MONOIDAL SEEDS OF THE CATEGORIES $\mathscr{C}_{w,v}$ OVER QUIVER HECKE ALGEBRAS

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Abstract. In this paper, when the quiver Hecke algebra R is symmetric, we present a new construction of quantum monoidal seeds $\mathscr{S}_{\underline{w},v}$ for $\mathscr{C}_{w,v}$ using the reflection functors $\mathscr{F}_i$ and the newly introduced operators $\mathscr{K}_i$. The monoidal seed $\mathscr{S}_{\underline{w},v}$ of $\mathscr{C}_{w,v}$ is obtained as a subseed of the monoidal seed $\mathscr{T}_{\underline{w},\mathsf{c}}$ of $\mathscr{C}_w$ constructed by applying $\mathscr{K}_i$ and $\mathscr{F}_i$ along the special KF sequence c determined by $\underline{w}$ and v. We further prove that the monoidal seed $\mathscr{S}_{\underline{w},v}$ coincides with the set of all prime factors of the determinantial modules $M(w_{\leqslant k}\Lambda_{i_k}, v_{\leqslant k}\Lambda_{i_k})$ for $k \in [1,r]^{w}_{v}$. Let $\mathscr{A}_{\underline{w},v}$ (resp. $\mathscr{U}_{\underline{w},v}$) denote the (resp. upper) quantum cluster algebra with non-invertible frozen variables associated with the initial seed $\Sigma_{\underline{w},v}$ determined by $\mathscr{S}_{\underline{w},v}$. We prove that the Grothendieck ring $K(\mathscr{C}_{w,v})$ of $\mathscr{C}_{w,v}$ lies between $\mathscr{A}_{\underline{w},v}$ and $\mathscr{U}_{\underline{w},v}$, and conjecture that $\mathscr{A}_{\underline{w},v} = \mathscr{U}_{\underline{w},v}$, which implies $\mathscr{C}_{w,v}$ provides the monoidal categorification of the cluster algebra $K(\mathscr{C}_{w,v})$ without the localization process.

Contents

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1. Introduction

Quiver Hecke algebras, introduced independently by Khovanov–Lauda and Rouquier ([39, 40, 48]), have occupied a central position in the study of the categorification of quantum groups (see [4, 19, 23] for example and references therein). Let C be a symmetrizable generalized Cartan matrix and let $U_q(\mathfrak{g})$ and R be the associated quantum group and quiver Hecke algebra respectively. It was shown in [39, 40, 48] that the *quantum unipotent coordinate ring* $A_q(\mathfrak{n})$, the dual of the positive half $U_q^+(\mathfrak{g})$, is categorified by the category R-gmod of finite-dimensional graded R-modules, which means that the Grothendieck ring $K(R\text{-gmod})$ is isomorphic to the integral form of $A_q(\mathfrak{n})$. Let W be the Weyl group associated with C. For any $w, v \in \mathsf{W}$ with $v \leqslant w$ in the *Bruhat order*, the subcategory $\mathscr{C}_{w,v}$ of R-gmod was introduced by the authors in [26] (see (2.2)). The category $\mathscr{C}_{w,v}$ is defined as a subcategory of the category $\mathscr{C}_w$

which provides a categorification of the quantum unipotent coordinate ring $A_q(\mathfrak{n}(w))$, regarded as a quantum deformation of the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent subgroup $N(w)$ associated with w. When $v = \mathsf{id}$, $\mathscr{C}_{w,v}$ coincides with the category $\mathscr{C}_w$. The category $\mathscr{C}_{w,v}$ categorifies the q-deformation $A_{w,v} \subset A_q(\mathfrak{n})$ of the *doubly-invariant algebra* ${}^{N'(w)}\mathbb{C}[N]^{N(v)}$, where N is the unipotent subgroup (see [26, Remark 2.19, Theorem 2.20] for details) and its localized category $\widetilde{\mathscr{C}}_{w,v}$ by central *determinantial modules* categorifies the coordinate ring $\mathbb{C}[\mathcal{R}_{w,v}]$ of the *open Richardson variety* $\mathcal{R}_{w,v}$ at the quantum level ([29]). The open Richardson varieties have deep connections to various areas of mathematics, including Kazhdan–Lusztig polynomials, total positivity and cluster algebras (see [37, 38, 44, 47, 42] and references therein).

Cluster algebras, introduced by Fomin and Zelevinsky [10], are commutative algebras equipped with distinguished generators grouped into *clusters* together with procedures called *mutations* to connect one cluster to neighbor clusters. A cluster algebra arises naturally from the open Richardson variety $\mathcal{R}_{w,v}$. It was shown by Leclerc ([42]) that, when C is of finite ADE type, the coordinate ring $\mathbb{C}[\mathcal{R}_{w,v}]$ contains a cluster algebra arising from the module category of the preprojective algebra determined by w and v. The initial seed $\Sigma^{\mathrm{Lec}}_{\underline{w},v}$ of this cluster algebra was constructed for each reduced expression $\underline{w}$ of w, and the cluster variables of $\Sigma^{\mathrm{Lec}}_{\underline{w},v}$ are obtained by taking irreducible factors of the generalized unipotent minors determined by $\underline{w}$ and v (see [42, Section 4.8.4]). Ménard later presented an algorithm to compute initial seeds for $\mathbb{C}[\mathcal{R}_{w,v}]$: starting from the seed of $\mathbb{C}[R_{w,\mathsf{id}}]$ associated with $\underline{w}$, one applies a suitable mutation sequence and then freezes and deletes certain cluster variables ([45]). This construction is generalized by Bao and Ye for a symmetrizable Kac-Moody $\mathfrak{g}$ in [1].

Leclerc conjectured that the cluster algebra $\mathscr{A}(\Sigma^{\mathrm{Lec}}_{\underline{w},v})$ with invertible frozen variables coincides with $\mathbb{C}[\mathcal{R}_{w,v}]$, and showed the conjecture in some special cases in [42]. In type A, Leclerc's conjecture was proved by Serhiyenko and Sherman-Bennett ([50]) by comparing with Ingermanson's result ([16]). In an arbitrary finite type, the conjecture was proved in the setting of *braid varieties* which are generalizations of open Richardson varieties (see [7, 11] and see also [8]). For the upper cluster algebra structure, it was proved that $\mathbb{C}[\mathcal{R}_{w,v}]$ coincides with the upper cluster algebra $\mathscr{U}(\Sigma^{\mathrm{Lec}}_{\underline{w},v})$ in [6] for ADE case, [7, 11] for finite type, and [1] for Kac-Moody case.

Monoidal categorification provides a powerful framework to study cluster algebras from the viewpoint of monoidal categories. The notion of monoidal categorification for cluster algebras was introduced by Hernandez-Leclerc in studying the Grothendieck ring of the category of finite-dimensional modules over quantum affine algebras ([15]). When C is of symmetric type, the category $\mathscr{C}_w$ provides a monoidal categorification of the quantum unipotent coordinate ring $A_q(\mathfrak{n}(w))$ ([19]). That is, every cluster monomial corresponds to a simple module in $\mathscr{C}_w$. Let $\underline{w}$ be a reduced expression of w of length $r = \ell(w)$. For each *admissible chain* $\mathfrak{C} = ([a_k, b_k])_{1 \leqslant k \leqslant r}$ of i-boxes associated with $\underline{w}$, one can construct a quantum monoidal seed $\mathscr{T}_{\mathfrak{C}}$ of $\mathscr{C}_w$ whose *cluster variable modules*, simple modules corresponding to cluster variables under the categorification, are given by the *determinantial modules* $\mathrm{M}_{\underline{w}}[a_k, b_k]$ for $k \in [1, r]$ (see [19, 25] and see also Section 4.1 for details). The determinantial modules $\mathrm{M}_{\underline{w}}[a_k, b_k]$ categorify the quantum unipotent minors in $A_q(\mathfrak{n}(w))$. In the case of $\mathfrak{C} = (\{1, k]\})_{1 \leqslant k \leqslant r}$, the monoidal seed $\mathscr{T}_{\mathfrak{C}}$ categorifies the seed of $A_q(\mathfrak{n}(w))$ given by Geiss-Leclerc-Schröer in [12, 13] under the categorification (see (4.1)). Since the algebra $\mathbb{C}[\mathcal{R}_{w,\mathsf{id}}]$ is isomorphic to the localized algebra $\widetilde{A_q(\mathfrak{n}(w))}|_{q=1}$ and the localized category $\widetilde{\mathscr{C}}_{w,v}$ categorifies the coordinate ring $\mathbb{C}[\mathcal{R}_{w,v}]$ of the open Richardson variety $\mathcal{R}_{w,v}$, the approach by Ménard and Bao-Ye to computing seeds for $\mathbb{C}[\mathcal{R}_{w,v}]$ provides monoidal seeds in $\mathscr{C}_{w,v}$ from those of $\mathscr{C}_w$. In [3], Bi showed that the monoidal seed obtained in this way is an initial seed of a cluster algebra contained in $K(\mathscr{C}_{w,v})$, and that every cluster monomial corresponds to a simple module in $\mathscr{C}_{w,v}$, based on the result of [19]. In particular, when $\mathfrak{g}$ is of finite ADE type, he showed that the seed of Leclerc is same as the seeds of Ménard and proved that the localized category $\widetilde{\mathscr{C}}_{w,v}$ provides a monoidal categorification of the cluster algebra $\mathbb{C}[\mathcal{R}_{w,v}]$. Although Ménard's algorithm and Bao-Ye's generalization show the existence of a monoidal seed of $\mathscr{C}_{w,v}$ as a *subseed* of a monoidal seed of $\mathscr{C}_w$, the algorithm does not provide a direct construction of cluster variable modules in a monoidal seed of $\mathscr{C}_{w,v}$ since it proceeds by applying a certain sequence of mutations.

In this paper, when the quiver Hecke algebra R is *symmetric*, we present a new construction of quantum monoidal seeds for $\mathscr{C}_{w,v}$ using the *reflection functors* $\mathscr{F}_i$ ([34, 36] and Section 2.5) and the newly introduced operators $\mathscr{K}_i$ (Section 2.8). For each reduced expression $\underline{w}$ of w, our new construction (Theorem 6.8) provides a quantum monoidal seed $\mathscr{S}_{\underline{w},v}$ of $\mathscr{C}_{w,v}$ whose cluster variable modules can be constructed by the crystal operators, since the operators $\mathscr{K}_i$ and $\mathscr{F}_i$ admit explicit descriptions in terms of crystals (Definition 2.18 and (2.3)). We further prove that the seed $\mathscr{S}_{\underline{w},v}$ coincides with the set of all prime factors of the determinantial modules $M(w_{\leqslant k}\Lambda_{i_k}, v_{\leqslant k}\Lambda_{i_k})$ for $k \in [1, r]^{\underline{w}}_{v}$ and that, when $\mathfrak{g}$ is of finite type ADE, $\mathscr{S}_{\underline{w},v}$ corresponds to the seed of Leclerc in [42, Corollary 4.4] under the categorification (Proposition 6.10). In contrast to the approaches of Ménard and Leclerc, our construction determines directly the cluster variable modules in terms of $\mathscr{K}_i$ and $\mathscr{F}_i$ without performing mutations or relying on the factorization of generalized unipotent minors into irreducible factors.

The operator $\mathscr{K}_i$ (resp. $\mathscr{K}_i^*$) is defined in the *symmetrizable setting* using the crystal operators $\widetilde{\mathrm{F}}_i$, $\widetilde{\mathrm{E}}_i$ (resp. $\widetilde{\mathrm{F}}_i^*$, $\widetilde{\mathrm{E}}_i^*$) together with the $\mathfrak{d}_i$ defined in (2.1) (see Definition

2.18). For any simple module M, the $\mathscr{K}_i(M)$ (resp. $\mathscr{K}_i^*(M)$) can be understood as the simple module on the $(\widetilde{\mathrm{F}}_i, \widetilde{\mathrm{E}}_i)$-generated (resp. $(\widetilde{\mathrm{F}}_i^*, \widetilde{\mathrm{E}}_i^*)$-generated) string through M which commutes with $\langle i\rangle$ *minimally* (see (2.13) and Lemma 2.19 (ii)), where $\langle i\rangle$ is the 1-dimensional simple $R(\alpha_i)$-module. We show that $\mathscr{K}_i$ and $\mathscr{K}_i^*$ preserve both of the *affreal* property (Lemma 2.19) and the commuting property (Lemma 2.20). In the case of simple modules satisfying $\varepsilon_i = 0$ (resp. $\varepsilon_i^* = 0$), we further investigate how the integer-valued invariants Λ change under the operators $\mathscr{K}_i$ (resp. $\mathscr{K}_i^*$), and prove that $\mathscr{K}_i$ (resp. $\mathscr{K}_i^*$) preserves both the integer-valued invariants $\mathfrak{d}$ and the primeness (see Lemma 2.22 and Lemma 2.24).

The operators $\mathscr{K}_i$ together with the reflection functor $\mathscr{F}_i$ lead to a new construction of monoidal seeds. *We assume that R is symmetric.* For any reduced expression $\underline{w} = (i_1, i_2, \ldots, i_r)$ of an element $w \in \mathsf{W}$ and any sequence $\mathsf{c} = (c_1, c_2, \ldots, c_r) \in \{\mathsf{k}, \mathsf{f}\}^r$ (that we call a *KF sequence*), we define the family of simple modules

$$\mathscr{T}_{\underline{w},\mathsf{c}} = (M_k)_{k\in[1,r]}$$

by applying $\mathscr{K}_i$ and $\mathscr{F}_i$ to one of $\langle i_k\rangle$ and the determinantial module $\mathrm{M}(w_{\geqslant k}\Lambda_{i_k}, \Lambda_{i_k})$ along the sequence c (see (5.5) for the definition). Proposition 5.7 says that the family $\mathscr{T}_{\underline{w},\mathsf{c}}$ is a monoidal seed of $\mathscr{C}_w$ in the *canonical mutation class* (see Remark 4.3). We remark that, in general, the seed $\mathscr{T}_{\underline{w},\mathsf{c}}$ differs from the monoidal seeds arising from *admissible chains* of *i-boxes* associated with $\underline{w}$ (see Example 5.8).

Let $w, v \in \mathsf{W}$ with $v \leqslant w$ and take $i \in I$ such that $s_i w < w$. One of the key observations (Proposition 3.1) is that,

(a) we have $\langle i\rangle \in \mathscr{C}_{w,v}$ and $\mathscr{K}_i$ sends simple modules in $\mathscr{C}_{s_i w,v}$ to simple modules in $\mathscr{C}_{w,v}$ whenever $v < s_i v$,

(b) $\mathscr{F}_i$ sends simple modules in $\mathscr{C}_{s_i w, s_i v}$ to simple modules in $\mathscr{C}_{w,v}$ whenever $s_i v < v$.

This leads us to a new construction of monoidal seeds for $\mathscr{C}_{w,v}$. For any reduced expression $\underline{w}$ of w, we define the subset $[1,r]^{\underline{w}}_v$ of the interval $[1,r]$ by using $\underline{w}$ and v (see (6.7)) and consider the monoidal seed $\mathscr{T}_{\underline{w},\mathsf{c}} = (M_k)_{k\in[1,r]}$ of $\mathscr{C}_w$ associated with $\underline{w}$ and the *special KF sequence* c determined by $[1,r]^{\underline{w}}_v$ (see (6.8)). We then define

$$\mathscr{S}_{\underline{w},v} = (M_k)_{k\in[1,r]^{\underline{w}}_v}$$

as a subset of $\mathscr{T}_{\underline{w},\mathsf{c}}$. In Theorem 6.8, we prove that $\mathscr{S}_{\underline{w},v}$ is a completely admissible monoidal seed (see Definition 4.1) in $\mathscr{C}_{w,v}$ by investigating the relationship between the two seeds in $\mathscr{C}_{s_i w}$ and $\mathscr{C}_{s_i w,v}$ under the operator $\mathscr{K}_i$ (see Section 6.1). Theorem 7.6 says that the mutation class of the seed $\mathscr{S}_{\underline{w},v}$ does not depend on the choice of reduced expressions $\underline{w}$. The set $\mathsf{J}^{\mathrm{fr}}_{\underline{w},v}$ of the frozen vertices of $\mathscr{S}_{\underline{w},v}$ is described explicitly in (6.10) and Lemma 6.6. Note that the number $|\mathsf{J}^{\mathrm{fr}}_{\underline{w},v}|$ is related to the coefficient of the second largest power in *Kazhdan-Lusztig R-polynomials* (see [9, Section 3.4] and Remark 6.7). It turns out that the seed $\mathscr{S}_{\underline{w},v}$ is equal to the seed of Leclerc (see Proposition 6.10). Hence [3, Theorem 5.11] implies that $\mathscr{S}_{\underline{w},v}$ is the same with the seed of Ménard and Bao-Ye's generalization when $\mathfrak{g}$ is symmetric. We illustrate the monoidal seeds $\mathscr{T}_{\underline{w},\mathsf{c}}$ and $\mathscr{S}_{\underline{w},v}$ with explicit examples (see Example 5.8 and 6.9) in the paper.

We also investigate the compatibility of $\mathscr{K}_i$ and $\mathscr{F}_i$ with *Laurent families* introduced in [31]. The Laurent family is a special commuting family of affreal simple modules which satisfies a categorical property analogous to the *Laurent phenomenon* (see Section 2.7). In Theorem 2.13, we give a formula for computing Λ in terms of g-vectors in the context of Laurent families. This can be viewed as a generalization of the formula expressed in terms of the *tropical invariant* by Cao ([5, Theorem 5.16]) from cluster variable modules to real simple modules (Remark 2.14). We show that $\mathscr{K}_i$ and $\mathscr{F}_i$ preserve Laurent families (Proposition 3.5), and prove that

$$\mathscr{A}_{\underline{w},v} \subset K(\mathscr{C}_{w,v}) \subset \mathscr{U}_{\underline{w},v},$$

where $\mathscr{A}_{\underline{w},v}$ and $\mathscr{U}_{\underline{w},v}$ are the quantum cluster algebra and the upper quantum cluster algebra with *non-invertible frozens* arising from the initial seed $\Sigma_{\underline{w},v}$ determined by $\mathscr{S}_{\underline{w},v}$, respectively (Theorem 8.2). We conjecture that $\mathscr{A}_{\underline{w},v} = \mathscr{U}_{\underline{w},v}$, which implies $\mathscr{C}_{w,v}$ provides the monoidal categorification of $K(\mathscr{C}_{w,v})$ *without the localization process* (see Conjecture 1).

This paper is organized as follows. In Section 2, we briefly review the necessary background including quantum groups, quiver Hecke algebras and Laurent families, and introduce the new operators $\mathscr{K}_i$ and $\mathscr{K}_i^*$. In Section 3, we investigate how $\mathscr{K}_i$ and $\mathscr{F}_i$ act on Laurent families in $\mathscr{C}_{w,v}$. Section 4 reviews the notion of monoidal categorification, and Section 5 explains how the monoidal seeds $\mathscr{T}_{\underline{w},\mathsf{c}}$ for $\mathscr{C}_w$ are constructed using $\mathscr{K}_i$ and $\mathscr{F}_i$. Section 6 introduces the new construction of monoidal seeds $\mathscr{S}_{\underline{w},v}$ of $\mathscr{C}_{w,v}$ in terms of $\mathscr{K}_i$ and $\mathscr{F}_i$, and Section 7 shows the mutation invariance of the monoidal seeds $\mathscr{S}_{\underline{w},v}$. Section 8 shows the cluster algebra structure on $\mathscr{C}_{w,v}$.

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Convention. Throughout this paper, we use the following convention.

(i) For a statement P, we set $\delta(P)$ to be 1 or 0 depending on whether P is true or not. In particular, $\delta_{i,j} = \delta(i = j)$ is the Kronecker delta.

(ii) For $a \in \mathbb{Z} \cup \{-\infty\}$ and $b \in \mathbb{Z} \cup \{\infty\}$ with $a \leqslant b$, we set

$$[a, b] = \{k \in \mathbb{Z} \mid a \leqslant k \leqslant b\}, \qquad [a, b) = \{k \in \mathbb{Z} \mid a \leqslant k < b\},$$
$$(a, b] = \{k \in \mathbb{Z} \mid a < k \leqslant b\}, \qquad (a, b) = \{k \in \mathbb{Z} \mid a < k < b\},$$

and call them *intervals*. When $a > b$, we understand them as empty sets.

(iii) For a monoid A and an index set K, we denote by $A^{\oplus K}$ the subset of A^K consisting of $\mathbf{a} = (a_k)_{k \in K}$ such that a_k coincides with the identity element except finitely many $k \in K$.

2. Preliminaries

2.1. **Quantum cluster algebras.** In this subsection, we briefly recall the definition of a quantum cluster algebra ([2]).

Let J be a finite index set with a decomposition $\mathsf{J} = \mathsf{J}^{\mathrm{ex}} \sqcup \mathsf{J}^{\mathrm{fr}}$. The elements in J^{ex} (resp. J^{fr}) are called *exchangeable* (resp. *frozen*) indices. Let q be an indeterminate. For a skew-symmetric integer-valued matrix $L = (l_{ij})_{i,j \in \mathsf{J}}$, a family $x = \{x_i\}_{i \in \mathsf{J}}$ in a skew field K over $\mathbb{Q}(q^{1/2})$ is *L-commuting* if it satisfies $x_i x_j = q^{l_{ij}} x_j x_i$ for $i, j \in \mathsf{J}$. We denote by $\mathcal{T}_x$ the $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of K generated by x, which is called the *quantum torus* associated with x. For any $\mathbf{a} = (a_i)_{i \in \mathsf{J}} \in \mathbb{Z}^{\mathsf{J}}$, we set

$$x(\mathbf{a}) := q^{\frac{1}{2} \sum_{i > j} a_i a_j l_{ij}} \overrightarrow{\prod_{i \in \mathsf{J}}} x_i^{a_i},$$

where the ordered product is taken in the decreasing order with respect to a total order on J. Note that $x(\mathbf{a})$ does not depend on the choice of a total order.

Let $\widetilde{B} = (b_{ij})_{i \in \mathsf{J}, j \in \mathsf{J}^{\mathrm{ex}}}$ be an *exchange matrix*, which is an integer-valued matrix whose principal part $(b_{ij})_{i,j \in \mathsf{J}^{\mathrm{ex}}}$ is skew-symmetric. We often write $\widetilde{B}_{ij}$ for the entry b_{ij}. We say that a pair $(L, \widetilde{B})$ is *compatible* if there is a positive integer d such that

$$\sum_{k \in \mathsf{J}} l_{ik} b_{kj} = d \delta_{i,j} \quad \text{for any } i \in \mathsf{J} \text{ and } j \in \mathsf{J}^{\mathrm{ex}}.$$

A triple $\Sigma = (\{x_k\}_{k \in \mathsf{J}}, L, \widetilde{B})$ is called a *quantum seed* in K if $\{x_k\}_{k \in \mathsf{J}}$ is an algebraically independent L-commuting family in K and $(L, \widetilde{B})$ is compatible.

