesgue measure, hence, $\rho_m(t^{1/m}\mathcal{M}) = t\rho_m(\mathcal{M})$ for all $t > 0$. Consequently, for every $t > 0$,

$$(\rho_m \otimes b_m)(t^{1/m}\mathcal{M} \times A) = t(\rho_m \otimes b_m)(\mathcal{M} \times A).$$

By Proposition 6.1, Ψ_m is m-stable in argument. Finally, set

$$\Psi(K) = \xi \mathbf{1}_{K \neq \varnothing} + \sum_{m=1}^{d} \Psi_m(K).$$

Since the components are independent, the Lévy measure of the non-negative infinitely divisible random variable $\Psi(K)$ is the sum of their one-dimensional Lévy measures. By (8.6), the Lévy measure of $\Psi(K)$ is

$$\sum_{m=0}^{d} V_m(K) b_m(dr) = \nu(\mathcal{C}_K \times dr).$$

Both $\Phi(K)$ and $\Psi(K)$ have vanishing deterministic part, hence Φ and Ψ are weakly equivalent. $\square$

The weak equivalence in Theorem 8.2 cannot in general be strengthened to equality of finite-dimensional distributions, as the following example shows.

Remark 8.3. Imposing only stationarity is not sufficient for the validity of Theorem 8.2. Let $\Phi(K) = \eta(K + A)$ where η is a Poisson point process and $A \in \mathcal{K}^d$ is not centrally symmetric. Then, for some $K \in \mathcal{K}^d$, $\Phi(K)$ does not have the same distribution as $\Phi(-K)$. However, each random valuation which is stable in argument has representation given by Theorem 8.1, and this representation is invariant in distribution with respect to the central symmetry transform. Therefore, while the valuation Φ is infinitely divisible, stationary, monotone, and has independent increments, it cannot be represented as the sum of stable random valuations.

Example 8.4. Let η be the stationary Poisson random measure on $\mathbb{R}^d$, that is,

$$\eta(K) = \sum_i \mathbf{1}_{x_i \in K},$$

where $\{x_i, i \geq 1\}$ is a homogeneous Poisson point process on $\mathbb{R}^d$. Let

$$\Phi(K) = \eta(K + B),$$

where B is the unit Euclidean ball. This is a stationary isotropic ID valuation with independent increments. The corresponding measure ν is supported by the family $\{B + x : x \in \mathbb{R}^d\}$. Below we construct a different valuation which shares with Φ all one-dimensional distributions.

For $m = 0, \dots, d$, let ρ_m be the measure on $A(d, d-m)$ normalised to satisfy (8.6). Let $b_m = \kappa_{d-m}\delta_1$, and consider the Poisson process $\{E_{m,i}, i \geq 1\}$ on $A(d, d-m)$ of intensity $\kappa_{d-m}\rho_m$. Define

$$\Psi_m(K) = \sum_i \mathbf{1}_{E_{m,i} \cap K \neq \varnothing}, \quad K \in \mathcal{K}^d,$$

which is clearly a σ-continuous stationary ID valuation with independent increments which is m-stable in argument for $m = 1, \dots, d$; and, for $m = 0$, $\Psi_0(K)$ is a Poisson random variable, which does not depend on $K \neq \varnothing$. Therefore, $\Psi_m(K)$ is a Poisson random variable of mean $\kappa_{d-m}V_m(K)$, independent for different m, and so $\Psi(K) = \Psi_0(K) + \cdots + \Psi_d(K)$ is Poisson with mean

$$\mathbf{E}\Psi(K) = \sum_{m=0}^{d} \kappa_{d-m}V_m(K) = V_d(K + B).$$

Hence, random variables $\Psi(K)$ and $\Phi(K)$ share the same distribution for all $K \in \mathcal{K}^d$. The finite-dimensional distributions of these two random valuations are clearly different, e.g., $\Phi(\{x\})$ and $\Phi(\{y\})$ are independent if the singletons x and y are at distance at least 2, while $\Psi(\{x\})$ and $\Psi(\{y\})$ are equal. Indeed, both values equal the random multiplicity of the unique element $\mathbb{R}^d$ of $A(d, d)$, which is precisely the valuation Ψ_0.

Example 8.5 (Poisson perimeter)**.** For $d = 2$, let $\nu = \mu_1 \otimes \delta_1$ be the product of the motion-invariant measure μ_1 on the affine Grassmannian $A(2, 1)$ normalised so that $\mu_1(\{L \in A(2, 1) : L \cap B \neq \varnothing\}) = 2\pi$ and the Dirac measure δ_1. Then the corresponding ID valuation $\Phi(K)$ with vanishing deterministic part equals the number of lines from the Poisson line process which hit K. The expectation of Φ is the perimeter of $K \in \mathcal{K}^2$.