For any $k \in \mathsf{J}^{\mathrm{ex}}$, the *mutation* $\mu_k(\Sigma)$ of a quantum seed $\Sigma = (\{x_k\}_{k \in \mathsf{J}}, L, \widetilde{B})$ is written as

$$\mu_k(\Sigma) := \big(\mu_k(x), \mu_k(L), \mu_k(\widetilde{B})\big),$$

where the mutation $\mu_k(\widetilde{B})$ (resp. $\mu_k(L)$) of $\widetilde{B}$ (resp. L) is described in [19, Section 5.2], and $\mu_k(x) = \{x'_i\}_{i\in \mathsf{J}}$ is defined as

$$x'_i := \begin{cases} x(\mathbf{a}') + x(\mathbf{a}'') & \text{if } i = k, \\ x_i & \text{if } i \neq k, \end{cases}$$

where $\mathbf{a}' = (a'_i)_{i\in\mathsf{J}}$ and $\mathbf{a}'' = (a''_i)_{i\in\mathsf{J}}$ are defined as

$$a'_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, b_{ik}) & \text{if } i \neq k, \end{cases} \quad \text{and} \quad a''_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, -b_{ik}) & \text{if } i \neq k. \end{cases}$$

We say two seeds Σ and Σ' are *mutation equivalent* if Σ' can be obtained from Σ by applying suitable mutations up to a permutation of the index set J. Note that mutation equivalence gives an equivalence relation on the set of all quantum seeds. The equivalence classes arising from this relation are called *mutation classes*.

Let $\Sigma = (\{x_k\}_{k\in\mathsf{J}}, L, \widetilde{B})$ be a quantum seed in a skew field K over $\mathbb{Q}[q^{\pm1/2}]$. The *quantum cluster algebra* $\mathscr{A}(\Sigma)$ associated with Σ is the $\mathbb{Z}[q^{\pm1/2}]$-subalgebra of K generated by all cluster variables in quantum seeds obtained from Σ by any finite sequence of mutations. For a quantum seed $\Sigma' = (\{x'_k\}_{k\in\mathsf{J}}, L', \widetilde{B}')$ mutation equivalent to Σ, the *quantum Laurent phenomenon* ([2]) says that $\mathscr{A}(\Sigma)$ is contained in the quantum torus $\mathcal{T}_{\Sigma'}$ generated by $\{x'_k\}_{k\in\mathsf{J}}$ inside K. We define $\mathscr{U}(\Sigma)$ to be the intersection of all quantum tori $\mathcal{T}_{\Sigma'}$ in K for all quantum seeds Σ' which are mutation equivalent to Σ. The algebra $\mathscr{U}(\Sigma)$ is called the *quantum upper cluster algebra* associated with Σ. Note that, by the definition, $\mathscr{A}(\Sigma)$ is a $\mathbb{Z}[q^{\pm1/2}]$-subalgebra of $\mathscr{U}(\Sigma)$.

2.2. **Quantum groups.** In this subsection, we briefly review quantum groups ([22, 43]).

A *Cartan datum* is a family $(\mathsf{C}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee, (\cdot,\cdot))$ consisting of a symmetrizable generalized *Cartan matrix* $\mathsf{C} = (c_{ij})_{i,j\in I}$, a free abelian group P called the *weight lattice*, its dual $\mathsf{P}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathsf{P}, \mathbb{Z})$ called the *co-weight lattice*, the set $\Pi := \{\alpha_i \in \mathsf{P} \mid i \in I\}$ of *simple roots*, the set $\Pi^\vee := \{h_i \in \mathsf{P}^\vee \mid i \in I\}$ of *simple coroots* and a $\mathbb{Q}$-valued symmetric bilinear form $(\cdot,\cdot)$ on P such that

(i) $\langle h_i, \alpha_j\rangle = a_{ij}$ for $i, j \in I$,
(ii) Π is linearly independent over $\mathbb{Q}$,
(iii) for each $i \in I$, there exists $\Lambda_i \in \mathsf{P}$ such that $\langle h_j, \Lambda_i\rangle = \delta_{j,i}$ for all $j \in I$.
(iv) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ and $\langle h_i, \lambda\rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$.

We call Λ_i the i-th *fundamental weight* for $i \in I$, and write $\mathsf{P}_+ := \{\Lambda \in \mathsf{P} \mid \langle h_i, \Lambda\rangle \geqslant 0 \text{ for any } i \in I\}$ and call it the *dominant weight lattice*. Let $\mathsf{Q} := \sum_{i\in I} \mathbb{Z}\alpha_i$ be the *root lattice* and let $\mathsf{Q}_+ := \sum_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha_i$ and $\mathsf{Q}_- := \sum_{i\in I} \mathbb{Z}_{\leqslant 0}\alpha_i$ be the positive and negative root lattice respectively. We set $\mathrm{ht}(\beta) := \sum_{i\in I} |a_i|$ for an element $\beta = \sum_{i\in I} a_i\alpha_i \in \mathsf{Q}$. We denote by Δ_+ (resp. Δ_-) the set of all positive roots (resp. negative roots).

We denote by $\mathsf{W} = \langle s_i \mid i \in I\rangle$ the *Weyl group* associated with C, where $s_i \in \mathrm{Aut}(\mathsf{P})$ is the i-th reflection defined by

$$s_i\lambda = \lambda - \langle h_i, \lambda\rangle\alpha_i \quad \text{for } \lambda \in \mathsf{P}.$$

For $w \in \mathsf{W}$, an expression $\underline{w} = s_{i_1}s_{i_2}\cdots s_{i_r}$ of w is *reduced* if r is minimal among all such expressions. For simplicity, we write $\underline{w} = (i_1, i_2, \ldots, i_r)$ if no confusion arises. For $w, v \in \mathsf{W}$, we write $v \leqslant w$ if there is a reduced expression of w containing a reduced expression of v as a subexpression. This partial order $\leqslant$ is called the *Bruhat order* on W.

Let q be an indeterminate. The *quantum group* $U_q(\mathfrak{g})$ associated with a Cartan datum $(\mathsf{C}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee, (\cdot,\cdot))$ is the associative algebra over $\mathbb{Q}(q)$ generated by e_i, f_i $(i \in I)$ and q^h $(h \in \mathsf{P}^\vee)$ satisfying certain defining relations (see [43] for details), and let $U_q^-(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by f_i (resp. e_i) for all $i \in I$. We also denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and by $U^-(\mathfrak{g})$ (resp. $U^+(\mathfrak{g})$) the corresponding half of $U(\mathfrak{g})$ generated by f_i (resp. e_i) for all $i \in I$.

A *crystal* is a set B with maps $\mathrm{wt}\colon B \to \mathsf{P}$, $\varphi_i, \varepsilon_i\colon B \to \mathbb{Z} \sqcup \{-\infty\}$ and $\widetilde{e}_i, \widetilde{f}_i\colon B \to B \sqcup \{0\}$ for $i \in I$ which satisfy certain conditions determined by the Cartan matrix C. We set $B(\infty)$ to be the *infinite crystal* of $U_q^-(\mathfrak{g})$. The $\mathbb{Q}(q)$-antiautomorphism $*$ on $U_q(\mathfrak{g})$ defined by

$$(e_i)^* = e_i, \qquad (f_i)^* = f_i, \qquad (q^h)^* = q^{-h},$$

gives another crystal structure $\widetilde{e}_i^*, \widetilde{f}_i^*, \varepsilon_i^*, \varphi_i^*$ on $B(\infty)$. For details on crystals, we refer the reader to [21, 22].

For any $\Lambda \in \mathsf{P}_+$, we denote by $V_q(\Lambda)$ the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight Λ and by $B(\Lambda)$ its crystal. Let u_Λ be the highest weight vector of $V_q(\Lambda)$. For any $w \in \mathsf{W}$, $\dim(V_q(\Lambda)_{w\Lambda}) = 1$ and let $u_{w\Lambda}$ be a generator of $V_q(\Lambda)_{w\Lambda}$, normalized suitably (see [21, § 12]). The vector $u_{w\Lambda}$ is called the *extremal vector* of $V_q(\Lambda)$ of weight $w\Lambda$. We also denote by $V(\Lambda)$ the highest weight $U(\mathfrak{g})$-module with highest weight Λ.

Lemma 2.1. *Let $w, v \in \mathsf{W}$. Then the following conditions are equivalent:*

(a) $v \leqslant w$,
(b) $u_{w\Lambda} \in U^-(\mathfrak{g})u_{v\Lambda}$ *for any* $\Lambda \in \mathsf{P}_+$,
(c) $u_{v\Lambda} \in U^+(\mathfrak{g})u_{w\Lambda}$ *for any* $\Lambda \in \mathsf{P}_+$,

Proof. We denote by $(-,-)$ the Shapovalov bilinear form on the highest weight module $V(\Lambda)$, and by $\sigma : U(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g})$ the anti-involution sending $e_i \mapsto f_i$ and $f_i \mapsto e_i$ for $i \in I$. Since the Shapovalov form is non-degenerate and $(ux, y) = (x, \sigma(u)y)$ for any $x, y \in V(\Lambda)$ and $u \in U(\mathfrak{g})$, it is easy to see that (b) and (c) are equivalent.

Since it was shown in [20, Corollary 3.2.2] that (a) implies (c), we will focus on proving that (c) implies (a). Let us show it by induction on $\ell(v) + \ell(w)$. Since it is trivial if $\ell(w) = 0$, let us assume that $\ell(w) > 0$. Take $i \in I$ such that $s_i w < w$. Let $\mathfrak{p}_i$ be the Lie subalgebra of $\mathfrak{g}$ generated by the Borel $\mathfrak{b}$ and f_i. Then $f_i u_{w\Lambda} = 0$ and $U^+(\mathfrak{g})u_{w\Lambda} = U(\mathfrak{p}_i)u_{w\Lambda}$.

Assume first that $s_i v < v$. Since $U(\mathfrak{p}_i)$ is invariant by $s_i = \exp(f_i)\exp(-e_i)\exp(f_i)$, we have

$$\begin{aligned} s_i\big(U^+(\mathfrak{g})u_{w\Lambda}\big) &= s_i\big(U(\mathfrak{p}_i)u_{w\Lambda}\big) = U(\mathfrak{p}_i)u_{w\Lambda} \\ &= U^+(\mathfrak{g})\mathbb{C}[f_i]u_{w\Lambda} = U^+(\mathfrak{g})u_{w\Lambda} \ni u_{v\Lambda}, \end{aligned}$$

which says that $u_{s_i v\Lambda} \in U^+(\mathfrak{g})u_{w\Lambda}$. Then the induction hypothesis implies $s_i v \leqslant w$. As $s_i v < v$ and $s_i w < w$, we conclude that $v \leqslant w$.

We now assume that $s_i v > v$. Since $e_i u_{v\Lambda} = 0$ and

$$u_{v\Lambda} \in U^+(\mathfrak{g})u_{w\Lambda} = U(\mathfrak{p}_i)u_{s_i w\Lambda} = \mathbb{C}[f_i](U^+(\mathfrak{g})u_{s_i w\Lambda} \cap \operatorname{Ker} e_i),$$

we have $u_{v\Lambda} \in U^+(\mathfrak{g})u_{s_i w\Lambda} \cap \operatorname{Ker} e_i$. By the induction hypothesis, we have $v \leqslant s_i w$, which implies $v \leqslant w$. □

2.3. **Quiver Hecke algebras.** Let $\mathbf{k}$ be a field. A *quiver Hecke datum* associated with $(\mathsf{C}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee, (\cdot,\cdot))$ is a family $\{Q_{i,j}(u,v)\}_{i,j\in I} \subset \mathbf{k}[u,v]$ of the form

$$Q_{i,j}(u,v) = \begin{cases} \displaystyle\sum_{p(\alpha_i,\alpha_i)+q(\alpha_j,\alpha_j)=-2(\alpha_i,\alpha_j)} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

such that $t_{i,j;-a_{ij},0} \in \mathbf{k}^\times$ and $Q_{i,j}(u,v) = Q_{j,i}(v,u)$ for any $i,j \in I$. For $\beta \in \mathsf{Q}_+$ with $\operatorname{ht}(\beta) = n$, define

$$I^\beta := \Big\{ \nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \sum_{k=1}^n \alpha_{\nu_k} = \beta \Big\}.$$

The *quiver Hecke algebra* $R(\beta)$ associated with $\{Q_{i,j}(u,v)\}_{i,j\in I}$ is the $\mathbb{Z}$-graded algebra over $\mathbf{k}$ generated by

$$\{e(\nu) \mid \nu \in I^\beta\}, \quad \{x_k \mid 1 \leqslant k \leqslant n\}, \quad \{\tau_t \mid 1 \leqslant t \leqslant n-1\}$$

satisfying certain defining relations determined by $\{Q_{i,j}(u,v)\}_{i,j\in I}$ (see [39, 40, 48] and see also [19, 26] for details). The $\mathbb{Z}$-grading of $R(\beta)$ is defined as

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau_t e(\nu)) = -(\alpha_{\nu_t}, \alpha_{\nu_{t+1}}).$$

The quiver Hecke algebra R is *symmetric* if C is symmetric and $Q_{i,j}(u,v)$ is a polynomial in $u-v$ for all $i,j \in I$.

For an $R(\beta)$-module M, we set $\mathrm{wt}(M) := -\beta \in -\mathsf{Q}_+$.

We set $R(\beta)$-gmod to be the category of finite-dimensional graded $R(\beta)$-modules, and write R-gmod:$=\bigoplus_{\beta\in\mathsf{Q}_+} R(\beta)$-gmod. *Throughout the paper, an $R(\beta)$-module means a graded $R(\beta)$-module and a homomorphism means a graded homomorphism.*

For $\beta,\gamma \in \mathsf{Q}_+$, we define $e(\beta,\gamma) := \sum_{\mu\in I^\beta,\nu\in I^\gamma} e(\mu,\nu) \in R(\beta+\gamma)$, where $e(\mu,\nu)$ denotes the concatenation $e(\mu_1,\ldots,\mu_m,\nu_1,\ldots,\nu_n)$ for $\mu = (\mu_1,\ldots,\mu_m)$ and $\nu = (\nu_1,\ldots,\nu_n)$. By using the algebra embedding $R(\beta)\otimes R(\gamma) \subset e(\beta,\gamma)R(\beta+\gamma)e(\beta,\gamma)$, we define

$$M \circ N := R(\beta+\gamma)e(\beta,\gamma) \otimes_{R(\beta)\otimes R(\gamma)} (M \otimes N)$$

for an $R(\beta)$-module M and an $R(\gamma)$-module N, and call it the *convolution product* of M and N. The abelian category R-gmod is a monoidal category with the convolution product as its tensor product.

Let q denote the *grading shift functor* given by $(qM)_k = M_{k-1}$ for a graded module $M = \bigoplus_{k\in\mathbb{Z}} M_k$. For graded $R(\beta)$-modules M and N, we write $\mathrm{Hom}_R(M,N)$ for the space of degree preserving homomorphisms, and define

$$\mathrm{H{\scriptsize OM}}_R(M,N) := \bigoplus_{k\in\mathbb{Z}} \mathrm{H{\scriptsize OM}}_R(M,N)_k \quad \text{where } \mathrm{H{\scriptsize OM}}_R(M,N)_k := \mathrm{Hom}_R(q^k M, N).$$

For a homomorphism $f \in \mathrm{H{\scriptsize OM}}_R(M,N)_a$, we set $\deg(f) := a$, and simply write $f\colon M \to N$ if it is not necessary to specify the grading. We also write $M \simeq N$ to indicate that M and N are isomorphic up to a grading shift if no confusion arises.

For $n \in \mathbb{Z}_{>0}$ and a simple module M, we write $M^{\circ n} := M \circ M^{\circ n-1}$, where we understand $M^{\circ 0}$ is the trivial module $\mathbf{1}$. We say that M and N *commute* if $M \circ N$ is isomorphic to $N \circ M$, and say that M and N *strongly commute* if $M \circ N$ is simple. Note that, if M and N strongly commute, then they commute. A simple $R(\beta)$-module M is *real* if $M^{\circ 2}$ is simple.

For R-modules X and Y, we set

$$X \nabla Y := \mathrm{hd}(X \circ Y) \quad \text{and} \quad X \Delta Y := \mathrm{soc}(X \circ Y)$$

where $\mathrm{hd}(M)$ (resp. $\mathrm{soc}(M)$) denotes the head (resp. socle) of a module M.

For $M \in R$-gmod, the dual $R(\beta)$-module $M^* := \mathrm{Hom}_{\mathbf{k}}(M, \mathbf{k})$ is defined by $(r \cdot f)(u) = f(\psi(r)u)$ for $r \in R(\beta)$, $u \in M$, $f \in M^*$, where ψ is the $\mathbf{k}$-algebra anti-involution on $R(\beta)$ fixing the generators. A simple $R(\beta)$-module M is called *self-dual* if there exists an isomorphism between M^* and M preserving the grading. For any simple module M, there exists a unique $n \in \mathbb{Z}$ such that $q^n M$ is self-dual.

For $i \in I$ and $M \in R(\beta)$-gmod, we set

$$\begin{aligned} \mathrm{E}_i M &= e(i,*)M, & \varepsilon_i(M) &= \max\{n \geqslant 0 \mid \mathrm{E}_i^n M \neq 0\}, \\ \mathrm{E}_i^* M &= e(*,i)M, & \varepsilon_i^*(M) &= \max\{n \geqslant 0 \mid \mathrm{E}_i^{*n} M \neq 0\}, \end{aligned}$$

where $e(i,*) = e(\alpha_i, \beta - \alpha_i)$ and $e(*,i) = e(\beta - \alpha_i, \alpha_i)$. We regard E_i and E_i^* as exact functors from $R(\beta)$-gmod to $R(\beta - \alpha_i)$-gmod. We denote by $\langle i \rangle$ a unique simple $R(\alpha_i)$-module of degree 0. For any simple module $X \in R(\beta)$-gmod, we define

$$\begin{aligned} \widetilde{\mathrm{F}}_i(X) &:= \langle i \rangle \nabla X, & \widetilde{\mathrm{E}}_i(X) &:= \mathrm{soc}(\mathrm{E}_i X) \simeq \mathrm{hd}(\mathrm{E}_i X), \\ \widetilde{\mathrm{F}}_i^*(X) &:= X \nabla \langle i \rangle, & \widetilde{\mathrm{E}}_i^*(X) &:= \mathrm{soc}(\mathrm{E}_i^* X) \simeq \mathrm{hd}(\mathrm{E}_i^* X). \end{aligned}$$

Note that $\widetilde{\mathrm{F}}_i(X)$ and $\widetilde{\mathrm{F}}_i^*(X)$ (resp. $\widetilde{\mathrm{E}}_i(X)$ and $\widetilde{\mathrm{E}}_i^*(X)$) are simple modules in $R(\beta + \alpha_i)$-gmod (resp. $R(\beta - \alpha_i)$-gmod). We write

$$\widetilde{\mathrm{E}}_i^{\max}(X) := \widetilde{\mathrm{E}}_i^{\varepsilon_i(X)}(X) \quad \text{and} \quad \widetilde{\mathrm{E}}_i^{*\max}(X) := (\widetilde{\mathrm{E}}_i^*)^{\varepsilon_i^*(X)}(X).$$

We remark that the operators $\widetilde{\mathrm{F}}_i$, $\widetilde{\mathrm{E}}_i$, $\widetilde{\mathrm{F}}_i^*$ and $\widetilde{\mathrm{E}}_i^*$ correspond to the crystal operators $\widetilde{f}_i$, $\widetilde{e}_i$, $\widetilde{f}_i^*$ and $\widetilde{e}_i^*$ under the crystal-theoretic categorification ([41]).

If $(i_1, i_2, \ldots, i_r) \in I^r$ satisfies the conditions,

(a) $(\alpha_{i_k}, \alpha_{i_{k+1}}) < 0$ for any $k \in [1, r-1]$,
(b) $i_k \neq i_{k+2}$ for any $k \in [1, r-2]$ such that $\langle h_{i_k}, \alpha_{i_{k+1}} \rangle = -1$,

then we denote by $\langle i_1, i_2, \ldots, i_r \rangle$ the 1-dimensional R-module, where x_t, τ_l act as 0 on $\langle i_1, i_2, \ldots, i_r \rangle$ and $e(i_1, i_2, \ldots, i_r)\langle i_1, i_2, \ldots, i_r \rangle = \langle i_1, i_2, \ldots, i_r \rangle$.

When $(\alpha_i, \alpha_j) < 0$, any simple module in $R(\alpha_i + \alpha_j)$-gmod is isomorphic to either $\langle i, j \rangle$ or $\langle j, i \rangle$. For $n \in \mathbb{Z}_{\geqslant 0}$, we simply write $\langle i^n \rangle := q_i^{n(n-1)/2} \langle i \rangle^{\circ n}$. It is a unique self-dual simple $R(n\alpha_i)$ module.

A pair (M, N) is Λ-*definable* if $\mathrm{H{\scriptstyle OM}}(M \circ N, N \circ M) = \mathbf{k}\,\mathbf{r}$ for some non-zero homomorphism $\mathbf{r}$, which is called the *R-matrix*. In this case, define

$$\Lambda(M, N) := \deg(\mathbf{r})$$

and

$$\widetilde{\Lambda}(M, N) := \frac{1}{2}\big(\Lambda(M, N) + (\mathrm{wt}(M), \mathrm{wt}(N))\big) \in \mathbb{Z}.$$

If a simple module M is *affreal*, which means that M is real and has an *affinization*, then (M,N) and (N,M) are Λ-definable for any simple N ([35]). When R is symmetric, any simple has an affinization ([19, Section 2.2]). Note that, when either M or N is an affreal simple module, M and N commute if and only if M and N strongly commute ([18]). If both of (M,N) and (N,M) are Λ-definable, then we say that (M,N) is $\mathfrak{d}$-*definable* and define

$$\mathfrak{d}(M,N) := \frac{1}{2}(\Lambda(M,N)+\Lambda(N,M)).$$

For $i \in I$ and a simple module M, we define

$$\mathfrak{d}_i(M) := \varepsilon_i(M) + \varepsilon_i^*(M) + \langle h_i, \mathrm{wt}(M)\rangle. \tag{2.1}$$

Note that $\frac{(\alpha_i,\alpha_i)}{2}\mathfrak{d}_i(M) = \mathfrak{d}(\langle i\rangle, M)$ ([29, Lemma 2.15]).

A sequence $\underline{L} = (L_1,\ldots,L_r)$ of simple modules is *almost affreal* if all L_i $(1 \leqslant i \leqslant r)$ are affreal except for at most one. A sequence $\underline{L} = (L_1,\ldots,L_r)$ is called Λ-definable if (L_i,L_j) is Λ-definable for any i,j such that $1 \leqslant i < j \leqslant r$. An almost affreal sequence is Λ-definable. An almost affreal sequence $\underline{L} = (L_1,\ldots,L_r)$ is said to be a *normal sequence* if the composition

$$\begin{aligned}\mathbf{r}_{\underline{L}} := \prod_{1\leqslant i<k\leqslant r}\mathbf{r}_{L_i,L_k} =& (\mathbf{r}_{L_{r-1},L_r})\circ\cdots(\mathbf{r}_{L_2,L_r}\circ\cdots\circ\mathbf{r}_{L_2,L_3})\circ(\mathbf{r}_{L_1,L_r}\circ\cdots\circ\mathbf{r}_{L_1,L_2})\\ &: \ q^{\sum_{1\leqslant i<k\leqslant r}\Lambda(L_i,L_k)}L_1\circ\cdots\circ L_r \longrightarrow L_r\circ\cdots L_1\end{aligned}$$

does not vanish. Note that, if (L,M,N) is normal, then the image of $\mathbf{r}_{L,M,N}$ is simple and coincides with $\mathrm{hd}(L\circ M\circ N)$ up to grading shifts and

(a) $\Lambda(L, M\nabla N) = \Lambda(L,M)+\Lambda(L,N)$,
(b) $\Lambda(L\nabla M, N) = \Lambda(L,N)+\Lambda(M,N)$

(see [24, Section 2.3] for details).

We say that a simple S is a *convolution factor* of a simple M if there exists a simple X such that $M \simeq S\circ X$. A simple S is called *prime* if $S \simeq X\circ Y$ for $X,Y \in R$-gmod, then $X\simeq \mathbf{1}$ or $Y\simeq\mathbf{1}$.

Lemma 2.2. *Let $i\in I$ and let M be a simple module in R-gmod. Then the following conditions are equivalent:*

(a) *$\langle i\rangle$ is a convolution factor of M,*
(b) *$\varepsilon_i(M) > 0$ and $\mathfrak{d}_i(\widetilde{\mathrm{E}}_i M) = 0$.*
(c) *$\varepsilon_i^*(M) > 0$ and $\mathfrak{d}_i(\widetilde{\mathrm{E}}_i^* M) = 0$.*
(d) *$\varepsilon_i^*\big(\widetilde{\mathrm{E}}_i^{\max} M\big) < \varepsilon_i^*(M)$,*

(e) $\varepsilon_i\big(\widetilde{\mathrm{E}}_i^{*\,\max} M\big) < \varepsilon_i(M)$,

Proof. Set $N := \widetilde{\mathrm{E}}_i^{\max}\widetilde{\mathrm{E}}_i^{*\,\max}(M) \simeq \widetilde{\mathrm{E}}_i^{*\,\max}\widetilde{\mathrm{E}}_i^{\max}(M)$. Applying [29, Proposition 2.16] to N, one can show that the conditions are equivalent to $\mathfrak{d}_i(N) < \varepsilon_i(M) + \varepsilon_i^*(M)$. □

2.4. **Categories $\mathscr{C}_w$ and $\mathscr{C}_{w,v}$.** In this section, we briefly recall the subcategories $\mathscr{C}_w$ and $\mathscr{C}_{w,v}$ of R-gmod ([19, 26]).

For an $R(\beta)$-module M, we define

$$\mathrm{W}(M) := \{\gamma \in \mathsf{Q}_+ \cap (\beta - \mathsf{Q}_+) \mid e(\gamma, \beta-\gamma)M \neq 0\},$$
$$\mathrm{W}^*(M) := \{\gamma \in \mathsf{Q}_+ \cap (\beta - \mathsf{Q}_+) \mid e(\beta-\gamma, \gamma)M \neq 0\}.$$

It is easy to see that $\mathrm{W}(M \circ N) = \mathrm{W}(M) + \mathrm{W}(N)$ and $\mathrm{W}^*(M \circ N) = \mathrm{W}^*(M) + \mathrm{W}^*(N)$.

By [51], we have

$$\mathrm{W}(M) \subset \mathrm{span}_{\mathbb{R}_{\geqslant 0}}(\mathrm{W}(M) \cap \Delta_+) \quad \text{for any simple module } M.$$

Here, for $A \subset \mathbb{R} \otimes \mathsf{Q}$, $\mathrm{span}_{\mathbb{R}_{\geqslant 0}}(A)$ denotes the smallest convex subset of $\mathbb{R} \otimes \mathsf{Q}$ which is stable by the multiplication by $\mathbb{R}_{>0}$ and contains $A \cup \{0\}$.

An ordered pair (M, N) of R-modules is called *unmixed* if

$$\mathrm{W}^*(M) \cap \mathrm{W}(N) \subset \{0\}.$$

Note that if (M, N) is unmixed, then it is Λ-definable and $\widetilde{\Lambda}(M, N) = 0$. An almost affreal triple (M, X, N) of simple modules M, X, N is normal if (M, N) is unmixed (see [26, Proposition 2.12] and [29, Corollary 2.13]).

For $w \in \mathsf{W}$, we define $\mathscr{C}_w$ to be the full subcategory of R-gmod consisting of modules M satisfying

$$\mathrm{W}(M) \subset \mathsf{Q}_+ \cap w\,\mathsf{Q}_-.$$

In a similar manner, for $v \in \mathsf{W}$, we define $\mathscr{C}_{*,v}$ to be the full subcategory of R-gmod consisting of modules N satisfying

$$\mathrm{W}^*(N) \subset \mathsf{Q}_+ \cap v\,\mathsf{Q}_+.$$

We now set $\mathscr{C}_{w,v}$ to be the full subcategory of R-gmod whose objects are contained in both of $\mathscr{C}_w$ and $\mathscr{C}_{*,v}$, i.e.,

$$\mathscr{C}_{w,v} = \mathscr{C}_w \cap \mathscr{C}_{*,v}. \tag{2.2}$$

We denote by $K(\mathscr{C}_w)$ and $K(\mathscr{C}_{w,v})$ the Grothendieck rings of $\mathscr{C}_w$ and $\mathscr{C}_{w,v}$, respectively. They are $\mathbb{Z}[q^{\pm 1}]$-algebras where the indeterminate q is induced by the grading shift functor. The category $\mathscr{C}_w$ gives a categorification of the quantum unipotent coordinate ring $A_q(\mathfrak{n}(w))$ ([39, 40, 48, 19]), i.e.,

$$K(\mathscr{C}_w) \simeq A_q(\mathfrak{n}(w)),$$

and $\mathscr{C}_{w,v}$ categorifies the quantization of the coordinate ring of the open *Richardson variety* $\mathcal{R}_{w,v}$ after localization ([26]). We sometimes write $K(\mathscr{C}_w)|_{q=1}$ and $K(\mathscr{C}_{w,v})|_{q=1}$ when we ignore the grading shift, i.e.,

$$K(\mathscr{C}_w)|_{q=1} := K(\mathscr{C}_w)/(q-1)K(\mathscr{C}_w) \quad \text{and} \quad K(\mathscr{C}_{w,v})|_{q=1} := K(\mathscr{C}_{w,v})/(q-1)K(\mathscr{C}_{w,v}).$$

2.5. **Reflection functors.** For each $i \in I$, we set $(R\text{-gmod})_i$ (resp. ${}_i(R\text{-gmod})$) to be the full subcategory of R-gmod consisting of modules killed by E_i (resp. E_i^*). In [34], we define the monoidal equivalence (cf. [36])

$$\mathscr{F}_i \colon (R\text{-gmod})_i \xrightarrow{\sim} {}_i(R\text{-gmod}).$$

called the *reflection functor* which categorifies Lusztig's braid symmetries ([43]). Since $\mathscr{F}_i$ categorifies the *Saito crystal reflection* ([49]) at the crystal level, $\mathscr{F}_i(M)$ for a simple module $M \in (R\text{-gmod})_i$ can be computed as follows:

$$\mathscr{F}_i(M) \simeq \widetilde{\mathrm{F}}_i^{\varphi_i^*(M)} \widetilde{\mathrm{E}}_i^{*\varepsilon_i^*(M)}(M). \tag{2.3}$$

The following proposition is proved (at least implicitly) in [51, § 2.3].

Proposition 2.3. *Let $i \in I$.*

(i) *For any $w \in \mathsf{W}$ such that $s_i w < w$, $\mathscr{F}_i$ induces an equivalence of monoidal categories*

$$\mathscr{F}_i \colon \mathscr{C}_{s_i w} \to \mathscr{C}_w \cap {}_i(R\text{-gmod})$$

(ii) *For any $v \in \mathsf{W}$ such that $v < s_i v$, $\mathscr{F}_i$ induces an equivalence of monoidal categories*

$$\mathscr{F}_i \colon \mathscr{C}_{*,v} \cap (R\text{-gmod})_i \to \mathscr{C}_{*,s_i v}.$$

Proof. Recall first a result of [51, § 2.3]. For convex preorder, we refer the reader to [51] (see also [26, Section 1.3]).

Let $\Delta_+^{\min} := \{\alpha \in \Delta_+ \mid \mathbb{Q}_{>0}\,\alpha \cap \Delta_+ = \mathbb{Z}_{>0}\,\alpha \cap \Delta_+\}$ be the set of minimal positive roots. For a convex order $\preccurlyeq$ on $\Delta_+^{\min}$, we extend it to the convex preorder on Δ_+ such that $m\alpha \preccurlyeq n\beta$ for any $m, n \in \mathbb{Z}_{>0}$ and $\alpha, \beta \in \Delta_+^{\min}$ with $\alpha \preccurlyeq \beta$.

For $i \in I$ and a convex order $\preccurlyeq$ on $\Delta_+^{\min}$ such that α_i is the smallest element of $\Delta_+^{\min}$ with respect to $\preccurlyeq$, let $\preccurlyeq^{s_i}$ be the convex order on $\Delta_+^{\min}$ defined by:

(a) α_i is the largest element of $\Delta_+^{\min}$ with respect to $\preccurlyeq^{s_i}$,
(b) for $\alpha, \beta \in \Delta_+^{\min} \setminus \{\alpha_i\}$, $\alpha \preccurlyeq^{s_i} \beta$ if and only if $s_i\alpha \preccurlyeq s_i\beta$.

For $\beta \in \Delta_+$, we denote by $(R\text{-gmod})_\beta^{\preccurlyeq}$ the full subcategory of R-gmod consisting of $M \in R$-gmod such that $\mathrm{W}(M) \cap \Delta_+ \subset \{\alpha \in \Delta_+ \mid \alpha \preccurlyeq \beta\}$. Similarly we denote by $(R\text{-gmod})_\beta^{*\preccurlyeq}$ the full subcategory of R-gmod consisting of $M \in R$-gmod such that $\mathrm{W}^*(M) \cap \{\alpha \in \Delta_+ \mid \alpha \preccurlyeq \beta\} = \varnothing$.

Now, take a convex order $\preccurlyeq$ on $\Delta_+^{\min}$ such that α_i is the smallest element of $\Delta_+^{\min}$ with respect to $\preccurlyeq$. Then by [51, § 2.3], we have the following:
for any $\beta \in \Delta_+ \setminus \{\alpha_i\}$, we have

(a) the reflection functor $\mathscr{F}_i$ gives an equivalence
$$(R\text{-gmod})_\beta^{\preccurlyeq^{s_i}} \xrightarrow{\sim} {}_iR\text{-gmod} \cap (R\text{-gmod})_\beta^{\preccurlyeq},$$

(b) the reflection functor $\mathscr{F}_i$ gives an equivalence
$$R\text{-gmod}_i \cap (R\text{-gmod})_\beta^{*\preccurlyeq^{s_i}} \xrightarrow{\sim} (R\text{-gmod})_\beta^{*\preccurlyeq}.$$

Now we shall prove (i) and (ii).
(i) Take a convex order $\preccurlyeq$ on $\Delta_+^{\min}$ such that α_i is the smallest and $\Delta_+ \cap w\Delta_- \preccurlyeq \Delta_+^{\min} \cap w\Delta_+$, and let β be the largest element of $\Delta_+ \cap w\Delta_-$. Then we have $(R\text{-gmod})_{s_i\beta}^{\preccurlyeq^{s_i}} = \mathscr{C}_{s_iw}$ and $(R\text{-gmod})_\beta^{\preccurlyeq} = \mathscr{C}_w$. Hence (a) implies (i).

(ii) Take a convex order $\preccurlyeq$ on $\Delta_+^{\min}$ such that α_i is the smallest and $\Delta_+ \cap s_iv\Delta_- \preccurlyeq \Delta_+^{\min} \cap s_iv\Delta_+$, and let β be the largest element of $\Delta_+ \cap s_iv\Delta_-$. Then we have $(R\text{-gmod})_{s_i\beta}^{*\preccurlyeq^{s_i}} = \mathscr{C}_{*,v}$ and $(R\text{-gmod})_\beta^{*\preccurlyeq} = \mathscr{C}_{*,s_iv}$. ☐

Lemma 2.4. *Let $w \in \mathsf{W}$ and let $i \in I$.*

(i) *Let M be a simple module in R-gmod. If $\mathfrak{d}_i(M) = 0$, then $\widetilde{\mathrm{E}}_i^{*\max} M \simeq \mathscr{F}_i(\widetilde{\mathrm{E}}_i^{\max} M)$.*
(ii) *If $s_iw < w$, then $\widetilde{\mathrm{E}}_i^{\max} N \in \mathscr{C}_{s_iw}$ for any simple module $N \in \mathscr{C}_w$.*

Proof. (i) We set $\beta := \mathrm{wt}(M)$ and
$$M' := \widetilde{\mathrm{E}}_i^{\max}(M), \quad M'' := \widetilde{\mathrm{E}}_i^{*\max}(M), \quad \text{and} \quad M_\circ := \widetilde{\mathrm{E}}_i^{\max}\widetilde{\mathrm{E}}_i^{*\max}(M).$$
We then write $M = \langle i \rangle^{\circ a} \nabla M' = M'' \nabla \langle i \rangle^{\circ b}$ where $a := \varepsilon_i(M)$ and $b := \varepsilon_i^*(M)$. Note that $\widetilde{\mathrm{E}}_i^{*\max} M' \simeq M_\circ$ and
$$\mathfrak{d}_i(M) = a + b + \langle h_i, \beta \rangle = 0.$$
Since $\mathrm{wt}(M_\circ) = \beta + (a + \varepsilon_i^*(M'))\alpha_i = \beta + (b + \varepsilon_i(M''))\alpha_i$, we have
$$a + \varepsilon_i^*(M') = b + \varepsilon_i(M''),$$
which implies
$$\varphi_i^*(M') = \varepsilon_i^*(M') + \langle h_i, \beta + a\alpha_i \rangle = \varepsilon_i^*(M') + a - b = \varepsilon_i(M'').$$

Hence, we have

$$\mathscr{F}_i(\widetilde{\mathrm{E}}_i^{\max} M) = \mathscr{F}_i(M') \simeq \widetilde{\mathrm{F}}_i^{\varphi_i^*(M)}(M_\circ) = \widetilde{\mathrm{F}}_i^{\varepsilon_i(M'')}(M_\circ) = M''.$$

(ii) Replacing N with $\widetilde{\mathrm{F}}_i^n N$ ($n \gg 0$), we may assume that $\mathfrak{d}_i(N) = 0$. As $N \in \mathscr{C}_w$, we have $L := \widetilde{\mathrm{E}}_i^{*\max}(N) \in \mathscr{C}_{w,s_i}$, which implies

$$\mathscr{F}_i^{-1}(L) \simeq \widetilde{\mathrm{E}}_i^{\max} N \in \mathscr{C}_{s_i w}$$

by (i) and Proposition 2.3. □

2.6. **Miscellaneous results.** In this subsection, we show several lemmas and propositions which will be used in the paper.

The following lemma can be obtained from [19, Lemma 3.2.11 and 3.2.12] by a straightforward modification to the setting of arbitrary quiver Hecke algebras.

Lemma 2.5 ([19, Lemma 3.2.11 and 3.2.12])**.** *Let $L, M, N \in R$-gmod be simple modules and assume that L is affreal.*

(i) *If L and M commute, then*

$$\Lambda(L, M) + \Lambda(L, N) = \Lambda(L, S) \quad \textit{for any simple quotient } S \textit{ of } M \circ N.$$

(ii) *If L and N commute, then*

$$\Lambda(M, L) + \Lambda(N, L) = \Lambda(S, L) \quad \textit{for any simple quotient } S \textit{ of } M \circ N.$$

We then have the following proposition.

Proposition 2.6. *Let $L, M, N \in R$-gmod be simple modules and assume that L is affreal.*

(i) *There exists a simple subquotient S of $M \circ N$ such that*

$$\Lambda(L, M) + \Lambda(L, N) = \Lambda(L, S).$$

Similarly, there exists a simple subquotient S' of $M \circ N$ such that

$$\Lambda(M, L) + \Lambda(N, L) = \Lambda(S', L).$$

(ii) *We have*

$$\Lambda(L, M) + \Lambda(L, N) = \max\{\Lambda(L, S) \mid S \textit{ is a composition factor of } M \circ N\},$$

$$\Lambda(M, L) + \Lambda(N, L) = \max\{\Lambda(S, L) \mid S \textit{ is a composition factor of } M \circ N\}.$$

Proof. (i) We focus on proving the first identity since the second can be shown in the same manner.

Let us take a positive integer n such that $X := L^{\circ n} \nabla M$ commutes with L (see [28, Corollary 3.18]). Take a simple quotient Y of $X \circ N$. Then Lemma 2.5 implies that

$$\Lambda(L, M) + \Lambda(L, N) = \Lambda(L, X) + \Lambda(L, N) = \Lambda(L, Y).$$

Since Y is a simple quotient of $L^{\circ n} \circ M \circ N$, there exists a simple subquotient S of $M \circ N$ such that $Y \simeq L^{\circ n} \nabla S$. Indeed, take a minimal submodule Z of $M \circ N$ such that the composition

$$L^{\circ n} \circ Z \to L^{\circ n} \circ M \circ N \twoheadrightarrow Y$$

does not vanish. Take a simple quotient S of Z. Then

$$L^{\circ n} \circ Z \twoheadrightarrow Y \text{ factors through } L^{\circ n} \circ S.$$

Hence we have

$$\Lambda(L, S) = \Lambda(L, L^{\circ n} \nabla S) = \Lambda(L, Y) = \Lambda(L, M) + \Lambda(L, N).$$

(ii) It follows from (i) and [19, Proposition 3.2.10]. □

Proposition 2.7. *Let (X, Y, Z) be an almost affreal normal sequence of simple modules.*

(i) *We assume that*
  (a) *X is affreal,*
  (b) *X commutes with $Y \nabla Z$,*
  Then X commutes with Y.

(ii) *We assume that*
  (a) *Z is affreal,*
  (b) *Z commutes with $X \nabla Y$,*
  Then Z commutes with Y.

Proof. Since (ii) can be proved in the same manner, we only prove (i). We may assume that Y and Z do not commute since the assertion is obvious otherwise. From the following commutative diagram

$$\begin{tikzcd}
X \circ Y \circ Z \arrow[r, "\mathbf{r}_{X,Y}"] \arrow[d, twoheadrightarrow] & Y \circ X \circ Z \arrow[r, "\mathbf{r}_{X,Z}"] & Y \circ Z \circ X \arrow[d, twoheadrightarrow] \\
X \circ (Y \nabla Z) \arrow[rr, "\sim"] & & (Y \nabla Z) \circ X,
\end{tikzcd}$$

we have

$$\operatorname{Ker}(X \circ Y \xrightarrow{\mathbf{r}_{X,Y}} Y \circ X) \circ Z \subset X \circ \operatorname{Ker}(Y \circ Z \twoheadrightarrow Y \nabla Z).$$

Thus the quasi-rigidity (see [18, Lemma 3.1] and see also [30, Section 6.3]) implies that there exists $S \subset Y$ such that

$$\operatorname{Ker}(X \circ Y \xrightarrow{\mathbf{r}_{X,Y}} Y \circ X) \subset X \circ S, \qquad S \circ Z \subset \operatorname{Ker}(Y \circ Z \twoheadrightarrow Y \nabla Z).$$

Since $\operatorname{Ker}(Y \circ Z \twoheadrightarrow Y \nabla Z) \neq Y \circ Z$, we have $S \neq Y$, which implies that $S = 0$. Hence the R-matrix $X \circ Y \xrightarrow{\mathbf{r}_{X,Y}} Y \circ X$ is injective, which means that it is an isomorphism (see [18, Section 3]). $\square$

Corollary 2.8. *Let M be an affreal simple module and N a simple module. Let $\{L_k\}_{k=1,2}$ be a commuting family of affreal simple modules.*

(i) *If M commutes with N and $\operatorname{hd}(N \circ L_1 \circ L_2)$, then M commutes with $N \nabla L_1$.*
(ii) *If M commutes with N and $\operatorname{hd}(L_2 \circ L_1 \circ N)$, then M commutes with $L_1 \nabla N$.*

Proof. We shall prove only (i) because (ii) can be shown in the same manner. Since M and N commute, $(M, N, L_1 \circ L_2)$ is a normal sequence. Hence (M, N, L_1, L_2) is also a normal sequence, and so is $(M, N \nabla L_1, L_2)$. Therefore, Proposition 2.7 implies the desired result. $\square$

Applying Corollary 2.8 to the setting $L_1{:=}\langle i^k \rangle$ and $L_2{:=}\langle i^{n-k} \rangle$, we have the following.

Corollary 2.9. *Let $i \in I$, let M be an affreal simple module and N a simple module. Suppose that M and N commute.*

(i) *If $n \in \mathbb{Z}_{\geqslant 0}$ and M commutes with $\widetilde{\mathrm{F}}_i^n(N)$, then M commutes with $\widetilde{\mathrm{F}}_i^k(N)$ for any $k \in [1, n]$.*
(ii) *If $n \in \mathbb{Z}_{\geqslant 0}$ and M commutes with $\widetilde{\mathrm{F}}_i^{*n}(N)$, then M commutes with $\widetilde{\mathrm{F}}_i^{*k}(N)$ for any $k \in [1, n]$.*

The following lemma is an analogue of [33, Lemma 2.23] for quiver Hecke algebras. We omit the proof since it follows from the same argument as in [33, Lemma 2.23] once the existence of affinizations is established. Note that the commutativity of ⓒ in the diagram of [33, Proof of Lemma 2.23] follows from the affinization of $M \nabla N$.

Lemma 2.10 (cf. [33, Lemma 2.23])**.** *Let X and Y be affreal simple modules such that $X \nabla Y$ admits an affinization. If $X \nabla Y$ commutes with X, then $X \nabla Y$ is an affreal simple module.*

2.7. **Laurent families.** In this section, we recall the notion of Laurent families introduced in [31] and study their properties that we need in this paper.