A Valuations with symmetric-independent increments

We now characterise ID valuations satisfying a weaker independence condition. We say that a random valuation Φ has *symmetric-independent increments* if the random variables $\Phi(K) - \Phi(K \cap L)$ and $\Phi(L) - \Phi(K \cap L)$ are independent for all $K, L \in \mathcal{K}^d$ such that $K \cup L$ is convex.

Lemma A.1. *If Φ is a random valuation with independent increments, then Φ has symmetric-independent increments.*

Proof. By assumption, $\Phi(K \cup L) - \Phi(L)$ and $\Phi(L) - \Phi(K \cap L)$ are independent. It remains to notice that $\Phi(K \cup L) - \Phi(L)$ is a.s. equal to $\Phi(K) - \Phi(K \cap L)$ by the additivity property of Φ. □

A function $g : \mathbb{R}^d \to \mathbb{R}_+$ is said to be *quasi-concave* if $\{x : g(x) \geq t\}$ is convex for all $t \in \mathbb{R}_+$. For any $K \in \mathcal{K}^d$ denote

$$g^\vee(K) = \sup\{g(x) : x \in K\},$$

and let $g^\vee(\varnothing) = 0$.

The function $g^\vee$ is a valuation. Indeed, let $K, L, K \cup L \in \mathcal{K}^d$ and assume that $g^\vee(K) \geq g^\vee(L)$. Then $g^\vee(K \cup L) = g^\vee(K)$, while quasi-concavity of g and convexity of $K \cup L$ yield $g^\vee(K \cap L) = g^\vee(L)$. Hence $g^\vee(K \cup L) + g^\vee(K \cap L) = g^\vee(K) + g^\vee(L)$.

Theorem A.2. *A monotone σ-continuous ID valuation Φ has symmetric-independent increments if and only if the outer measure of its Lévy measure Λ is supported by valuations ψ each of which is of the form*

$$\psi(K) = g^\vee(K)$$

for an upper semicontinuous quasi-concave function $g : \mathbb{R}^d \to \mathbb{R}_+$.

Proof. By Lemma 4.3, Λ is supported by ψ such that

$$\psi(K) = \psi(K \cap L) \quad \text{or} \quad \psi(L) = \psi(K \cap L) \tag{A.1}$$

for all $K, L \in \mathcal{K}^d$ with $K \cup L \in \mathcal{K}^d$. Furthermore, all ψ from the support of Λ are monotone and σ-continuous. Taking into account monotonicity of ψ, we have that

$$\psi(K \cup L) = \psi(K) + \psi(L) - \psi(K \cap L) = \max(\psi(K), \psi(L)).$$

Define $g(x) = \psi(\{x\})$, $x \in \mathbb{R}^d$. Consider any hyperplane H which intersects the relative interior of K and so splits K into two convex sets K_1 and K_2 whose union is K. Assume that $\psi(K_1) = \psi(K)$. Splitting K_1 further, we obtain a sequence K_n, $n \geq 1$, which shrinks to a point x and such that $\psi(K_n) = \psi(K)$ for all n. By σ-continuity, $g(x) = \psi(\{x\}) = \psi(K)$ for some $x \in K$. Since $\psi(K) \geq \psi(\{y\}) = g(y)$ for all $y \in K$ by monotonicity, we obtain that $g(x) = g^\vee(K)$.

If $x_n \to x$, pass to a subsequence along which $g(x_n) \to \limsup_{n\to\infty} g(x_n)$. Let K_n be the closed convex hull of $\{x\}$ and $\{x_j, j \geq n\}$. Then $\psi(K_n) = g^\vee(K_n) \geq \sup_{j\geq n} g(x_j)$. By σ-continuity,

$$g(x) = \psi(\{x\}) \geq \limsup_{n\to\infty} g(x_n),$$

so g is upper semicontinuous.

Consider the segment $[x, y]$ and a point z in its relative interior. Then

$$g^\vee([x, y]) + g(z) = g^\vee([x, z]) + g^\vee([z, y]).$$

Without loss of generality assume that $g^\vee([x, y]) = g^\vee([z, y])$. Then $g(z) = g^\vee([x, z]) \geq g(x)$. Thus, $g(z) \geq \min(g(x), g(y))$, and so g is quasi-concave.

In the other direction, if g is quasi-concave, then (A.1) holds, so that Φ has symmetric-independent increments by Lemma 4.3. Indeed, assume that $g^\vee(K\cap L) = t$. Then $g^\vee(K) \geq t$ and $g^\vee(L) \geq t$. If both inequalities are strict, choose s such that $t < s < \min\{g^\vee(K), g^\vee(L)\}$. Then $g(x) > s$ for some $x \in K$ and $g(y) > s$ for some $y \in L$. By quasi-concavity $g(z) > s$ for all z from the segment $[x, y]$. However, this segment intersects $K \cap L$, and we arrive at a contradiction. □