2.7.1. *Definition of Laurent families.* Let $\mathcal{C}$ be a full subcategory of R-gmod such that $\mathcal{C}$ has $\mathbf{1}$ and is stable under taking convolution products, subquotients, extensions and grading shifts, and let K be an index set. A family $\mathcal{M} = \{M_k\}_{k\in K} \subset \mathcal{C}$ of affreal simple modules is called a *commuting family* if M_k is an affreal simple module for any $k \in K$ and

$$M_i \circ M_j \simeq M_j \circ M_i \qquad \text{(up to a grading shift) for any } i, j \in K.$$

For $\mathbf{a} = (a_k)_{k\in K} \in \mathbb{Z}_{\geqslant 0}^{\oplus K}$, the module $\mathcal{M}(\mathbf{a})$ is defined as follows:

$$\begin{aligned} &\mathcal{M}(\mathbf{a}) \text{ is self-dual,} \\ &\mathcal{M}(\mathbf{a}) \simeq \mathop{\circ}_{k\in K} (M_k^{\circ a_k}) \text{ up to a grading shift.} \end{aligned} \tag{2.4}$$

Then we have

$$\mathcal{M}(\mathbf{a}) \circ \mathcal{M}(\mathbf{b}) \simeq q^{\lambda(\mathbf{a},\mathbf{b})}\mathcal{M}(\mathbf{a}+\mathbf{b}), \tag{2.5}$$

where $\lambda(\mathbf{a},\mathbf{b}) = -\sum_{j,k\in K} a_j b_k \widetilde{\Lambda}(M_j, M_k)$ with $\mathbf{b} = (b_k)_{k\in K}$.

Definition 2.11 ([31])**.** Let $\mathcal{M} = \{M_k\}_{k\in K}$ be a commuting family of affreal simple modules.

(i) A simple module X is called an $\mathcal{M}$-*monomial* if X is isomorphic to $\mathcal{M}(\mathbf{a})$ for some $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{\oplus K}$.

(ii) $\mathcal{M}$ is *independent* if $\mathbf{a} = \mathbf{b}$ whenever $\mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b})$.

(iii) $\mathcal{M}$ is a *quasi-Laurent family* in $\mathcal{C}$ if it is independent and satisfies the following condition:

$$\begin{aligned} &\text{if a simple module } X \in \mathcal{C} \text{ commutes with } M_k \text{ for any } k \in K, \text{ then there exist} \\ &\mathbf{a}, \mathbf{b} \in \mathbb{Z}_{\geqslant 0}^{\oplus K} \text{ such that } X \circ \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b}). \end{aligned} \tag{2.6}$$

(iv) $\mathcal{M}$ is a *Laurent family* if it is independent and satisfies the following condition (2.7) instead of (2.6):

$$\begin{aligned} &\text{any simple module } X \in \mathcal{C} \text{ which commutes with } M_k \text{ for any } k \in K \text{ is an} \\ &\mathcal{M}\text{-monomial.} \end{aligned} \tag{2.7}$$

2.7.2. *g-vectors.* Let $\mathcal{M} = \{M_k \mid k \in K\}$ be a quasi-Laurent family in the category $\mathcal{C}$.

Let $\Lambda_{\mathcal{M}} := (\Lambda(M_i, M_j))_{i,j\in K}$, i.e., $q^{\Lambda(M_i,M_j)}[M_i][M_j] = [M_j][M_i]$, and let $\mathcal{T}_{\mathcal{M}}$ be the quantum torus generated by $\{[M_k]\}_{k\in K}$.

We extend the definition of $[\mathcal{M}(\mathbf{a})]$ for any $\mathbf{a} \in \mathbb{Z}^{\oplus K}$ so that (2.5) holds for any $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^{\oplus K}$. For any simple module $X \in \mathcal{C}$, there exist some $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{Z}_{\geqslant 0}^{\oplus K}$ such that

$$\mathcal{M}(\mathbf{a}) \nabla X \simeq \mathcal{M}(\mathbf{b}) \quad \text{and} \quad X \nabla \mathcal{M}(\mathbf{c}) \simeq \mathcal{M}(\mathbf{d}).$$

We define

$$\mathrm{g}^{\mathrm{L}}_{\mathcal{M}}(X) := \mathbf{b} - \mathbf{a} \quad \text{and} \quad \mathrm{g}^{\mathrm{R}}_{\mathcal{M}}(X) := \mathbf{d} - \mathbf{c}. \tag{2.8}$$

By [31, Lemma 3.9], $\mathrm{g}^{\mathrm{L}}_{\mathcal{M}}$ and $\mathrm{g}^{\mathrm{R}}_{\mathcal{M}}$ are well-defined.

By [31, Corollary 3.4], for any module $X \in \mathcal{C}$, there exist $\mathbf{a}, \mathbf{b}_1, \ldots, \mathbf{b}_t \in \mathbb{Z}_{\geqslant 0}^{\oplus K}$ such that

$$[X \circ \mathcal{M}(\mathbf{a})] = \sum_{s=1}^{t} q^{c_s}[\mathcal{M}(\mathbf{b}_s)] \qquad \text{for some } c_s \in \mathbb{Z}. \tag{2.9}$$

Then the following lemma is straightforward.

Lemma 2.12. *The correspondence $[X] \mapsto \sum_{s=1}^{t} q^{c_s}[\mathcal{M}(\mathbf{b}_s)][\mathcal{M}(\mathbf{a})]^{-1} \in \mathcal{T}_{\mathcal{M}}$ gives a well-defined injective $\mathbb{Z}[q, q^{-1}]$-algebra homomorphism*

$$\varphi_{\mathcal{M}} \colon K(\mathcal{C}) \rightarrowtail \mathcal{T}_{\mathcal{M}}. \tag{2.10}$$

Theorem 2.13. *Let $X, L \in \mathcal{C}$ be simple modules such that L is affreal, and write*

$$\varphi_{\mathcal{M}}([X]) = \sum_{k} q^{a_k}[\mathcal{M}(\mathbf{a}_k)] \qquad \textit{for some } a_k \in \mathbb{Z} \textit{ and } \mathbf{a}_k \in \mathbb{Z}^{\oplus K}.$$

Then we have

$$\Lambda(L, X) = \max_k\{\Lambda(L, \mathcal{M}(\mathbf{a}_k))\} = \max_k\{\mathrm{g}^{\mathrm{R}}_{\mathcal{M}}(L) \cdot \Lambda_{\mathcal{M}} \cdot \mathbf{a}_k^T\},$$
$$\Lambda(X, L) = \max_k\{\Lambda(\mathcal{M}(\mathbf{a}_k), L)\} = \max_k\{\mathbf{a}_k \cdot \Lambda_{\mathcal{M}} \cdot (\mathrm{g}^{\mathrm{L}}_{\mathcal{M}}(L))^T\},$$

where v^T is the transpose of a vector v.

Proof. Replacing X with $X \circ \mathcal{M}(\mathbf{c})$ for $\mathbf{c} \gg 0$, we may assume that $\mathbf{a}_k \in \mathbb{Z}_{\geqslant 0}^{\oplus K}$. Applying Proposition 2.6 to (2.9), we have

$$\Lambda(L, X) = \max_k\{\Lambda(L, \mathcal{M}(\mathbf{a}_k))\} \quad \text{and} \quad \Lambda(X, L) = \max_k\{\Lambda(\mathcal{M}(\mathbf{a}_k), L)\}. \tag{2.11}$$

We write $\mathcal{M}(\mathbf{b}) \nabla L \simeq \mathcal{M}(\mathbf{c})$ for some $\mathbf{b} = (b_i)_{i\in K}, \mathbf{c} = (c_i)_{i\in K} \in \mathbb{Z}_{\geqslant 0}^{\oplus K}$. Note that $\mathsf{g}^{\mathrm{L}}_{\mathcal{M}}(L) = \mathbf{c} - \mathbf{b}$. For any $\mathbf{a} = (a_i)_{i\in K} \in \mathbb{Z}^{\oplus K}$, we have

$$\begin{aligned}\Lambda(\mathcal{M}(\mathbf{a}), \mathcal{M}(\mathbf{c})) &= \sum_{i,j\in K} a_i \Lambda(M_i, M_j) c_j,\\ \Lambda(\mathcal{M}(\mathbf{a}), \mathcal{M}(\mathbf{b}) \nabla L) &= \Lambda(\mathcal{M}(\mathbf{a}), \mathcal{M}(\mathbf{b})) + \Lambda(\mathcal{M}(\mathbf{a}), L)\\ &= \sum_{i,j\in K} a_i \Lambda(M_i, M_j) b_j + \Lambda(\mathcal{M}(\mathbf{a}), L),\end{aligned}$$

which implies

$$\Lambda(\mathcal{M}(\mathbf{a}), L) = \sum_{i,j\in K} a_i \Lambda(M_i, M_j)(c_j - b_j) = \mathbf{a} \cdot \Lambda_{\mathcal{M}} \cdot (\mathsf{g}^{\mathrm{L}}_{\mathcal{M}}(L))^T.$$

Hence, we have $\Lambda(X, L) = \max_k\{\mathbf{a}_k \cdot \Lambda_{\mathcal{M}} \cdot (\mathsf{g}^{\mathrm{L}}_{\mathcal{M}}(L))^T\}$ by (2.11).

The case for $\mathsf{g}^{\mathrm{R}}_{\mathcal{M}}(L)$ can be proved in the same manner. $\square$

Remark 2.14. Suppose that $\mathcal{M}$ is a monoidal seed of $\mathcal{C}$ (see § 4.1). In this case, we have the following.

(i) $\mathsf{g}^{\mathrm{L}}_{\mathcal{M}}(X)$ and $\mathsf{g}^{\mathrm{R}}_{\mathcal{M}}(X)$ are the *extended g-vectors* of a simple module X with respect to $\mathcal{M}$ introduced in [24, 31].

(ii) Let $\widetilde{B}$ be the exchange matrix associated with $\mathcal{M}$. The formula in Theorem 2.13 can be written as follows:

$$\begin{aligned}\Lambda(X, L) &= \max_k\{\mathbf{a}_k \cdot \Lambda_{\mathcal{M}} \cdot (\mathsf{g}^{\mathrm{L}}_{\mathcal{M}}(L))^T\}\\ &= \mathsf{g}^{\mathrm{L}}_{\mathcal{M}}(X) \cdot \Lambda_{\mathcal{M}} \cdot (\mathsf{g}^{\mathrm{L}}_{\mathcal{M}}(L))^T + \max_k\{(\mathbf{a}_k - \mathsf{g}^{\mathrm{L}}_{\mathcal{M}}(X)) \cdot \Lambda_{\mathcal{M}} \cdot (\mathsf{g}^{\mathrm{L}}_{\mathcal{M}}(L))^T\}\end{aligned} \tag{2.12}$$

Since $\mathbf{a}_k$ is larger than $\mathsf{g}^{\mathrm{L}}_{\mathcal{M}}(X)$ in the *dominance order*, we can write $\mathbf{a}_k - \mathsf{g}^{\mathrm{L}}_{\mathcal{M}}(X) = (\widetilde{B}v)^T$ for some column vector v with non-negative entries ([24, Lemma 3.6]). Since the pair $(-\Lambda_{\mathcal{M}}, \widetilde{B})$ is compatible with $d = 2$, the formula (2.12) can be rewritten in terms of the F-polynomial, which coincides with Cao's formula expressed in terms of the *tropical invariant* introduced in [5, Theorem 5.16]. Therefore, Theorem 2.13 may be regarded as a generalization of Cao's formula [5, Theorem 5.16] for cluster variable modules to the case of real simple modules.

2.7.3. *Stability of Laurent families.* We shall show that the notion of (quasi)-Laurent families is stable under a mutation-like procedure.

Lemma 2.15. *Let $\mathcal{M} = \{M_k\}_{k\in K}$ and $\mathcal{M}' = \{M'_k\}_{k\in K}$ be commuting families of affreal simple modules. Let $s \in K$. Assume that*

(a) *$M_k \simeq M'_k$ for $k \in K \setminus \{s\}$,*

(b) *there exist $\mathcal{M}$-monomials U and V and an exact sequence*

$$0 \longrightarrow U \longrightarrow M_s \circ M'_s \longrightarrow V \longrightarrow 0.$$

If $\mathcal{M}$ is independent, then $\mathcal{M}'$ is also independent.

Proof. Let $\mathbf{a} = (a_k)_{k\in K},\ \mathbf{b} = (b_k)_{k\in K} \in \mathbb{Z}_{\geqslant 0}^{\oplus K}$. Assuming that $\mathcal{M}'(\mathbf{a}) = \mathcal{M}'(\mathbf{b})$, let us show that $\mathbf{a} = \mathbf{b}$. We may assume that $a_s \leqslant b_s$. Let $\{\mathbf{e}_k \mid k \in K\}$ be the natural basis of $\mathbb{Z}_{\geqslant 0}^{\oplus K}$. Then we have

$$\begin{aligned}
[\mathcal{M}'(\mathbf{a})] \cdot [M_s]^{b_s} &= [\mathcal{M}'(\mathbf{a} - a_s\mathbf{e}_s)] \cdot ([U] + [V])^{a_s} \cdot [M_s]^{b_s - a_s} \quad \text{and} \\
[\mathcal{M}'(\mathbf{b})] \cdot [M_s]^{b_s} &= [\mathcal{M}'(\mathbf{b} - b_s\mathbf{e}_s)] \cdot ([U] + [V])^{b_s}.
\end{aligned}$$

Hence we have

$$[\mathcal{M}(\mathbf{a} + (b_s - 2a_s)\mathbf{e}_s)] \cdot ([U] + [V])^{a_s} = [\mathcal{M}(\mathbf{b} - b_s\mathbf{e}_s)] \cdot ([U] + [V])^{b_s}.$$

Then $\mathbf{a} = \mathbf{b}$ follows from the independence of $\mathcal{M}$. $\square$

Proposition 2.16. *Let $\mathcal{M} = \{M_k\}_{k\in K}$ and $\mathcal{M}' = \{M'_k\}_{k\in K}$ be commuting families of affreal simple modules in $\mathcal{C}$. Let $s \in K$. Assume that*

(a) *$M_k \simeq M'_k$ for $k \in K \setminus \{s\}$,*

(b) *there exist $\{M_k \mid k \in K \setminus \{s\}\}$-monomials U and V and an exact sequence*

$$0 \longrightarrow U \longrightarrow M_s \circ M'_s \longrightarrow V \longrightarrow 0.$$

Then, $\mathcal{M}'$ is a quasi-Laurent family in $\mathcal{C}$ if $\mathcal{M}$ is a quasi-Laurent family in $\mathcal{C}$.

Proof. By the preceding lemma, $\mathcal{M}'$ is independent.

Let X be a simple module in $\mathcal{C}$ which commutes with all M'_k. Let us show that there exists $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{\oplus K}$ such that $X \circ \mathcal{M}'(\mathbf{a})$ is an $\mathcal{M}'$-monomial. By [28, Corollary 3.18], there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that any simple subquotient of $X \circ M_s^n$ commutes with M_s. Since M_k $(k \in K \setminus \{s\})$ commutes with X and M_s, M_k commutes with any simple subquotient of $X \circ M_s^n$. Hence any simple subquotient of $X \circ M_s^n$ commutes with all members of $\mathcal{M}$. Since $\mathcal{M}$ is a quasi-Laurent family, there exists $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{\oplus K}$ such that $\mathcal{M}(\mathbf{a}) \circ S$ is an $\mathcal{M}$-monomial for any simple subquotient S of $X \circ M_s^n$. Hence there exist finitely many $\mathbf{b}(t) \in \mathbb{Z}_{\geqslant 0}^{\oplus K}$ such that

$$[X] \cdot [M_s]^n \cdot [\mathcal{M}(\mathbf{a})] = \sum_t [\mathcal{M}(\mathbf{b}(t))].$$

Thus replacing $\mathbf{a}$ with $\mathbf{a} + n\mathbf{e}_s$, we have

$$[X] \cdot [\mathcal{M}(\mathbf{a})] = \sum_t [\mathcal{M}(\mathbf{b}(t))] \quad \text{for some } \mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{\oplus K} \text{ and finitely many } \mathbf{b}(t) \in \mathbb{Z}_{\geqslant 0}^{\oplus K}.$$

Let us take $m \in \mathbb{Z}$ such that $m \geqslant a_s$ and $m \geqslant \mathbf{b}(t)_s$ for any t, where $\mathbf{b}(t)_s$ is the s-th component of $\mathbf{b}(t)$. Set $\bar{\mathbf{a}} = \mathbf{a} - a_s\mathbf{e}_s$ and $\bar{\mathbf{b}}(t) = \mathbf{b}(t) - \mathbf{b}(t)_s\mathbf{e}_s$. Multiplying $[M'_s]^m$, we obtain

$$[X] \cdot [\mathcal{M}(\bar{\mathbf{a}})] \cdot ([U] + [V])^{a_s} \cdot [M'_s]^{m-a_s} = \sum_t [\mathcal{M}(\bar{\mathbf{b}}(t))] \cdot (U+V)^{\mathbf{b}(t)_s} \cdot [M'_s]^{m-\mathbf{b}(t)_s}$$

Note that the right-hand side is a sum of $\mathcal{M}'$-monomials. Hence a component $[X] \cdot [\mathcal{M}(\bar{\mathbf{a}})] \cdot [U]^{a_s} \cdot [M'_s]^{m-a_s}$ of the left-hand side is an $\mathcal{M}'$-monomial. □

Proposition 2.17. *Let $\mathcal{M} = \{M_k\}_{k \in K}$ and $\mathcal{M}' = \{M'_k\}_{k \in K}$ be commuting families of affreal simple modules in $\mathcal{C}$. Let $s \in K$. Assume that*

(a) *$M_k \simeq M'_k$ for ant $k \in K \setminus \{s\}$,*

(b) *there exist $\{M_k \mid k \in K \setminus \{s\}\}$-monomials U and V and an exact sequence*

$$0 \longrightarrow U \longrightarrow M_s \circ M'_s \longrightarrow V \longrightarrow 0$$

such that U and V have have no common convolution factor other than $\mathbf{1}$.

Then, $\mathcal{M}'$ is a Laurent family in $\mathcal{C}$ if $\mathcal{M}$ is a Laurent family.

Proof. By the preceding lemma, $\mathcal{M}'$ is independent.

Let X be a simple module in $\mathcal{C}$ which commutes with all M'_k. Let us show that X is an $\mathcal{M}'$-monomial.

By the same argument as the one in the proof of the preceding proposition, there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that any simple subquotient of $X \circ M_s^n$ commutes with all members of $\mathcal{M}$. Hence there exist finitely many $\mathbf{b}(t) \in \mathbb{Z}_{\geqslant 0}^{\oplus K}$ such that

$$[X] \cdot [M_s]^n = \sum_t [\mathcal{M}(\mathbf{b}(t))].$$

Set $\bar{\mathbf{b}}(t) = \mathbf{b}(t) - \mathbf{b}(t)_s\mathbf{e}_s$. Multiplying $[M'_s]^n$, we obtain

$$\begin{aligned}[X] \cdot ([U] + [V])^n = &\sum_{\mathbf{b}(t)_s \geqslant n} [\mathcal{M}(\mathbf{b}(t) - n\mathbf{e}_s)] \cdot (U+V)^n \\ &\qquad + \sum_{\mathbf{b}(t)_s < n} [\mathcal{M}(\bar{\mathbf{b}}(t)] \cdot (U+V)^{\mathbf{b}(t)_s} [M'_s]^{n-\mathbf{b}(t)_s}.\end{aligned}$$

The right-hand side is a sum of $\mathcal{M}$-monomials and $\mathcal{M}'$-monomials. Hence $[X] \cdot [U]^n$ is either an $\mathcal{M}$-monomial or an $\mathcal{M}'$-monomial. Note that an $\mathcal{M}$-monomial which commutes with M'_s is an $\mathcal{M}'$-monomial because $\mathfrak{d}(M_s, M'_s) > 0$. Since $X \circ U^n$ commutes with M'_s, we conclude that $[X] \cdot [U]^n$ is an $\mathcal{M}'$-monomial. Similarly, $[X] \cdot [V]^n$ is also an $\mathcal{M}'$-monomial. Set $U^{\circ n} \simeq \mathcal{M}(\mathbf{a})$ and $V^{\circ n} \simeq \mathcal{M}(\mathbf{a}')$. Then $a_s = a'_s = 0$ and $a_k a'_k = 0$ for any $k \in K$.

Thus there exist $\mathbf{b}$ and $\mathbf{b}'$ such that

$$X \circ \mathcal{M}'(\mathbf{a}) \simeq \mathcal{M}'(\mathbf{b}) \quad \text{and} \quad X \circ \mathcal{M}'(\mathbf{a}') \simeq \mathcal{M}'(\mathbf{b}').$$

Hence we have $X \circ \mathcal{M}'(\mathbf{a}) \circ \mathcal{M}'(\mathbf{a}') \simeq \mathcal{M}'(\mathbf{a}') \circ \mathcal{M}'(\mathbf{b}) \simeq \mathcal{M}'(\mathbf{a}) \circ \mathcal{M}'(\mathbf{b}')$, which implies that

$$\mathbf{a}' + \mathbf{b} = \mathbf{a} + \mathbf{b}'.$$

Now, let us show that $a_k \leqslant b_k$ for any k. If $a_k = 0$, it is trivial. Otherwise, we have $a'_k = 0$, which implies $b_k = a_k + b'_k \geqslant a_k$.

Thus we obtain $X \simeq \mathcal{M}'(\mathbf{b} - \mathbf{a})$. □

2.8. **The operators $\mathscr{K}_i$ and $\mathscr{K}_i^*$.** In this subsection, we introduce and study new operators $\mathscr{K}_i$ and $\mathscr{K}_i^*$ on simple modules, which will be crucially used to construct seeds of $\mathscr{C}_{w,v}$.

Definition 2.18. For any $i \in I$ and a simple $M \in R$-gmod, we define

$$\mathscr{K}_i(M) := \widetilde{\mathrm{F}}_i^{a} \left(\widetilde{\mathrm{E}}_i^{\max}(M) \right) \quad \text{and} \quad \mathscr{K}_i^*(M) := \widetilde{\mathrm{F}}_i^{*\,b} \left(\widetilde{\mathrm{E}}_i^{*\max}(M) \right),$$

where $a = \mathfrak{d}_i(\widetilde{\mathrm{E}}_i^{\max}(M))$ and $b = \mathfrak{d}_i(\widetilde{\mathrm{E}}_i^{*\max}(M))$.