B Moments of random valuations

Let Φ be an integrable random valuation, that is, $\Phi(K)$ is integrable for all $K \in \mathcal{K}^d$. Then its expectation is a deterministic valuation $\varphi = \mathbf{E}\Phi$, which is translation invariant if Φ is stationary and is monotone if Φ is monotone. The dominated convergence theorem implies that φ is σ-continuous if Φ is monotone and σ-continuous. Assume that Φ is square-integrable and define its covariance function by

$$C(K, L) = \mathbf{E}\big[\Phi(K)\Phi(L)\big] - \mathbf{E}\Phi(K)\mathbf{E}\Phi(L), \quad K, L \in \mathcal{K}^d.$$

Since

$$0 = \mathbf{E}\big[\Phi(M)\big(\Phi(K \cup L) + \Phi(K \cap L) - \Phi(K) - \Phi(L)\big)\big],$$

for $K, L, M \in \mathcal{K}^d$ with $K \cup L \in \mathcal{K}^d$, we have that

$$C(M, K \cup L) + C(M, K \cap L) - C(M, K) - C(M, L) = 0. \tag{B.1}$$

Thus, the covariance function is additive in each of its arguments and so is a *bivaluation*. Denote

$$\sigma^2(K) = C(K, K) = \mathbf{E}\Phi(K)^2 - (\mathbf{E}\Phi(K))^2, \quad K \in \mathcal{K}^d.$$

Proposition B.1. *Let $\Phi : \mathcal{K}^d \to \mathbb{R}$ be a square-integrable stochastic process such that $K \mapsto \mathbf{E}\Phi(K)$ is a deterministic valuation. Then Φ is a random valuation with uncorrelated increments if and only if*

$$C(K, L) = \sigma^2(K \cap L), \quad K, L \in \mathcal{K}^d,\ K \cup L \in \mathcal{K}^d, \tag{B.2}$$

and $\sigma^2(\cdot)$ is a monotone valuation.

Proof. Without loss of generality assume that Φ is centred, that is, $\mathbf{E}\Phi(K) = 0$ for all K.

Necessity. If $K, L \in \mathcal{K}^d$ and $K \cup L \in \mathcal{K}^d$, then Lemma A.1 (adjusted for uncorrelated increments) implies that $\Phi(K) - \Phi(K \cap L)$ and $\Phi(L) - \Phi(K \cap L)$ are uncorrelated. Noticing that $\Phi(K) - \Phi(K \cap L)$ and $\Phi(K \cap L)$ are also uncorrelated, we have

$$\begin{aligned} C(K, L) &= \mathbf{E}\big[\Phi(K)\Phi(L)\big] \\ &= \mathbf{E}\big[(\Phi(K) - \Phi(K \cap L) + \Phi(K \cap L))(\Phi(L) - \Phi(K \cap L) + \Phi(K \cap L))\big] \\ &= \sigma^2(K \cap L), \quad K, L, K \cup L \in \mathcal{K}^d. \end{aligned}$$

Then (B.1) with $M = K \cup L$ becomes

$$\sigma^2(K \cup L) + \sigma^2(K \cap L) - \sigma^2(K) - \sigma^2(L) = 0,$$

meaning that $\sigma^2 : \mathcal{K}^d \to \mathbb{R}_+$ is a non-negative valuation. For $L \subset K$,

$$\sigma^2(K) - \sigma^2(L) = \mathbf{E}[(\Phi(K) - \Phi(L) + \Phi(L))^2] - \mathbf{E}\Phi(L)^2 = \mathbf{E}(\Phi(K) - \Phi(L))^2 \geq 0,$$

so that σ^2 is monotone.

Sufficiency. Let K, L be an admissible pair. Then

$$\begin{aligned} &\mathrm{Var}(\Phi(K \cup L) + \Phi(K \cap L) - \Phi(K) - \Phi(L)) \\ &\quad = \sigma^2(K \cup L) + \sigma^2(K \cap L) + \sigma^2(K) + \sigma^2(L) + 2\sigma^2(K \cap L) - 2\sigma^2(K) - 2\sigma^2(L) - 2\sigma^2(K \cap L) \\ &\quad = \sigma^2(K \cup L) + \sigma^2(K \cap L) - \sigma^2(K) - \sigma^2(L) = 0. \end{aligned}$$

For every admissible pait K, L, the valuation identity holds almost surely. Let $N \subset M \subset L \subset K$ be convex compact sets. Then

$$\mathbf{E}[(\Phi(K) - \Phi(L))(\Phi(M) - \Phi(N))] = \sigma^2(M) - \sigma^2(N) - \sigma^2(M) + \sigma^2(N) = 0. \qquad \square$$

Corollary B.2. *If two square-integrable valuations with independent increments share the same one-dimensional distributions, then their covariance functions coincide for all admissible pairs $K, L \in \mathcal{K}^d$.*

The values of $C(K, L)$ on convex bodies K and L whose union is not convex are in general not functionals of $K \cap L$. For instance, assume that $C(K, L) = \sigma^2(K \cap L)$ for all $K, L \in \mathcal{K}^d$. Then, for disjoint K and L, we would have $C(K, L) = 0$.

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