Note that

$$\mathscr{K}_i(M) \simeq \widetilde{\mathrm{F}}_i^{\mathfrak{d}_i(M)}(M) \quad \text{if } \langle i \rangle \text{ is not a convolution factor of } M.$$

Thanks to [29, Proposition 2.16], $\langle i \rangle$ is not a convolution factor of $\mathscr{K}_i(M)$ and $\mathscr{K}_i^*(M)$, and

$$\begin{aligned} \mathfrak{d}_i(\mathscr{K}_i(M)) &= 0, & \mathscr{K}_i\big(\mathscr{K}_i(M)\big) &= \mathscr{K}_i(M), \\ \mathfrak{d}_i(\mathscr{K}_i^*(M)) &= 0, & \mathscr{K}_i^*\big(\mathscr{K}_i^*(M)\big) &= \mathscr{K}_i^*(M). \end{aligned} \tag{2.13}$$

For a family $S = \{M_k\}_{k\in K}$ of simple modules, we write $\mathscr{K}_i(S) := \{\mathscr{K}_i(M_k)\}_{k\in K}$ and $\mathscr{K}_i^*(S) := \{\mathscr{K}_i^*(M_k)\}_{k\in K}$.

We now show several properties of $\mathscr{K}_i$ and $\mathscr{K}_i^*$ in the following lemmas, which will play a crucial role in constructing seeds in $\mathscr{C}_{w,v}$.

Lemma 2.19. *Let $M \in R$-gmod be a simple module.*

(i) *If M is affreal, then $\mathscr{K}_i(M)$ and $\mathscr{K}_i^*(M)$ are affreal.*
(ii) *The following conditions are equivalent:*
  (a) *$\mathfrak{d}_i(M) = 0$ and $\langle i \rangle$ is not a convolution factor of M,*
  (b) *$\mathscr{K}_i(M) \simeq M$,*
  (c) *$\mathscr{K}_i^*(M) \simeq M$.*

(iii) *If* $M \in (R\text{-gmod})_i$, *then* $\mathscr{K}_i(M) \simeq \mathscr{K}_i^{*}\mathscr{F}_i(M)$
(iv) *If* $M \in (R\text{-gmod})_i \cap {}_i(R\text{-gmod})$, *then* $\mathscr{K}_i(M) \simeq \mathscr{F}_i(M)$.

Proof. (i) Since the case for $\mathscr{K}_i^*$ can be proved by the same argument, we only prove the case for $\mathscr{K}_i$.

Let $(\mathsf{M}, z_{\mathsf{M}})$ be an affinization of M. Thanks to [28, Lemma 3.3], we may assume that $\varepsilon_i(M) = 0$. We set $\Lambda := \sum_{i\in I} \varepsilon_i^*(M)\Lambda_i \in \mathsf{P}_+$ and define $a_\Lambda := \{a_{\Lambda,i}(t_i)\}_{i\in I}$ by

$$a_{\Lambda,i}(t_i) := \mathscr{E}_i^*(\mathsf{M})(t_i) \in \mathbb{Z}[z_{\mathsf{M}}, t_i] \qquad \text{for any } i \in I,$$

where $\mathscr{E}_i^*(\mathsf{M})$ is the lift of $\varepsilon_i^*(M)$ defined in [30, Definition 9.1] (see also [28, Definition 3.4]). We write $\beta := \mathrm{wt}(M) \in \mathsf{Q}_-$, $A := \mathbb{Z}[z_{\mathsf{M}}]$, and denote by $R_A^{a_\Lambda}$ the cyclotomic quiver Hecke algebra defined in [28, (1.2)]. By the construction, M is contained in $R_A^{a_\Lambda}(\Lambda+\beta)\text{-gMod} \cap \operatorname{Ker} \mathrm{E}_i$. Since we have

$$n := \langle h_i, \Lambda+\beta\rangle = \langle h_i, \Lambda\rangle + \langle h_i, \beta\rangle = \mathfrak{d}_i(M),$$

the following category equivalence (see [48, Lemma 4.14] and [17])

$$R_A^{a_\Lambda}(\Lambda+\beta)\text{-Mod} \cap \operatorname{Ker} \mathrm{E}_i \xrightarrow{\ \mathrm{F}_i^{\Lambda(n)}\ } R_A^{a_\Lambda}(s_i(\Lambda+\beta))\text{-Mod} \cap \operatorname{Ker} \mathrm{F}_i^{\Lambda}$$

implies that $\mathrm{F}_i^{\Lambda(n)}(\mathsf{M})$ is an affinization of $\mathrm{F}_i^{\Lambda(n)}(M) = \mathscr{K}_i(M)$, where F_i^{Λ} is the functor over $R_A^{a_\Lambda}$ categorifying the action of f_i on $V(\Lambda)$ (see [28, (1.3)]).

The reality of $\mathscr{K}_i(M)$ follows by applying Lemma 2.10 to the setting $X = \langle i^{\mathfrak{d}_i(M)}\rangle$ and $Y = M$.

(ii) Set $N := \widetilde{\mathrm{E}}_i^{\max}\widetilde{\mathrm{E}}_i^{*\max}(M) \simeq \widetilde{\mathrm{E}}_i^{*\max}\widetilde{\mathrm{E}}_i^{\max}(M)$. By [29, Proposition 2.16], the conditions are equivalent to $\mathfrak{d}_i(N) = \varepsilon_i(M) + \varepsilon_i^*(M)$.

(iii) Let $M \in (R\text{-gmod})_i$ and write $M' := \widetilde{\mathrm{E}}_i^{*\max}(M)$. Then we have

$$\begin{aligned}
\mathfrak{d}_i(M) &= \varepsilon_i^*(M) + \langle h_i, \mathrm{wt}(M)\rangle = \varphi_i^*(M),\\
\mathfrak{d}_i(M') &= \langle h_i, \mathrm{wt}(M')\rangle = 2\varepsilon_i^*(M) + \langle h_i, \mathrm{wt}(M)\rangle = \mathfrak{d}_i(M) + \varepsilon_i^*(M),
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathscr{K}_i^{*}\mathscr{F}_i(M) &= \mathscr{K}_i^{*}\left(\widetilde{\mathrm{F}}_i^{\varphi_i^*(M)}(M')\right) = \widetilde{\mathrm{F}}_i^{*\varepsilon_i^*(M)}\widetilde{\mathrm{F}}_i^{\varphi_i^*(M)}(M')\\
&= \widetilde{\mathrm{F}}_i^{\varphi_i^*(M)}\left(\widetilde{\mathrm{F}}_i^{*\varepsilon_i^*(M)}(M')\right) = \widetilde{\mathrm{F}}_i^{\mathfrak{d}_i(M)}(M)\\
&= \mathscr{K}_i(M),
\end{aligned}$$

where the third equality follows from [29, Theorem 2.17].

(iv) It can be proved by the same argument in (iii). $\square$

Lemma 2.20. *Let M and N be simple modules and one of them is affreal. If M and N commute, then*

(i) *$\mathscr{K}_i(M)$ and $\mathscr{K}_i(N)$ commute and $\mathscr{K}_i(M \circ N) \simeq \mathscr{K}_i(M) \circ \mathscr{K}_i(N)$,*
(ii) *$\mathscr{K}_i^*(M)$ and $\mathscr{K}_i^*(N)$ commute and $\mathscr{K}_i^*(M \circ N) \simeq \mathscr{K}_i^*(M) \circ \mathscr{K}_i^*(N)$.*

Proof. (i) Note that $\widetilde{\mathrm{E}}_i^{\max}(M \circ N) \simeq (\widetilde{\mathrm{E}}_i^{\max} M) \circ (\widetilde{\mathrm{E}}_i^{\max} N)$. Hence replacing M and N with $\widetilde{\mathrm{E}}_i^{\max} M$ and $\widetilde{\mathrm{E}}_i^{\max} N$, we may assume that $\varepsilon_i(M) = \varepsilon_i(N) = 0$.

Let $m = \mathfrak{d}_i(M)$ and $n = \mathfrak{d}_i(N)$. Since $m + n = \mathfrak{d}_i(M \circ N)$, we have

$$
\begin{aligned}
\mathscr{K}_i(M \circ N) = \langle i \rangle^{\circ(m+n)} \nabla (M \circ N) &\simeq \big(\langle i \rangle^{\circ(m+n)} \nabla M\big) \nabla N \\
&\simeq (\langle i \rangle^{\circ n} \nabla \mathscr{K}_i(M)) \nabla N \simeq (\mathscr{K}_i(M) \nabla \langle i \rangle^{\circ n}) \nabla N \\
&\simeq \mathscr{K}_i(M) \nabla (\langle i \rangle^{\circ n} \nabla N) \\
&\simeq \mathscr{K}_i(M) \nabla \mathscr{K}_i(N).
\end{aligned}
$$

In the same manner, we have $\mathscr{K}_i(N \circ M) \simeq \mathscr{K}_i(N) \nabla \mathscr{K}_i(M)$. As $M \circ N \simeq N \circ M$, we have

$$\mathscr{K}_i(M) \nabla \mathscr{K}_i(N) \simeq \mathscr{K}_i(N) \nabla \mathscr{K}_i(M).$$

Then Lemma 2.19 (i) says that one of $\mathscr{K}_i(M)$ and $\mathscr{K}_i(N)$ is affreal, which implies the assertion by [18, Corollary 3.9] and [35, Proposition 2.10].

(ii) It can be proved in the same manner as above. $\square$

Lemma 2.21. *Let M be an affreal simple module, N a simple module and $i \in I$. Let $a \in \mathbb{Z}$ such that $0 \leqslant a \leqslant \mathfrak{d}_i(N)$. We assume that M and N commute.*

(i) *If M and $\widetilde{\mathrm{F}}_i^{*a} N$ commute, then $\mathscr{K}_i(M)$ and $\widetilde{\mathrm{F}}_i^{\mathfrak{d}_i(N)-a} N$ commute.*
(ii) *If M and $\widetilde{\mathrm{F}}_i^{a} N$ commute, then $\mathscr{K}_i^*(M)$ and $\widetilde{\mathrm{F}}_i^{*\,\mathfrak{d}_i(N)-a} N$ commute.*

Proof. Since (ii) can be proved in the same manner, we focus on (i).

We may assume that $\langle i \rangle$ is not a convolution factor of M. Set $n := \mathfrak{d}_i(N) - a \in \mathbb{Z}_{\geqslant 0}$ and $m := \mathfrak{d}_i(M)$. Since $(\widetilde{\mathrm{F}}_i^n N, \mathscr{K}_i M, \langle i^a \rangle)$ is normal by the simplicity of $\mathscr{K}_i M \circ \langle i^a \rangle$, we have

$$
\begin{aligned}
\widetilde{\mathrm{F}}_i^{*a}\big(\widetilde{\mathrm{F}}_i^n N \nabla \mathscr{K}_i M\big) &\simeq \mathrm{hd}\big(\widetilde{\mathrm{F}}_i^n N \circ \mathscr{K}_i M \circ \langle i^a \rangle\big) \\
&\simeq \mathrm{hd}\big((\widetilde{\mathrm{F}}_i^n N) \circ \langle i^a \rangle \circ \mathscr{K}_i M\big) \\
&\simeq \mathrm{hd}\big(\widetilde{\mathrm{F}}_i^n (N \nabla \langle i^a \rangle) \circ \mathscr{K}_i M\big) \\
&\simeq \mathrm{hd}\big(\mathscr{K}_i (N \nabla \langle i^a \rangle) \circ \mathscr{K}_i M\big)
\end{aligned}
$$

$$\underset{*}{\simeq} \mathscr{K}_i\big((\widetilde{\mathrm{F}}_i^{*a} N) \circ M\big),$$

where $\underset{*}{\simeq}$ follows from Lemma 2.20. On the other hand, we have

$$\begin{aligned}\mathscr{K}_i M \nabla \widetilde{\mathrm{F}}_i^n N &\simeq \mathrm{hd}\big(\mathscr{K}_i M \circ \langle i^n \rangle \circ N\big) \simeq \mathrm{hd}\big(\langle i^n \rangle \circ \mathscr{K}_i M \circ N\big)\\ &\simeq \mathrm{hd}\big(\langle i^{n+m} \rangle \circ M \circ N\big),\end{aligned}$$

which implies

$$\begin{aligned}\widetilde{\mathrm{F}}_i^{*a}\big(\mathscr{K}_i M \nabla \widetilde{\mathrm{F}}_i^n N\big) &\underset{*}{\simeq} \mathrm{hd}\big(\langle i^{n+m} \rangle \circ M \circ N \circ \langle i^a \rangle\big)\\ &\simeq \mathrm{hd}\big(\langle i^{n+m} \rangle \circ M \circ (\widetilde{\mathrm{F}}_i^{*a} N)\big)\\ &\simeq \mathscr{K}_i(M \circ \widetilde{\mathrm{F}}_i^{*a} N).\end{aligned}$$

Here $\underset{*}{\simeq}$ follows the fact that $\langle i^b \rangle \circ X \circ \langle i^c \rangle$ has a simple head for any simple X if $b + c \leqslant \mathfrak{d}_i(X)$ ([29, Lemma 2.17]). In conclusion, we have

$$\widetilde{\mathrm{F}}_i^{*a}\big(\widetilde{\mathrm{F}}_i^n N \nabla \mathscr{K}_i M\big) \simeq \widetilde{\mathrm{F}}_i^{*a}\big(\mathscr{K}_i M \nabla \widetilde{\mathrm{F}}_i^n N\big),$$

which implies that $\widetilde{\mathrm{F}}_i^n N \nabla \mathscr{K}_i M \simeq \mathscr{K}_i M \nabla \widetilde{\mathrm{F}}_i^n N$. Thus we have the assertion. □

Lemma 2.22. *Let M and N be simple modules. Assume that one of them is affreal.*

(i) *If $\varepsilon_i(N) = 0$, then we have*

$$\widetilde{\mathrm{F}}_i^{\mathfrak{d}_i(M)+\mathfrak{d}_i(N)}(M \nabla N) \simeq \mathscr{K}_i(M) \nabla \mathscr{K}_i(N) \quad \textit{and}$$
$$\Lambda\big(\mathscr{K}_i(M), \mathscr{K}_i(N)\big) = \Lambda(M,N) + \mathfrak{d}_i(M) \cdot \Lambda(\langle i \rangle, N) - \mathfrak{d}_i(N) \cdot \Lambda(\langle i \rangle, M).$$

In particular, we have

$$\mathscr{K}_i(M) \nabla \mathscr{K}_i(N) \simeq \langle i^a \rangle \circ \mathscr{K}_i(M \nabla N) \quad \textit{where } a = \mathfrak{d}_i(M) + \mathfrak{d}_i(N) - \mathfrak{d}_i(M \nabla N).$$

(ii) *If $\varepsilon_i^*(M) = 0$, then we have*

$$\widetilde{\mathrm{F}}_i^{*\,\mathfrak{d}_i(M)+\mathfrak{d}_i(N)}(M \nabla N) \simeq \mathscr{K}_i^*(M) \nabla \mathscr{K}_i^*(N) \quad \textit{and}$$
$$\Lambda\big(\mathscr{K}_i^*(M), \mathscr{K}_i^*(N)\big) = \Lambda(M,N) + \mathfrak{d}_i(N) \cdot \Lambda(M, \langle i \rangle) - \mathfrak{d}_i(M) \cdot \Lambda(N, \langle i \rangle).$$

In particular, we have

$$\mathscr{K}_i^*(M) \nabla \mathscr{K}_i^*(N) \simeq \mathscr{K}_i^*(M \nabla N) \circ \langle i^a \rangle \quad \textit{where } a = \mathfrak{d}_i(M) + \mathfrak{d}_i(N) - \mathfrak{d}_i(M \nabla N).$$

(iii) *If $\varepsilon_i(M) = \varepsilon_i(N) = 0$, then we have*

$$\mathfrak{d}\big(\mathscr{K}_i(M), \mathscr{K}_i(N)\big) = \mathfrak{d}(M,N).$$

(iv) *If* $\varepsilon_i^*(M) = \varepsilon_i^*(N) = 0$, *then we have*

$$\mathfrak{d}\big(\mathscr{K}_i^*(M), \mathscr{K}_i^*(N)\big) = \mathfrak{d}(M, N).$$

Proof. (i) We set $m := \mathfrak{d}_i(M)$ and $n := \mathfrak{d}_i(N)$. Since $(\langle i \rangle, N)$ is an unmixed pair, the triple $(\langle i^{m+n} \rangle, M, N)$ is a normal sequence by [29, Corollary 2.13]. Hence $\langle i^{m+n} \rangle \circ M \circ N$ has a simple head and

$$\begin{aligned} \mathrm{hd}\big(\langle i^{m+n} \rangle \circ M \circ N\big) &\simeq \mathrm{hd}\big(\langle i^n \rangle \circ \mathscr{K}_i(M) \circ N\big) \simeq \mathrm{hd}\big(\mathscr{K}_i(M) \circ \langle i^n \rangle \circ N\big) \\ &\simeq \mathscr{K}_i(M) \nabla \mathscr{K}_i(N). \end{aligned}$$

The normality of $(\langle i \rangle, M, N)$ tells us that $\Lambda\big(\mathscr{K}_i(M), N\big) = \Lambda(\langle i^m \rangle, N) + \Lambda(M, N)$, which implies

$$\begin{aligned} \Lambda\big(\mathscr{K}_i(M), \mathscr{K}_i(N)\big) &= \Lambda\big(\mathscr{K}_i(M), \langle i^n \rangle \nabla N\big) = \Lambda\big(\mathscr{K}_i(M), \langle i^n \rangle\big) + \Lambda\big(\mathscr{K}_i(M), N\big) \\ &= -n\Lambda\big(\langle i \rangle, \mathscr{K}_i(M)\big) + \Lambda(\langle i^m \rangle, N) + \Lambda(M, N) \\ &= -n\Lambda(\langle i \rangle, M) + m\Lambda(\langle i \rangle, N) + \Lambda(M, N), \end{aligned}$$

where the second and third equality follow from the fact that $\mathscr{K}_i(M)$ and $\langle i \rangle$ commute. Since $\mathfrak{d}_i(M) + \mathfrak{d}_i(N) \geqslant \mathfrak{d}_i(M \nabla N)$ (see [19, Section 3.2] and see also [27, Proposition 4.2]), we have the last assertion.

(ii) It can be proved in the same manner as (i).

(iii) and (iv) follow from (i) and (ii) respectively. □

For a module $M \in R$-gmod, we denote by $\ell(M)$ the *composition length of* M.

Lemma 2.23. *Let* M *and* N *be simple modules such that one of them is affreal and* $\ell(M \circ N) \leqslant 2$.

(i) *If* $\varepsilon_i(M) = \varepsilon_i(N) = 0$, *then we have either*
  (a) $\mathscr{K}_i(M) \nabla \mathscr{K}_i(N) \simeq \mathscr{K}_i(M \nabla N)$ *and* $\mathfrak{d}_i(M) + \mathfrak{d}_i(N) = \mathfrak{d}_i(M \nabla N)$, *or*
  (b) $\mathscr{K}_i(N) \nabla \mathscr{K}_i(M) \simeq \mathscr{K}_i(N \nabla M)$ *and* $\mathfrak{d}_i(M) + \mathfrak{d}_i(N) = \mathfrak{d}_i(N \nabla M)$.

(ii) *If* $\varepsilon_i^*(M) = \varepsilon_i^*(N) = 0$, *then we have either*
  (a) $\mathscr{K}_i^*(M) \nabla \mathscr{K}_i^*(N) \simeq \mathscr{K}_i^*(M \nabla N)$ *and* $\mathfrak{d}_i(M) + \mathfrak{d}_i(N) = \mathfrak{d}_i(M \nabla N)$, *or*
  (b) $\mathscr{K}_i^*(N) \nabla \mathscr{K}_i^*(M) \simeq \mathscr{K}_i^*(N \nabla M)$ *and* $\mathfrak{d}_i(M) + \mathfrak{d}_i(N) = \mathfrak{d}_i(N \nabla M)$.

Proof. We focus on proving (i) since (ii) can be shown in the same manner.

If $\ell(M \circ N) = 1$, then the assertion is obvious. Hence, we may assume that $\ell(M \circ N) = 2$.

Then, we have the short exact sequence

$$0 \to N \nabla M \to M \circ N \to M \nabla N \to 0.$$

By Proposition 2.6, we have either

- $\Lambda(M \nabla N, \langle i \rangle) = \Lambda(M, \langle i \rangle) + \Lambda(N, \langle i \rangle)$ or
- $\Lambda(N \nabla M, \langle i \rangle) = \Lambda(M, \langle i \rangle) + \Lambda(N, \langle i \rangle)$.

Since the proof for the other case is similar, we assume the first case, i.e.,

$$\Lambda(M \nabla N, \langle i \rangle) = \Lambda(M, \langle i \rangle) + \Lambda(N, \langle i \rangle). \tag{2.14}$$

We write $m = \mathfrak{d}_i(M)$ and $n = \mathfrak{d}_i(N)$. The triple $(\langle i \rangle, M, N)$ is normal as the pair $(\langle i \rangle, N)$ is unmixed. Thus, by (2.14), we have

$$\begin{aligned}
\mathfrak{d}_i(M \nabla N) &= \frac{1}{(\alpha_i, \alpha_i)}\big(\Lambda(\langle i \rangle, M \nabla N) + \Lambda(M \nabla N, \langle i \rangle)\big) \\
&= \frac{1}{(\alpha_i, \alpha_i)}\big(\Lambda(\langle i \rangle, M) + \Lambda(\langle i \rangle, N) + \Lambda(M, \langle i \rangle) + \Lambda(N, \langle i \rangle)\big) \\
&= \mathfrak{d}_i(M) + \mathfrak{d}_i(N).
\end{aligned}$$

Hence the result follows from Lemma 2.22. □

Lemma 2.24. *Let M be a prime simple module.*

(i) *If $\varepsilon_i(M) = 0$, then $\mathscr{K}_i(M)$ is prime.*
(ii) *If $\varepsilon_i^*(M) = 0$, then $\mathscr{K}_i^*(M)$ is prime.*

Proof. As (ii) can be proved in the same manner, we prove only (i).

Let $\mathscr{K}_i(M) \simeq X \circ Y$. Then we have

$$M \simeq \widetilde{\mathrm{E}}_i^{\max} \mathscr{K}_i(M) \simeq \widetilde{\mathrm{E}}_i^{\max} X \circ \widetilde{\mathrm{E}}_i^{\max} Y.$$

This implies that either $\widetilde{\mathrm{E}}_i^{\max} X$ or $\widetilde{\mathrm{E}}_i^{\max} Y$ is isomorphic to $\mathbf{1}$, say $\widetilde{\mathrm{E}}_i^{\max} X \simeq \mathbf{1}$. Then we have $X \simeq \langle i^n \rangle$ for some $n \in \mathbb{Z}_{\geqslant 0}$, which implies $n = 0$ by Lemma 2.19 (ii). □

Lemma 2.25. *Let $i \in I$ and let M and N be affreal simple modules such that $\varepsilon_i(M) = \varepsilon_i(N) = 0$. Assume further that M and N have no common convolution factor other than $\mathbf{1}$. Then $\mathscr{K}_i(M)$ and $\mathscr{K}_i(N)$ have no common convolution factor other than $\mathbf{1}$.*

Proof. Assume that S is a convolution factor of $\mathscr{K}_i(M)$ and $\mathscr{K}_i(N)$. Then $\widetilde{\mathrm{E}}_i^{\max}(S)$ is a convolution factor of M and N. Hence $\widetilde{\mathrm{E}}_i^{\max}(S) \simeq \mathbf{1}$ and hence $S \simeq \langle i^n \rangle$ for some $n \in \mathbb{Z}_{\geqslant 0}$. Since S is a convolution factor of $\mathscr{K}_i(M)$, we have $n = 0$. □

3. Laurent families in $\mathscr{C}_{w,v}$

In this section, we investigate how the operations $\mathscr{K}_i$ and $\mathscr{F}_i$ act on Laurent families in the category $\mathscr{C}_{w,v}$.

3.1. **Stability for $\mathscr{C}_{w,v}$.** Let $w, v \in \mathsf{W}$. From the definitions of W and W*, we have

$$\mathrm{W}(X) \cup \mathrm{W}(Y) \subset \mathrm{W}(X \circ Y) \quad \text{and} \quad \mathrm{W}^*(X) \cup \mathrm{W}^*(Y) \subset \mathrm{W}^*(X \circ Y)$$

for *non-zero* modules X and Y in R-gmod. Thus we obtain the following:

$$\begin{aligned} &X \circ Y \in \mathscr{C}_w \text{ implies that } X \text{ and } Y \text{ belong to } \mathscr{C}_w, \\ &X \circ Y \in \mathscr{C}_{*,v} \text{ implies that } X \text{ and } Y \text{ belong to } \mathscr{C}_{*,v}. \end{aligned} \tag{3.1}$$

Proposition 3.1. *Let $w, v \in \mathsf{W}$ such that $s_i w < w$.*

(i) $\widetilde{\mathrm{E}}_i^{\max}$ *sends any simple in $\mathscr{C}_{w,v}$ to a simple in $\mathscr{C}_{s_i w,v}$.*
(ii) *If $s_i v < v$, then $\mathscr{F}_i \colon \mathscr{C}_{s_i w, s_i v} \to \mathscr{C}_{w,v}$ is an equivalence.*
(iii) *$\mathscr{K}_i$ sends any simple in either $\mathscr{C}_{s_i w}$ or $\mathscr{C}_w$ to a simple in $\mathscr{C}_w$.*
(iv) *$\mathscr{K}_i$ sends any simple in $\mathscr{C}_{*,v}$ to a simple in $\mathscr{C}_{*,v}$ whenever $s_i v > v$.*

Proof. By the assumption, we have $\langle i \rangle \in \mathscr{C}_w$.

(i) follows from Lemma 2.4 and the fact that $\mathscr{C}_{*,v}$ is stable by $\widetilde{\mathrm{E}}_i$.

(ii) follows from Proposition 2.3.

(iii) If M is a simple in $\mathscr{C}_{s_i w}$, then the statement follows from Proposition 2.3 and Lemma 2.19 (iii). We now assume that M is a simple in $\mathscr{C}_w$. If $\mathfrak{d}_i(M) = 0$, then

$$M \simeq \langle i^n \rangle \circ \mathscr{K}_i(M) \qquad \text{for some } n \in \mathbb{Z}_{\geqslant 0},$$

which gives the assertion by (3.1). If $\mathfrak{d}_i(M) > 0$, then the assertion follows from $\mathscr{K}_i(M) \simeq \widetilde{\mathrm{F}}_i^{\mathfrak{d}_i(M)} M$.

(iv) follows from the fact that $\mathscr{C}_{*,v}$ is stable by $\widetilde{\mathrm{F}}_i$. □

Lemma 3.2. *Let $i \in I$ and $w, v \in \mathsf{W}$ such that $v \leqslant w$.*

(i) *Assume that $s_i w < w$. Then we have*
  (a) $\varepsilon_i\big(\mathrm{M}(w\Lambda, v\Lambda)\big) \leqslant -\langle h_i, w\Lambda \rangle$ *for any $\Lambda \in \mathsf{P}_+$,*
  (b) $\varepsilon_i\big(\mathrm{M}(w\Lambda, v\Lambda)\big) = -\langle h_i, w\Lambda \rangle$ *for any $\Lambda \in \mathsf{P}_+$ if and only if $v \leqslant s_i w$.*
(ii) *Assume that $v < s_i v$. Then we have*
  (a) $\varepsilon_i^*\big(\mathrm{M}(w\Lambda, v\Lambda)\big) \leqslant \langle h_i, v\Lambda \rangle$ *for any $\Lambda \in \mathsf{P}_+$,*
  (b) $\varepsilon_i^*\big(\mathrm{M}(w\Lambda, v\Lambda)\big) = \langle h_i, v\Lambda \rangle$ *for any $\Lambda \in \mathsf{P}_+$ if and only if $s_i v \leqslant w$.*

Proof. Since the proof of (ii) is similar, we prove only (i). Since (a) and "if" part of (b) are well known (see [19, Lemma 9.1.4 and Lemma 9.1.5] for example), we will focus on proving the "only if" part of (b).

Assume that $\varepsilon_i\big(\mathrm{M}(w\Lambda, v\Lambda)\big) = -\langle h_i, w\Lambda\rangle$ for any $\Lambda \in \mathsf{P}_+$. The isomorphism class of $\mathrm{M}(w\Lambda, v\Lambda)$ corresponds to the element of $A_q(\mathfrak{n})$ given by $U_q^+(\mathfrak{g}) \ni a \mapsto (au_{w\Lambda}, u_{v\Lambda}) \in \mathbf{k}$. Putting $n := -\langle h_i, w\Lambda\rangle$, the element $\mathrm{E}_i^{(n)}\mathrm{M}(w\Lambda, v\Lambda)$ corresponds to

$$U_q^+(\mathfrak{g}) \ni a \longmapsto (ae_i^{(n)}u_{w\Lambda}, u_{v\Lambda}) = (au_{s_iw\Lambda}, u_{v\Lambda}).$$

Since it does not vanish, we have $u_{v\Lambda} \in U_q^+(\mathfrak{g})u_{s_iw\Lambda}$, which implies $v \leqslant s_iw$ by Lemma 2.1 $\square$

In the following theorem, we give a criterion on w and v to determine whether $\langle i\rangle$ is central in $\mathscr{C}_{w,v}$.

Theorem 3.3. *Let $i \in I$ and let $v, w \in \mathsf{W}$ such that*

$$v \leqslant w, \qquad v < s_iv \quad \textit{and} \quad s_iw < w.$$

Then the following conditions are equivalent:

(a) *$\langle i\rangle$ is a convolution factor of $\mathrm{M}(w\Lambda, v\Lambda)$ for some $\Lambda \in \mathsf{P}_+$,*
(b) *$\langle i\rangle$ lies in the center of $\mathscr{C}_{w,v}$, namely any simple in $\mathscr{C}_{w,v}$ commutes with $\langle i\rangle$,*
(c) *$s_iv \not\leqslant s_iw$.*

Proof. We have $v \leqslant s_iw$. Note also that $\langle i\rangle$ is contained in $\mathscr{C}_{w,v}$ by the assumption.

(a) $\Rightarrow$ (b) follows from [29, Theorem 4.4].

(b) $\Rightarrow$ (c) Assuming that

$$s_iv \leqslant s_iw, \tag{3.2}$$

we shall show that there exists a simple in $\mathscr{C}_{w,v}$ which does not commute with $\langle i\rangle$ by using induction on $\big(\ell(w), \ell(w)-\ell(v)\big)$. Here, the induction is taken with respect to the lexicographic order on $\mathbb{Z}^2_{\geqslant 0}$, i.e., $(a,b) > (c,d)$ if and only if either $(a > c)$ or ($a = c$ and $b > d$).

By (3.2), we have $\ell(s_iw) > 0$. Hence there exists $j \in I$ such that $s_iws_j < s_iw$, which implies

$$ws_j < w \quad \text{and} \quad s_iws_j < ws_j. \tag{3.3}$$

We now consider the following cases.

(i) Suppose that $vs_j < v$. Since $vs_j < v < s_iv$, we have $vs_j < s_ivs_j$ and $s_ivs_j < s_iv$. By (3.2) and (3.3), we have

$$vs_j \leqslant ws_j \quad \text{and} \quad s_ivs_j \leqslant s_iws_j.$$

By the induction hypothesis, $\langle i\rangle$ is not in the center of $\mathscr{C}_{ws_j,vs_j}$. There is an equivalence of monoidal categories $\widetilde{\mathscr{C}}_{w,v} \simeq \widetilde{\mathscr{C}}_{ws_j,vs_j}$ which sends $\langle i\rangle$ to $\langle i\rangle$, where

$\widetilde{\mathscr{C}}_{w,v}$ and $\widetilde{\mathscr{C}}_{ws_j,vs_j}$ are the localized categories ([32, Theorem 4.8]). Hence $\langle i\rangle$ is also not in the center of $\mathscr{C}_{w,v}$.

(ii) Suppose that $v < vs_j$.

(ii-1) Assume $s_iv < s_ivs_j$. By (3.2) and (3.3), we have

$$vs_j \leqslant w \quad \text{and} \quad s_ivs_j \leqslant s_iw.$$

The induction hypothesis implies that $\langle i\rangle$ is not in the center of $\mathscr{C}_{w,vs_j} \subset \mathscr{C}_{w,v}$.

(ii-2) Assume that $s_ivs_j < s_iv$. Since

$$v < vs_j, \quad s_i(vs_j) < vs_j \quad \text{and} \quad s_ivs_j < s_iv = (s_ivs_j)s_j,$$

we have $s_ivs_j \leqslant v$, which implies

$$v = s_ivs_j.$$

This tells us that $v\alpha_j = \alpha_i$, i.e., $h_j = v^{-1}h_i$, and

$$\mathrm{M}(w\Lambda_j, s_iv\Lambda_j) = \mathrm{M}(w\Lambda_j, vs_j\Lambda_j) \in \mathscr{C}_{w,vs_j} \subset \mathscr{C}_{w,v}.$$

We have

$$\langle h_i, s_iv\Lambda_j\rangle = -\langle v^{-1}h_i, \Lambda_j\rangle = -\langle h_j, \Lambda_j\rangle = -1.$$

Hence, we have

$$\begin{aligned}\mathfrak{d}_i\big(\mathrm{M}(w\Lambda_j, s_iv\Lambda_j)\big) = \varepsilon_i\big(\mathrm{M}(w\Lambda_j, s_iv\Lambda_j)\big) &+ \varepsilon_i^*\big(\mathrm{M}(w\Lambda_j, s_iv\Lambda_j)\big)\\ &+\langle h_i, w\Lambda_j - s_iv\Lambda_j\rangle.\end{aligned}$$

By Lemma 3.2, we have $\varepsilon_i\big(\mathrm{M}(w\Lambda_j, s_iv\Lambda_j)\big) = -\langle h_i, w\Lambda_j\rangle$, and $v < s_iv$ implies that $\varepsilon_i^*\big(\mathrm{M}(w\Lambda_j, s_iv\Lambda_j)\big) = 0$. Hence we obtain $\mathfrak{d}_i\big(\mathrm{M}(w\Lambda_j, s_iv\Lambda_j)\big) = \langle h_i, -s_iv\Lambda_j\rangle = 1$. Thus, $\mathrm{M}(w\Lambda_j, s_iv\Lambda_j)$ does not commute with $\langle i\rangle$ since $\mathfrak{d}_i\big(\mathrm{M}(w\Lambda_j, s_iv\Lambda_j)\big) = 1$.

(c) $\Rightarrow$ (a) Assume that $s_iv \nleqslant s_iw$. As $v \leqslant w$, $v < s_iv$ and $s_iw < w$, we have $s_iv \leqslant w$. Set

$$C := \mathrm{M}(w\Lambda, v\Lambda) \quad \text{and} \quad n := \varepsilon_i(C) = -\langle h_i, w\Lambda\rangle.$$

By Lemma 3.2, there exists $\Lambda \in \mathsf{P}_+$ such that

$$\varepsilon_i\big(\widetilde{\mathrm{E}}_i^{*\max}(C)\big) = \varepsilon_i\big(\mathrm{M}(w\Lambda, s_iv\Lambda)\big) < n = \varepsilon_i(C).$$

Hence Lemma 2.2 says that $\langle i\rangle$ is a convolution factor of C. $\square$

Lemma 3.4. *Let $w, v \in \mathsf{W}$ and $i \in I$. Assume that $v < w$, $s_i w < w$, and $v < s_i v$. Then for $\Lambda \in \mathsf{P}_+$ we have*

$$\mathrm{M}(w\Lambda, v\Lambda) \simeq \langle i^n \rangle \circ \mathscr{K}_i(\mathrm{M}(s_i w\Lambda, v\Lambda)) \quad \text{for some } n \in \mathbb{Z}_{\geqslant 0}.$$

Proof. Since $\mathrm{M}(s_i w\Lambda, v\Lambda) \simeq \widetilde{\mathrm{E}}_i^{\max}(\mathrm{M}(w\Lambda, v\Lambda))$, we have

$$\mathscr{K}_i(\mathrm{M}(w\Lambda, v\Lambda)) = \mathscr{K}_i(\mathrm{M}(s_i w\Lambda, v\Lambda)).$$

Note that $\langle i \rangle \in \mathscr{C}_{w,v}$ and $\mathfrak{d}_i(\mathrm{M}(w\Lambda, v\Lambda)) = 0$ ([29, Theorem 4.4]). By Lemma 2.19 (ii), we conclude that

$$\mathrm{M}(w\Lambda, v\Lambda) \simeq \langle i^n \rangle \circ \mathscr{K}_i(\mathrm{M}(w\Lambda, v\Lambda)) \simeq \langle i^n \rangle \circ \mathscr{K}_i(\mathrm{M}(s_i w\Lambda, v\Lambda))$$

for some $n \in \mathbb{Z}_{\geqslant 0}$. □

3.2. **Stability of Laurent families.** In this subsection, we investigate properties of $\mathscr{K}_i$ and $\mathscr{F}_i$ related to Laurent families. Recall the notion of Laurent families given in Section 2.7.

The following proposition says that $\mathscr{F}_i$ and $\mathscr{K}_i$ are compatible with Laurent families.

Proposition 3.5. *Let $w, v \in \mathsf{W}$ and $i \in I$ such that $v < w$, $v < s_i v$ and $s_i w < w$, and let $\mathcal{M} = \{M_k\}_{k \in K}$ be a Laurent family in $\mathscr{C}_{s_i w, v}$. Then, we have*

(i) *$\mathscr{F}_i(\mathcal{M})$ is a Laurent family in $\mathscr{C}_{w, s_i v}$,*

(ii) *$\mathcal{M}' := \{\langle i \rangle\} \cup \mathscr{K}_i(\mathcal{M})$ is a Laurent family in $\mathscr{C}_{w,v}$.*

Proof. (i) is obvious since $\mathscr{F}_i \colon \mathscr{C}_{s_i w, v} \to \mathscr{C}_{w, s_i v}$ is an equivalence of monoidal categories by Proposition 3.1 (ii).

(ii) Set $K' := \{0\} \sqcup K$ and

$$M'_0 := \langle i \rangle \quad \text{and} \quad M'_k := \mathscr{K}_i(M_k)$$

so that $\mathcal{M}' = \{M'_k\}_{k \in K'}$. Note that $\widetilde{\mathrm{E}}_i^{\max} M'_k \simeq M_k$ for any $k \in K$.

Suppose that $\mathbf{a}' = (a_k)_{k \in K'}$ and $\mathbf{b}' = (b_k)_{k \in K'}$ satisfy $\mathcal{M}'(\mathbf{a}') \simeq \mathcal{M}'(\mathbf{b}')$. Applying $\widetilde{\mathrm{E}}_i^{\max}$, we obtain $\mathop{\circ}\limits_{k \in K} {M_k}^{\circ a_k} \simeq \mathop{\circ}\limits_{k \in K} {M_k}^{\circ b_k}$. Hence we obtain $a_k = b_k$ for any $k \in K$ by the assumption, which implies $a_0 = b_0$. This shows that $\mathcal{M}'$ is independent.

Let $X \in \mathscr{C}_{w,v}$ be a simple module commuting with M'_k for any $k \in K'$. By Proposition 3.1 (i) and [26, Lemma 3.1], $\widetilde{\mathrm{E}}_i^{\max} X \in \mathscr{C}_{s_i w, v}$ commutes with every $\widetilde{\mathrm{E}}_i^{\max} M'_k \simeq M_k$ for $k \in K$. Hence the assumption says that there exists $\mathbf{a} = (a_k)_{k \in K}$ such that $\widetilde{\mathrm{E}}_i^{\max} X \simeq \mathop{\circ}\limits_{k \in K} {M_k}^{\circ a_k}$. Since X commutes with $\langle i \rangle$, there exists a_0 such that

$$X \simeq \langle i^{a_0} \rangle \circ \mathscr{K}_i(\widetilde{\mathrm{E}}_i^{\max} X) \simeq \mathop{\circ}\limits_{k \in K'} {M'_k}^{\circ a_k},$$

which completes the proof. □

Let $\mathcal{M}$ be a commuting family of affreal simple modules in $\mathscr{C}_{w,v}$. Let us consider the following condition on $\mathcal{M}$:

$$\text{(3.4)}\qquad \begin{array}{l}\text{Any } M \in \mathcal{M} \text{ satisfies exactly one of the following conditions:}\\ \quad \text{(a) there exists a simple module in } \mathscr{C}_{w,v} \text{ which does not commute with } M,\\ \quad \text{(b) } M \text{ is a convolution factor of } \mathrm{M}(w\Lambda, v\Lambda) \text{ for some } \Lambda \in \mathsf{P}_+.\end{array}$$

Proposition 3.6. *Let $i \in I$ and let $w, v \in \mathsf{W}$ such that $v < w$, $v < s_i v$, and $s_i w < w$. Let $\mathcal{M}$ be a commuting family of affreal simple modules in $\mathscr{C}_{s_i w,v}$ satisfying (3.4).*

(i) *$\mathscr{F}_i(\mathcal{M}) \subset \mathscr{C}_{w,s_i v}$ also satisfies (3.4),*

(ii) *$\{\langle i\rangle\} \cup \mathscr{K}_i(\mathcal{M}) \subset \mathscr{C}_{w,v}$ also satisfies (3.4).*

Proof. (i) It follows from Proposition 3.1 (ii) and the fact that $\mathscr{F}_i\big(\mathrm{M}(s_i w\Lambda, v\Lambda)\big) \simeq \mathrm{M}(w\Lambda, s_i v\Lambda)$ for any $\Lambda \in \mathsf{P}_+$.

(ii) It follows from Theorem 3.3 and Lemma 3.4 together with the following: for an affreal simple M and a simple N in $\mathscr{C}_{s_i w,v}$, $\mathscr{K}_i(M)$ and $\mathscr{K}_i(N)$ do not strongly commute if M and N do not strongly commute by Lemma 2.22 (iii). □

4. Monoidal categorifications

From now until the end of the paper, we assume that R is symmetric. Hence the Cartan matrix C is symmetric, and we choose a bilinear form $(\cdot\,,\cdot)$ on the weight lattice P such that $(\alpha_i, \alpha_i) = 2$ for all $i \in I$. Note that until now the theory of monoidal categorification by quiver Hecke algebras is available only in the symmetric case.

4.1. Monoidal seeds.

In this subsection, we briefly review the monoidal categorification of $\mathscr{C}_w$ ([19]).

We first recall the notion of monoidal categorification. Let $\mathcal{C}$ be a full subcategory of R-gmod which has $\mathbf{1}$ and is stable under taking convolution products, extensions, subquotients and grading shifts.

Definition 4.1 ([19])**.** Let J be an index set with a decomposition $\mathsf{J} = \mathsf{J}^{\mathrm{ex}} \sqcup \mathsf{J}^{\mathrm{fr}}$.

(i) A triple $\Sigma = (\{M_i\}_{i\in\mathsf{J}}, L, \widetilde{B})$ is a *quantum monoidal seed* in $\mathcal{C}$ if it satisfies the following:
  - (a) $\{M_i\}_{i\in\mathsf{J}}$ is a commuting family of real simple modules in $\mathcal{C}$,
  - (b) $L = (l_{ij})_{i,j\in\mathsf{J}}$ is an integer-valued skew-symmetric matrix,
  - (c) $\widetilde{B} = (b_{ij})_{i\in\mathsf{J}, j\in\mathsf{J}^{\mathrm{ex}}}$ is an exchange matrix,
  - (d) $M_i \circ M_j \simeq q^{l_{ij}} M_j \circ M_i$ for any $i, j \in \mathsf{J}$, i.e., $\Lambda(M_j, M_i) = l_{ij}$,
  - (e) $(L, \widetilde{B})$ is compatible with $d = 2$ (see Section 2.1),

(f) $\sum_{i\in \mathsf{J}} b_{i,k}\mathrm{wt}(M_i) = 0$ for all $k \in \mathsf{J}^{\mathrm{ex}}$.

(ii) A quantum monoidal seed $\Sigma = (\{M_i\}_{i\in\mathsf{J}}, L, \widetilde{B})$ in $\mathcal{C}$ admits a *mutation* in direction $k \in \mathsf{J}^{\mathrm{ex}}$ if

(a) there exists a real simple module $M'_k \in \mathcal{C}$ such that $\mathfrak{d}(M_k, M'_k) = 1$ and an exact sequence

$$0 \to q^a \mathop{\circ}_{b_{ik}>0} M_i^{\circ b_{ik}} \to M_k \circ M'_k \to q^b \mathop{\circ}_{b_{ik}<0} M_i^{\circ(-b_{ik})} \to 0,$$

where the exponents a, b are determined by Λ (see [19, Definition 6.2.3] for more details on the exponents a, b),

(b) the triple $\mu_k(\Sigma) = (\{M_i\}_{i\in\mathsf{J}\setminus\{k\}} \cup \{M'_k\}, \mu_k(L), \mu_k(\widetilde{B}))$ is a quantum monoidal seed in $\mathcal{C}$.

(iii) If a quantum monoidal seed Σ admits successive mutations in all directions, then we say that Σ is *completely admissible*.

(iv) The category $\mathcal{C}$ is called a *monoidal categorification* of a quantum cluster algebra A over $\mathbb{Z}[q^{\pm 1/2}]$ if

(a) the Grothendieck ring $\mathbb{Z}[q^{\pm1/2}] \otimes_{\mathbb{Z}[q^{\pm1}]} K(\mathcal{C})$ is isomorphic to A,

(b) there exists a completely admissible quantum monoidal seed $\Sigma = (\{M_i\}_{i\in\mathsf{J}}, L, \widetilde{B})$ in $\mathcal{C}$ such that $(\{q^{-(\mathrm{wt}(M_i),\mathrm{wt}(M_i))/4}[M_i]\}_{i\in\mathsf{J}}, L, \widetilde{B})$ is a quantum seed of A.

(v) Let $\Sigma = (\{M_i\}_{i\in\mathsf{J}}, L, \widetilde{B})$ in $\mathcal{C}$ be a quantum monoidal seed in $\mathcal{C}$. A simple object X of $\mathcal{C}$ is called a Σ-*monomial* if $X \simeq \mathop{\circ}_{k\in\mathsf{J}} M_k^{\circ n_k}$ for some $(n_k)_{k\in\mathsf{J}} \in \mathbb{Z}_{\geqslant 0}^{\mathsf{J}}$.

By [24, Lemma 3.2], the exchange matrix $\widetilde{B}$ can be computed by using the matrix Λ and $\{\mathrm{wt}(M_s)\}_{s\in\mathsf{J}}$. Hence $\{M_s\}_{s\in\mathsf{J}}$ along with the decomposition $\mathsf{J} = \mathsf{J}^{\mathrm{ex}} \sqcup \mathsf{J}^{\mathrm{fr}}$ determines the exchange matrix $\widetilde{B}$. For a monoidal seed $\Sigma = (\{M_s\}_{s\in\mathsf{J}}, -\Lambda, \widetilde{B})$, it suffices to write

$$\mathscr{S} = (M_s)_{s\in\mathsf{J}}$$

to denote the seed Σ instead of specifying the entire triple. From now on, we simply write $\mathscr{S} = (M_s)_{s\in\mathsf{J}}$ for the monoidal seed Σ instead of writing the whole triple, and simply call it a seed instead of calling a quantum monoidal seed. We sometimes write $\mathscr{S}_{\mathrm{ex}} := (M_s)_{s\in\mathsf{J}^{\mathrm{ex}}}$ and $\mathscr{S}_{\mathrm{fr}} := (M_s)_{s\in\mathsf{J}^{\mathrm{fr}}}$.

The following proposition immediately follows from Proposition 2.17.

Proposition 4.2. *Let $\mathscr{S} = (M_k)_{k\in\mathsf{J}}$ be a seed of $\mathcal{C}$, and let $\mathscr{S}' = (M'_k)_{k\in\mathsf{J}}$ be the mutation of $\mathscr{S}$ at $t \in \mathsf{J}^{\mathrm{ex}}$. Assume that $\mathscr{S}$ is a Laurent family. Then $\mathscr{S}'$ is also a Laurent family.*

We now focus on the category $\mathscr{C}_w$. Let $w \in \mathsf{W}$ and choose a reduced expression of w

$$\underline{w} = (i_1, i_2, \dots, i_r).$$

For an index $1 \leqslant k \leqslant r$ and $j \in I$, we set

$$k_{\underline{w}}(j)^+ := \min\big(\{t \in [1,r] \mid t \geqslant k,\ i_t = j\} \sqcup \{+\infty\}\big),$$
$$k_{\underline{w}}(j)^- := \max\big(\{t \in [1,r] \mid t \leqslant k,\ i_t = j\} \sqcup \{-\infty\}\big),$$

and

$$k_{\underline{w}}^+ := \min\big(\{t \in [1,r] \mid t > k,\ i_t = i_k\} \sqcup \{+\infty\}\big),$$
$$k_{\underline{w}}^- := \max\big(\{t \in [1,r] \mid t < k,\ i_t = i_k\} \sqcup \{-\infty\}\big).$$

We often drop $\underline{w}$ in the above notations when no confusion arises.

An interval $[a,b] \subseteq [1,r]$ is called an *i-box of $\underline{w}$* if $a \leqslant b$ and $i_a = i_b$. From any interval $[a,b] \subseteq [1,r]$, we obtain two i-boxes

$$[a,b\} := [a, b(i_a)^-] \quad \text{and} \quad \{a,b] := [a(i_b)^+, b].$$

A chain $\mathfrak{C} = ([a_t,b_t])_{1\leqslant t\leqslant r}$ of i-boxes of $\underline{w}$ is called *admissible* if, for each $t = 1,\dots,r$, the union

$$[\tilde{a}_t, \tilde{b}_t] := \bigcup_{1\leqslant j\leqslant t} [a_j, b_j]$$

is an interval of length t and either

$$[a_t,b_t] = [\tilde{a}_t, \tilde{b}_t\} \quad \text{or} \quad [a_t,b_t] = \{\tilde{a}_t, \tilde{b}_t].$$

For any i-box $[a,b]$, the corresponding *determinantial module* is defined by

$$\mathrm{M}_{\underline{w}}[a,b] := \mathrm{hd}(\mathrm{S}_b \circ \mathrm{S}_{b^-} \circ \cdots \circ \mathrm{S}_{a^+} \circ \mathrm{S}_a),$$

where $\mathrm{S}_k := \mathscr{F}_{i_1}\mathscr{F}_{i_2}\dots\mathscr{F}_{i_{k-1}}(\langle i_k\rangle)$ is the k-th *cuspidal module* for $k \in [1,r]$. We simply write $\mathrm{M}[a,b]$ for $\mathrm{M}_{\underline{w}}[a,b]$ if no confusion arises.

For each admissible chain $\mathfrak{C} = (\mathfrak{c}_k)_{k\in[1,r]}$ of i-boxes, we set

$$\mathscr{T}_{\mathfrak{C}} := (\mathrm{M}(\mathfrak{c}_k))_{k\in\mathsf{J}} \quad \text{with } \mathsf{J} = [1,r].$$

where $\mathsf{J}^{\mathrm{fr}} = \{k \in [1,r] \mid \ \mathfrak{c}_k = [1(i)^+, r(i)^-] \text{ for some } i \in I\}$. Define

$$\Lambda_{\mathfrak{C}} := (\Lambda(\mathrm{M}(\mathfrak{c}_i), \mathrm{M}(\mathfrak{c}_j)))_{i,j\in\mathsf{J}}$$

and $\widetilde{B}_{\mathfrak{C}}$ to be the exchange matrix determined by $\mathfrak{C}$ in [25, Section 3]. It was shown in [19, Section 11.2] (see also [25]) that the triple $\Sigma_{\mathfrak{C}} = (\mathscr{T}_{\mathfrak{C}}, -\Lambda_{\mathfrak{C}}, \widetilde{B}_{\mathfrak{C}})$ is a quantum monoidal seed of $\mathscr{C}_w$, and $\mathscr{C}_w$ provides a *monoidal categorification* of $A_q(\mathfrak{n}(w))$.

This tells us that every cluster monomial corresponds to a real simple module under the monoidal categorification. Such a simple module is called a *cluster monomial module*. When the cluster monomial is a cluster variable (resp. frozen variable), we call it a *cluster variable module* (resp. *frozen variable module*).

Remark 4.3. Even if we choose different reduced expressions and different admissible chains, the corresponding seeds are connected by mutations and permutations ([33, 25]). Hence the mutation class of $\Sigma_{\mathfrak{C}}$ does not depend on the choices of reduced expressions and admissible chains. We call it the *canonical mutation class of* $\mathscr{C}_w$.

The following monoidal seed arising from the admissible chain $\mathfrak{C} = (\{1, k])_{k\in[1,r]}$ is called the *GLS seed* of $\mathscr{C}_w$:

$$\mathscr{T}_{\underline{w},\mathrm{GLS}} := (\mathrm{M}_{\underline{w}}\{1, k])_{k\in[1,r]} \tag{4.1}$$

(see [12, 13] and see also [19, Section 11]).

5. Construction of monoidal seeds in $\mathscr{C}_w$

In this section, we give a method to construct a seed of $\mathscr{C}_w$, which may not be obtained by the method of admissible chains of i-boxes.

5.1. Operators $\mathscr{K}_i$ and $\mathscr{F}_i$.

By [24, Theorem 4.10], we have the following lemma.

Lemma 5.1. *Let $\mathscr{T} = (M_k)_{k\in\mathsf{J}}$ be a seed of $\mathscr{C}_w$ in the canonical mutation class, and let $t \in \mathsf{J}^{\mathrm{ex}}$.*

(i) *Let $\mathscr{T}' = (M'_k)_{k\in\mathsf{J}}$ be a seed of $\mathscr{C}_w$ in the canonical mutation class. If $M_k \simeq M'_k$ for $k \in \mathsf{J}\setminus\{t\}$ and $M_t \not\simeq M'_t$, then $\mathscr{T}' = \mu_t(\mathscr{T})$.*

(ii) *Suppose that a simple module $X \in \mathscr{C}_w$ satisfies:*
  (a) *X commutes M_k for any $k \in \mathsf{J}\setminus\{t\}$,*
  (b) *$\mathfrak{d}(X, M_t) = 1$,*
  (c) *$X \nabla M_t$ and $M_t \nabla X$ have no common convolution factor other than $\mathbf{1}$.*
  Then we have $X \simeq \mu_t(M_t)$.

Proof. (i) Set $\mathscr{T}'' = \mu_t(\mathscr{T}) = (M''_k)_{k\in\mathsf{J}}$. Since M'_t commutes with M_k for any $k \in \mathsf{J}\setminus\{t\}$, [24, Theorem 4.10] implies that M'_t is a $\mathscr{T}$-monomial or a $\mathscr{T}''$-monomial. Since $[M'_t]$ is prime, $M'_t = M''_t$, which implies that $\mathscr{T}' = \mathscr{T}''$.

(ii) Let $\mathscr{T}' = \mu_t(\mathscr{T})$ and let $M'_t = \mu_t(M_t)$. From the assumption, we have the following two exact sequences:

$$\begin{aligned} 0 \longrightarrow M'_t \nabla M_t \longrightarrow & M_t \circ M'_t \longrightarrow M_t \nabla M'_t \longrightarrow 0, \\ 0 \longrightarrow X \nabla M_t \longrightarrow & M_t \circ X \longrightarrow M_t \nabla X \longrightarrow 0. \end{aligned} \tag{5.1}$$

Combining [24, Theorem 4.10] with (a), we know that X is a $\mathscr{T}'$-monomial or a $\mathscr{T}$-monomial. Then (b) implies that X is a $\mathscr{T}'$-monomial, i.e., $X \simeq M_t'^{\circ n} \circ Y$ for some $n \in \mathbb{Z}_{\geqslant 0}$ and a $(\mathscr{T} \setminus \{M_t\})$-monomial Y. By (b), we have $n = 1$. Hence $X \nabla M_t \simeq Y \circ (M_t' \nabla M_t)$ and $M_t \nabla X \simeq Y \circ (M_t \nabla M_t')$. Then (c) implies that $Y \simeq \mathbf{1}$. □

Proposition 5.2. *Let $i \in I$ and $w \in \mathsf{W}$ such that $s_i w < w$. For a seed $\mathscr{T}$ of $\mathscr{C}_{s_i w}$ in the canonical mutation class, we define*

$$\widetilde{\mathscr{T}} := \{\mathrm{M}(w\Lambda_i, \Lambda_i)\} \cup \mathscr{F}_i(\mathscr{T}).$$

Then we have the following.

(i) *$\widetilde{\mathscr{T}}$ is a seed of $\mathscr{C}_w$ in the canonical mutation class.*

(ii) *Moreover, we have*

$$\widetilde{\mathscr{T}}_{\mathrm{ex}} = \begin{cases} \{\mathrm{M}(w\Lambda_i, s_i\Lambda_i)\} \cup \mathscr{F}_i(\mathscr{T}_{\mathrm{ex}}) & \text{if } s_i w > s_i, \\ \mathscr{F}_i(\mathscr{T}_{\mathrm{ex}}) & \text{otherwise,} \end{cases}$$

$$\widetilde{\mathscr{T}}_{\mathrm{fr}} = \begin{cases} \{\mathrm{M}(w\Lambda_i, \Lambda_i)\} \cup \big(\mathscr{F}_i(\mathscr{T}_{\mathrm{fr}}) \setminus \{\mathrm{M}(w\Lambda_i, s_i\Lambda_i)\}\big) & \text{if } s_i w > s_i, \\ \mathscr{F}_i(\mathscr{T}_{\mathrm{fr}}) \cup \{\mathrm{M}(w\Lambda_i, \Lambda_i) = \langle i \rangle\} & \text{otherwise.} \end{cases}$$

(iii) *If $\mathscr{T}'$ is a mutation of $\mathscr{T}$, then $\widetilde{\mathscr{T}'}$ is also a mutation of $\widetilde{\mathscr{T}}$.*

Proof. Since it is obvious when $\ell(w) = 1$, we assume that $\ell(w) > 1$.

Let $r = \ell(w)$ and take a reduced sequence $\underline{w} = (i_1, i_2, \ldots, i_r)$ of w with $i_1 = i$, and $\underline{s_i w} = (i_2, \ldots, i_r)$ of $s_i w$. Take a seed $\mathscr{T}_\circ = \big\{\mathrm{M}_{\underline{s_i w}}[k, r-1]\big\}_{k \in [1, r-1]}$ of $\mathscr{C}_{s_i w}$ which is in the canonical mutation class. Then

$$\widetilde{\mathscr{T}_\circ} = \{\mathrm{M}(w\Lambda_i, \Lambda_i\} \cup \mathscr{F}_i(\mathscr{T}_\circ) = \{\mathrm{M}_{\underline{w}}[k, r\}\}_{k \in [1, r]}$$

is a seed of $\mathscr{C}_w$. Note that $\widetilde{\mathscr{T}_\circ}$ is in the canonical mutation class.

Since $\mathscr{F}_i$ is a monoidal functor from $\mathscr{C}_{s_i w}$ to $\mathscr{C}_w$, taking $\widetilde{\mathscr{T}}$ is compatible with mutations. Since $\mathscr{T}$ is connected to $\mathscr{T}_\circ$ via mutations, we have (i) and (iii) .

If $s_i w \not> s_i$, then $\mathrm{M}(w\Lambda_i, s_i\Lambda_i)$ is trivial. Suppose that $s_i w > s_i$. Then $\mathrm{M}(s_i w\Lambda_i, \Lambda_i)$ is frozen in $\mathscr{C}_{s_i w}$ but $\mathrm{M}(w\Lambda_i, s_i\Lambda_i) = \mathscr{F}_i(\mathrm{M}(s_i w\Lambda_i, \Lambda_i))$ ([30, Lemma 10.6]) is exchangeable in $\mathscr{C}_w$. Thus we have (ii). □

Proposition 5.3. *Let $i \in I$ and $w \in \mathsf{W}$ such that $s_i w < w$. For a seed $\mathscr{T}$ of $\mathscr{C}_{s_i w}$ in the canonical mutation class, we define*

$$\widetilde{\mathscr{T}} := \{\langle i \rangle\} \cup \mathscr{K}_i(\mathscr{T}).$$

Then we have the following.

(i) $\widetilde{\mathscr{T}}$ *is a seed of* $\mathscr{C}_w$ *in the canonical mutation class.*
(ii) *Moreover, we have*

$$\widetilde{\mathscr{T}}_{\mathrm{ex}} = \begin{cases} \{\langle i \rangle\} \cup \mathscr{K}_i(\mathscr{T}_{\mathrm{ex}}) & \text{if } s_i w > s_i, \\ \mathscr{K}_i(\mathscr{T}_{\mathrm{ex}}) & \text{otherwise,} \end{cases}$$

$$\widetilde{\mathscr{T}}_{\mathrm{fr}} = \begin{cases} \mathscr{K}_i(\mathscr{T}_{\mathrm{fr}}) & \text{if } s_i w > s_i, \\ \{\langle i \rangle\} \cup \mathscr{K}_i(\mathscr{T}_{\mathrm{fr}}) & \text{otherwise.} \end{cases}$$

(iii) *If* $\mathscr{T}'$ *is a mutation of* $\mathscr{T}$*, then* $\widetilde{\mathscr{T}'}$ *is also a mutation of* $\widetilde{\mathscr{T}}$*.*

Proof. We may assume that $\ell(w) > 1$ since the statement is trivial otherwise.

Let $r = \ell(w)$ and take a reduced sequence $\underline{w} = (i_1, i_2, \ldots, i_r)$ of w with $i_1 = i$, and $\underline{s_i w} = (i_2, \ldots, i_r)$ of $s_i w$. Taking the seed

$$\mathscr{T}_\circ = \mathscr{T}_{\underline{s_i w},\mathrm{GLS}} = (\mathrm{M}_{\underline{s_i w}}\{1,k])_{k\in[1,r-1]} \tag{5.2}$$

of $\mathscr{C}_{s_i w}$, we have

$$\widetilde{\mathscr{T}_\circ} = \{\langle i \rangle\} \cup \mathscr{K}_i(\mathscr{T})) = (\mathrm{M}_{\underline{w}}\{1,k])_{k\in[1,r]} = \mathscr{T}_{\underline{w},\mathrm{GLS}} \tag{5.3}$$

which is a seed of $\mathscr{C}_w$ in the canonical mutation class.

We now consider a seed $\mathscr{T}$ of $\mathscr{C}_{s_i w}$, and $\mathscr{T}'$ is a mutation of $\mathscr{T}$. Suppose that $\widetilde{\mathscr{T}} = (\langle i \rangle, \mathscr{K}_i(\mathscr{T}))$ is a seed of $\mathscr{C}_w$. Let $M' \in \mathscr{T}'$ be the mutation of $M \in \mathscr{T}$ and set

$$X := M \nabla M' \quad \text{and} \quad Y := M' \nabla M.$$

Since M' is a mutation of M, $\mathfrak{d}(M, M') = 1$, X and Y are $\mathscr{T}$-monomials, and X and Y do not have a common convolution factor. Moreover M' commutes with any member of $\mathscr{T} \setminus \{M\}$.

Since the value ε_i of any module in $\mathscr{C}_{s_i w}$ is zero, we have

(a) if $\mathscr{K}_i(U) \simeq \mathscr{K}_i(V)$ for some simple modules $U, V \in \mathscr{C}_{s_i w}$, then $U \simeq V$,
(b) $\mathfrak{d}\big(\mathscr{K}_i(M), \mathscr{K}_i(M')\big) = 1$ by Lemma 2.22 (iii),
(c) $\mathscr{K}_i(M) \nabla \mathscr{K}_i(M') \simeq \langle i^m \rangle \circ \mathscr{K}_i(X)$ and $\mathscr{K}_i(M') \nabla \mathscr{K}_i(M) \simeq \langle i^n \rangle \circ \mathscr{K}_i(Y)$ for some $m, n \in \mathbb{Z}_{\geqslant 0}$ by Lemma 2.22 (i),
(d) $\mathscr{K}_i(X)$ and $\mathscr{K}_i(Y)$ are $\mathscr{K}_i(\mathscr{T})$-monomials and they have no common convolution factor other than $\mathbf{1}$.
(e) $\mathscr{K}_i(M')$ commutes with any member of $\mathscr{K}_i(\mathscr{T}) \setminus \{\mathscr{K}_i(M)\}$ by Lemma 2.22 (iii).

Since $\mathscr{K}_i(M')$ commutes with $\langle i \rangle$ and has no $\langle i \rangle$ as a factor by Lemma 2.19 (ii), either of m and n in (c) is zero. Lemma 5.1 (ii) says that $\mathscr{K}_i(M')$ is a mutation of $\mathscr{K}_i(M)$, i.e., $\widetilde{\mathscr{T}'}$ is a mutation of $\widetilde{\mathscr{T}}$. This implies (i) and (iii) by (5.2) and (5.3).

If $s_i w \not> s_i$, then $\langle i \rangle$ is a central object in $\mathscr{C}_w$ (up to a multiple of q), which says that $\langle i \rangle$ is frozen. If $s_i w > s_i$, then $\langle i \rangle$ is not central, i.e., not frozen. Hence we have (ii) $\square$

Proposition 5.4. *Let $i \in I$ and $w \in \mathsf{W}$ such that $s_i < s_i w < w$. Let $\mathscr{T}$ be a seed of $\mathscr{C}_{s_i w}$ in the canonical mutation class, and set $\widetilde{\mathscr{T}} := \{\langle i \rangle\} \cup \mathscr{K}_i(\mathscr{T})$, a seed of $\mathscr{C}_w$.*

- (i) *Let $Z \in \mathscr{C}_w$ be the mutation of $\langle i \rangle \in \widetilde{\mathscr{T}}$. Then we have*
  - (a) $\mathfrak{d}_i(Z) = 1$,
  - (b) $Y := \widetilde{\mathrm{E}}_i^{\max} Z \in \mathscr{C}_{s_i w}$ *is a $\mathscr{T}$-monomial,*
  - (c) $X := \widetilde{\mathrm{F}}_i^{*} Y$ *is a $\mathscr{T}$-monomial,*
  - (d) *there is an exact sequence*

$$0 \longrightarrow \mathscr{K}_i(X) \longrightarrow \langle i \rangle \circ Z \longrightarrow \mathscr{K}_i(Y) \longrightarrow 0.$$

- (ii) *Conversely, if X and Y are $\mathscr{T}$-monomials such that*
  - (a) $X \simeq \widetilde{\mathrm{F}}_i^{*} Y$,
  - (b) *X and Y have no common convolution factor,*

  then $\widetilde{\mathrm{F}}_i^{\mathfrak{d}_i(X)} Y \simeq \widetilde{\mathrm{E}}_i^{*} \mathscr{K}_i(X)$ is a mutation of $\langle i \rangle \in \widetilde{\mathscr{T}}$.

Proof. Note that $\widetilde{\mathscr{T}}$ is a seed of $\mathscr{C}_w$ in the canonical mutation class by Proposition 5.3.

(i) Since Z is a mutation of $\langle i \rangle$, we have $\mathfrak{d}_i(Z) = 1$ and there exist $\mathscr{T}$-monomials X and Y and an exact sequence

$$0 \longrightarrow \mathscr{K}_i(X) \longrightarrow \langle i \rangle \circ Z \longrightarrow \mathscr{K}_i(Y) \longrightarrow 0.$$

Note that $\varepsilon_i(X) = \varepsilon_i(Y) = 0$ since $X, Y \in \mathscr{C}_{s_i w}$. Hence we have $Y \simeq \widetilde{\mathrm{E}}_i^{\max} Z$ and $X \simeq \widetilde{\mathrm{E}}_i^{\max} \widetilde{\mathrm{F}}_i^{*} Z$. Since $\mathfrak{d}_i(Z) > 0$, we have $\widetilde{\mathrm{E}}_i^{\max} \widetilde{\mathrm{F}}_i^{*} Z \simeq \widetilde{\mathrm{F}}_i^{*} \widetilde{\mathrm{E}}_i^{\max} Z \simeq \widetilde{\mathrm{F}}_i^{*} Y$.

(ii) Set $Z = \widetilde{\mathrm{F}}_i^{\mathfrak{d}_i(X)} Y$. If $\mathfrak{d}_i(Y) = 0$, then $X \simeq Y \circ \langle i \rangle$, which contradicts (b). It follows that $\mathfrak{d}_i(Y) = \mathfrak{d}_i(X) + 1$ by [29, Proposition 2.16 (i)]. Thus we have $\mathfrak{d}_i(Z) = 1$, $\widetilde{\mathrm{F}}_i(Z) \simeq \mathscr{K}_i(Y)$ and $\widetilde{\mathrm{F}}_i^{*}(Z) \simeq \mathscr{K}_i(X)$. Hence we have an exact sequence

$$0 \longrightarrow \mathscr{K}_i(X) \longrightarrow \langle i \rangle \circ Z \longrightarrow \mathscr{K}_i(Y) \longrightarrow 0.$$

Applying Lemma 2.21 (i) to the setting $M = S$, $N = Y$ and $a = 1$ for any $S \in \mathscr{T}$, we obtain that Z commutes with $\mathscr{K}_i(S)$. By Lemma 2.25, $\mathscr{K}_i(X)$ and $\mathscr{K}_i(Y)$ have no common convolution factor other than $\mathbf{1}$. Then the assertion follows from Lemma 5.1 (ii). $\square$

5.2. **Seed construction for $\mathscr{C}_w$.** In this subsection, we present a new construction of monoidal seeds of $\mathscr{C}_w$.

Let $w \in \mathsf{W}$ and fix a reduced sequence of w

$$\underline{w} = (i_1, i_2, \dots, i_r)$$

throughout this subsection. Recall the notations $k^+_{\underline{w}}$, $k^-_{\underline{w}}$ given in Section 4. We set $w_{\geqslant k} := s_{i_k} s_{i_{k+1}} \cdots s_{i_r}$ for $k \in [1, r]$.

Let $\mathsf{c} = (c_1, \dots, c_r) \in \{\mathsf{k}, \mathsf{f}\}^r$. The sequence c is called a *KF sequence* for $\mathscr{C}_w$. We define

$$\mathscr{T}_{\underline{w},\mathsf{c}} = (M_k)_{k \in [1,r]}, \tag{5.4}$$

where M_k is given by

$$M_k := \begin{cases} \mathscr{G}_1 \cdots \mathscr{G}_{k-1}(\langle i_k \rangle) & \text{if } c_k = \mathsf{k}, \\ \mathscr{G}_1 \cdots \mathscr{G}_{k-1}\big(\mathrm{M}(w_{\geqslant k}\Lambda_{i_k}, \Lambda_{i_k})\big) & \text{if } c_k = \mathsf{f}, \end{cases} \tag{5.5}$$

and

$$\mathscr{G}_t := \begin{cases} \mathscr{K}_{i_t} & \text{if } c_t = \mathsf{k}, \\ \mathscr{F}_{i_t} & \text{if } c_t = \mathsf{f}. \end{cases}$$

Remark 5.5. Let $\mathsf{c}' = (c'_s)_{s \in [1,r]}$ be another KF sequence which satisfies $c'_s = c_s$ for any $s \in [1, r]$ such that $(i_s)^+ \leqslant r$. Then $\mathscr{T}_{\underline{w},\mathsf{c}'} = \mathscr{T}_{\underline{w},\mathsf{c}}$.

Lemma 5.6. *The following conditions on $s \in [1, r]$ are equivalent:*

(a) *s is frozen,*
(b) *$s = \min\{k \in \mathcal{F} \mid i_k = i_s\}$ where $\mathcal{F} = \{k \in [1, r] \mid \mathsf{c}_k = \mathsf{f}\} \cup \{k \in [1, r] \mid k^+ = \infty\}$.*
(c) *$M_s \simeq \mathrm{M}(w\Lambda_{i_s}, \Lambda_{i_s})$.*

Proof. By Remark 5.5, we may assume that $\mathcal{F} = \{k \in [1, r] \mid \mathsf{c}_k = \mathsf{f}\}$. Then the result follows from $\mathscr{K}_i\big(\mathrm{M}(w'\Lambda_j, \Lambda_j)\big) \simeq \mathrm{M}(s_i w'\Lambda_j, \Lambda_j)$ if $w' < s_i w'$. $\square$

We set

$$\mathsf{J} := [1, r], \qquad \mathsf{J}^{\mathrm{fr}} := \{k \in \mathsf{J} \mid k \text{ is frozen}\}, \qquad \mathsf{J}^{\mathrm{ex}} := \mathsf{J} \setminus \mathsf{J}^{\mathrm{fr}} \tag{5.6}$$

where the frozen condition is given in Lemma 5.6. Thanks to Propositions 5.2 and 5.3, we have the following.

Proposition 5.7. *The family $\mathscr{T}_{\underline{w},\mathsf{c}}$ with $\mathsf{J} = \mathsf{J}^{\mathrm{ex}} \sqcup \mathsf{J}^{\mathrm{fr}}$ is a seed of $\mathscr{C}_w$ in the canonical mutation class.*

Example 5.8. Let R be a symmetric quiver Hecke algebra of finite type A_3 and let $w := w_0$ be the longest element in the Weyl group W. Let us take

$$\underline{w} = (i_1, \ldots, i_6) = (1, 2, 1, 3, 2, 1) \quad \text{and} \quad \mathsf{c} = (c_1, \ldots, c_6) = (\mathsf{k}, \mathsf{f}, \mathsf{k}, \mathsf{f}, \mathsf{k}, \mathsf{k}).$$

Using (2.3), Definition 2.18 and [19, Lemma 9.1.5] (see also [26, Lemma 1.7, Proposition 4.1]), the corresponding seed $\mathscr{T}_{\underline{w},\mathsf{c}} = (M_k)_{k\in[1,r]}$ can be computed as

(i) $M_1 = \langle 1 \rangle$,
(ii) $M_2 = \mathscr{K}_1(\mathrm{M}(w_{\geqslant 2}\Lambda_2, \Lambda_2)) = \mathscr{K}_1(\langle 2, 1, 3, 2 \rangle) = \langle 2, 1, 3, 2 \rangle$,
(iii) $M_3 = \mathscr{K}_1\mathscr{F}_2(\langle 1 \rangle) = \mathscr{K}_1(\langle 2, 1 \rangle) = \langle 2, 1 \rangle$,
(iv) $M_4 = \mathscr{K}_1\mathscr{F}_2\mathscr{K}_1(\mathrm{M}(w_{\geqslant 4}\Lambda_3, \Lambda_3)) = \mathscr{K}_1\mathscr{F}_2\mathscr{K}_1(\langle 3 \rangle) = \mathscr{K}_1(\langle 2, 3 \rangle) = \langle 1, 2, 3 \rangle$,
(v) $M_5 = \mathscr{K}_1\mathscr{F}_2\mathscr{K}_1\mathscr{F}_3(\langle 2 \rangle) = \mathscr{K}_1\mathscr{F}_2(\langle 1, 3, 2 \rangle) = \langle 2, 1, 3 \rangle$,
(vi) $M_6 = \mathscr{K}_1\mathscr{F}_2\mathscr{K}_1\mathscr{F}_3\mathscr{K}_2(\langle 1 \rangle) = \mathscr{K}_1\mathscr{F}_2\mathscr{K}_1(\langle 3, 2, 1 \rangle) = \mathscr{K}_1(\langle 3, 2, 1 \rangle) = \langle 3, 2, 1 \rangle$,

where $\langle 2, 1, 3, 2 \rangle = \langle 2, 1 \rangle \nabla \langle 3, 2 \rangle$ and $\langle 2, 1, 3 \rangle = \langle 2, 1 \rangle \nabla \langle 3 \rangle$ (see Section 2.3). Note that M_1, M_3, M_4, M_6 are 1-dimensional and M_2, M_5 are 2-dimensional. The exchangeable and frozen indices are given as

$$\mathsf{J}^{\mathrm{ex}} = \{1, 3, 5\} \quad \text{and} \quad \mathsf{J}^{\mathrm{fr}} = \{2, 4, 6\}.$$

The seed $\mathscr{T}_{\underline{w},\mathsf{c}}$ can be depicted as follows:

$$\begin{tikzcd}
M_3 \arrow[r] & M_5 \arrow[r] \arrow[dr] & M_1 \arrow[dl] \\
\boxed{M_6} \arrow[u] & \boxed{M_4} \arrow[u] & \boxed{M_2}
\end{tikzcd}$$

where the boxed ones are frozen. We remark that this seed $\mathscr{T}_{\underline{w},\mathsf{c}}$ does not coincide with any seed $\mathscr{T}_{\mathfrak{C}}$ arising from an admissible chain $\mathfrak{C}$ of i-boxes with respect to the reduced expression $\underline{w}$.

6. Construction of monoidal seeds in $\mathscr{C}_{w,v}$

In this section, we introduce a new construction of monoidal seeds of $\mathscr{C}_{w,v}$ using $\mathscr{K}_i$ and $\mathscr{F}_i$.

6.1. **Pair of seeds.** Let $w, v \in \mathsf{W}$ such that $v \leqslant w$.

We first prepare the following two lemmas in order to prove the main theorem of this subsection.

Lemma 6.1. *Let $\Lambda \in \mathsf{P}_+$. If the determinantial module $\mathrm{M}(w\Lambda, \Lambda)$ is contained in $\mathscr{C}_{w,v}$, then we have $v\Lambda = \Lambda$.*

Proof. By [26, Proposition 4.8] (see also [46, Proposition 3.52]), we have

$$\mathrm{M}(w\Lambda, v\Lambda) \nabla \mathrm{M}(v\Lambda, \Lambda) \simeq \mathrm{M}(w\Lambda, \Lambda).$$

Since $\mathrm{M}(w\Lambda, \Lambda)$ is contained in $\mathscr{C}_{w,v}$, we have $\mathrm{M}(v\Lambda, \Lambda) = \mathbf{1}$, which yields the assertion. □

Lemma 6.2. *Let $i \in I$ with $s_i w < w$. Let M be a simple module in $\mathscr{C}_w$ such that $\mathfrak{d}_i(M) = 1$ and let N be a real simple in $\mathscr{C}_w$ such that $N \simeq \mathscr{K}_i(N)$. Then there exists a simple subquotient S of $M \circ N$ such that $\mathfrak{d}_i(S) = 1$.*

Proof. Assume that $\mathfrak{d}_i(S) \neq 1$ for any simple subquotient S of $M \circ N$. Since any simple subquotient S of $M \circ N$ satisfies $\mathfrak{d}_i(S) \leqslant \mathfrak{d}_i(M) + \mathfrak{d}_i(N) = 1$ (see [19, Proposition 3.2.10]), any simple subquotient S of $M \circ N$ commutes with $\langle i \rangle$. This means that $\langle i \rangle \circ S$ is simple. As $\mathscr{K}_i(M) \circ N$ is a quotient of $\langle i \rangle \circ M \circ N$, $\langle i \rangle$ is a convolution factor of any simple subquotient of $\mathscr{K}_i(M) \circ N$. Hence $[\mathscr{K}_i(M)] \cdot [N]$ is divisible by $[\langle i \rangle]$ in $K(\mathscr{C}_w)|_{q=1}$, which is the polynomial algebra of the variables including $[\langle i \rangle]$. Hence, $[\mathscr{K}_i(M)]$ or $[N]$ is divisible by $[\langle i \rangle]$. Therefore, [24, Theorem 4.1] implies that $\langle i \rangle$ is a convolution factor of $\mathscr{K}_i(M)$ or N, which is a contradiction by Lemma 2.19 (ii). □

The following lemma, a consequence of (3.1), is one of the main ingredients to construct a seed of $\mathscr{C}_{w,v}$.

Lemma 6.3. *Let $w, v \in \mathsf{W}$ such that $v \leqslant w$. Let $\mathscr{T} = \big((M_k)_{k \in \mathsf{K}}, \widetilde{B}\big)$ with $\mathsf{K} = \mathsf{K}^{\mathrm{ex}} \sqcup \mathsf{K}^{\mathrm{fr}}$ be a completely admissible seed in R-gmod. Let $\mathsf{J} = \mathsf{J}^{\mathrm{ex}} \sqcup \mathsf{J}^{\mathrm{fr}}$ satisfies*

(a) $\mathsf{J} \subset \mathsf{K}$ *and* $\mathsf{J}^{\mathrm{ex}} \subset \mathsf{K}^{\mathrm{ex}}$,
(b) $M_k \in \mathscr{C}_{w,v}$ *for any* $k \in \mathsf{J}$,
(c) $\widetilde{B}|_{(\mathsf{K}\setminus\mathsf{J}) \times \mathsf{J}^{\mathrm{ex}}} = 0$.

Then $\mathscr{S} = \big((M_k)_{k \in \mathsf{J}}, \widetilde{B}|_{\mathsf{J} \times \mathsf{J}^{\mathrm{ex}}}\big)$ is a completely admissible seed in $\mathscr{C}_{w,v}$.

Proof. It is trivial that $\mathscr{S}$ is completely admissible in R-gmod. Hence it is enough to show that the conditions (a)–(c) are satisfied after mutations.

Let $s \in \mathsf{J}^{\mathrm{ex}}$ and $\mathscr{T}' = \big((M'_k)_{k \in \mathsf{K}}, \widetilde{B}'\big)$ be the mutation of $\mathscr{T}$ at s. Then it is easy to see that $\widetilde{B}'|_{(\mathsf{K}\setminus\mathsf{J}) \times \mathsf{J}^{\mathrm{ex}}} = 0$. Since $M_s \nabla M'_s$ and $M'_s \nabla M_s$ are $\mathscr{S}$-monomials by (c), $M_s \circ M'_s$ also belongs to $\mathscr{C}_{w,v}$. Hence M'_s also belongs to $\mathscr{C}_{w,v}$ by (3.1). Hence $\mathscr{S}' = \big((M'_k)_{k \in \mathsf{J}}, \widetilde{B}'|_{\mathsf{J} \times \mathsf{J}^{\mathrm{ex}}}\big)$ is a seed in $\mathscr{C}_{w,v}$.

Thus we obtain the desired result. □

Let $i \in I$ and assume that

$$v < s_i v \quad \text{and} \quad s_i w < w.$$

Let $\mathscr{T} = \left(\{M_k\}_{k\in \mathsf{K}}, \widetilde{B}_\circ\right)$ be a seed of $\mathscr{C}_{s_i w}$ in the canonical mutation class, where $\widetilde{B}_\circ := (b_{j,k})_{(j,k)\in \mathsf{K}\times\mathsf{K}^{\mathrm{ex}}}$ is an exchange matrix. We define

$$\mathscr{S} := \mathscr{T} \cap \mathscr{C}_{s_i w,v} \quad \text{and} \quad \mathsf{J} := \{k \in \mathsf{K} \mid M_k \in \mathscr{S}\},$$

and we assume that

$$\mathscr{S} \text{ satisfies (3.4) with } \mathscr{C}_{s_i w,v} \text{ instead of } \mathscr{C}_{w,v}.$$

Set

$$\begin{aligned} &\mathscr{S}_{\mathrm{ex}} := \{M \in \mathscr{S} \mid M \text{ satisfies (a) in (3.4)}\}, \quad \mathsf{J}^{\mathrm{ex}} := \{k \in \mathsf{J} \mid M_k \in \mathscr{S}_{\mathrm{ex}}\}, \\ &\mathscr{S}_{\mathrm{fr}} := \{M \in \mathscr{S} \mid M \text{ satisfies (b) in (3.4)}\}, \quad \mathsf{J}^{\mathrm{fr}} := \{k \in \mathsf{J} \mid M_k \in \mathscr{S}_{\mathrm{fr}}\}. \end{aligned} \tag{6.1}$$

Hence we have

$$\mathscr{S}_{\mathrm{ex}} \subset \mathscr{T}_{\mathrm{ex}} \quad \text{and} \quad \mathscr{S} \cap \mathscr{T}_{\mathrm{fr}} \subset \mathscr{S}_{\mathrm{fr}},$$

where the second inclusion follows from Lemma 6.1.

Define

$$\widetilde{\mathscr{S}} := \{\langle i\rangle\} \cup \mathscr{K}_i(\mathscr{S}) \quad \text{and} \quad \widetilde{\mathscr{T}} := \{\langle i\rangle\} \cup \mathscr{K}_i(\mathscr{T}),$$

and set $\widetilde{\mathsf{J}} := \{k_0\} \sqcup \mathsf{J}$, $\widetilde{\mathsf{K}} := \{k_0\} \sqcup \mathsf{K}$ and

$$\widetilde{M}_{k_0} := \langle i\rangle, \qquad \widetilde{M}_k := \mathscr{K}_i(M_k) \quad \text{for } k \in \mathsf{K}.$$

Note that $\widetilde{\mathscr{S}} \subset \widetilde{\mathscr{T}}$ and $\widetilde{\mathscr{S}}$ satisfies (3.4) in $\mathscr{C}_{w,v}$ by Proposition 3.6. We now define $\widetilde{\mathscr{S}}_{\mathrm{ex}}$, $\widetilde{\mathscr{S}}_{\mathrm{fr}}$, $\widetilde{\mathsf{J}}^{\mathrm{ex}}$ and $\widetilde{\mathsf{J}}^{\mathrm{fr}}$ in the same manner as (6.1). Theorem 3.3 tells us that

$$\widetilde{\mathsf{J}}^{\mathrm{ex}} = \begin{cases} \{k_0\} \cup \mathsf{J}^{\mathrm{ex}} & \text{if } s_i v \leqslant s_i w, \\ \mathsf{J}^{\mathrm{ex}} & \text{otherwise,} \end{cases} \quad \text{and} \quad \widetilde{\mathsf{J}}^{\mathrm{fr}} = \widetilde{\mathsf{J}} \setminus \widetilde{\mathsf{J}}^{\mathrm{ex}}.$$

Note that $\widetilde{\mathscr{S}}_{\mathrm{ex}} \subset \widetilde{\mathscr{T}}_{\mathrm{ex}}$ and $\widetilde{\mathscr{S}} \cap \widetilde{\mathscr{T}}_{\mathrm{fr}} \subset \widetilde{\mathscr{S}}_{\mathrm{fr}}$.

Proposition 5.3 says that there is an exchange matrix $\widetilde{B} := (\widetilde{b}_{j,k})_{(j,k)\in\widetilde{\mathsf{K}}\times\widetilde{\mathsf{K}}^{\mathrm{ex}}}$ such that the pair $(\widetilde{\mathscr{T}}, \widetilde{B})$ is a seed of $\mathscr{C}_w$. Moreover, by Proposition 5.3 (iii) together with the seeds (5.2) and (5.3), we have

$$\widetilde{B}|_{\mathsf{K}\times\mathsf{K}_{\mathrm{ex}}} = \widetilde{B}_\circ. \tag{6.2}$$

We further assume that

$$\begin{aligned} &\mathscr{S} \text{ is a Laurent family in } \mathscr{C}_{s_i w,v}, \\ &\widetilde{B}_\circ|_{(\mathsf{K}\setminus\mathsf{J})\times\mathsf{J}^{\mathrm{ex}}} = 0. \end{aligned} \tag{6.3}$$

Note that (6.3) implies the pair $(\mathscr{S}, \widetilde{B}|_{\mathsf{J}\times\mathsf{J}^{\mathrm{ex}}})$ is a completely admissible seed in $\mathscr{C}_{s_iw,v}$ by Lemma 6.3.

Theorem 6.4.

(i) *We have*

$$\widetilde{B}|_{(\widetilde{\mathsf{K}}\setminus\widetilde{\mathsf{J}})\times\widetilde{\mathsf{J}}^{\mathrm{ex}}} = 0.$$

(ii) *The pair* $\big(\widetilde{\mathscr{S}}, \widetilde{B}|_{\widetilde{\mathsf{J}}\times\widetilde{\mathsf{J}}^{\mathrm{ex}}}\big)$ *is a completely admissible seed in* $\mathscr{C}_{w,v}$.

Proof. Since (ii) follows from (i) by Lemma 6.3, we focus on proving (i). We may assume that

$$s_iv \leqslant s_iw \tag{6.4}$$

since otherwise $k_0 \in \widetilde{\mathsf{J}}^{\mathrm{fr}}$ by Theorem 3.3 and the assertion follows directly from (6.2) and (6.3). In this case, we have $\widetilde{\mathsf{J}}^{\mathrm{ex}} = \{k_0\} \cup \mathsf{J}^{\mathrm{ex}}$. By (6.2) and (6.3), it is enough to show that

$$\widetilde{B}|_{(\widetilde{\mathsf{K}}\setminus\widetilde{\mathsf{J}})\times\{k_0\}} = 0.$$

Let us first show that $\widetilde{\mathscr{S}}$ admits a mutation at k_0.

Set $C := \mathop{\circ}\limits_{k\in\mathsf{J}} \widetilde{M}_k$. Note that C is an $\widetilde{\mathscr{S}}$-monomial. By (6.4) and Theorem 3.3, there exists a simple module $M \in \mathscr{C}_{w,v}$ such that $\mathfrak{d}_i(M) > 0$. Replacing M with $\widetilde{\mathrm{F}}_i^{\mathfrak{d}_i(M)-1}M$, we may assume that $\mathfrak{d}_i(M) = 1$ from the beginning. Assume that $\mathfrak{d}(C, M) > 0$. Then Lemma 6.2 implies that there exists a simple subquotient M' of $M \circ C$ such that $\mathfrak{d}_i(M') = 1$. Hence M' is a simple module in $\mathscr{C}_{w,v}$ such that $\mathfrak{d}_i(M') = 1$ and $\mathfrak{d}(C, M') < \mathfrak{d}(C, M)$ by [19, Corollary 4.1.2] (see also [28, Corollary 3.18]). Arguing by induction on $\mathfrak{d}(C, M)$, we assume from the outset that there exists a simple $M \in \mathscr{C}_{w,v}$ such that

$$\mathfrak{d}_i(M) = 1 \quad \text{and} \quad M \text{ commutes with } C.$$

Hence M commutes with $\widetilde{M}_k$ for any $k \in \mathsf{J}$.

As $\langle i\rangle$ and M commute with every $\widetilde{M}_k$, the modules $\langle i\rangle \nabla M$ and $M \nabla \langle i\rangle$ commute with $\widetilde{M}_k$ for all $k \in \widetilde{\mathsf{J}}$. Proposition 3.5 and (6.3) say that $\widetilde{\mathscr{S}}$ is a Laurent family in $\mathscr{C}_{w,v}$, which implies that there exist $\mathbf{a} = (a_k)_{k\in\widetilde{\mathsf{J}}}, \mathbf{b} = (b_k)_{k\in\widetilde{\mathsf{J}}} \in \mathbb{Z}_{\geqslant 0}^{\oplus\widetilde{\mathsf{J}}}$ such that

$$M \nabla \langle i\rangle \simeq \widetilde{\mathscr{S}}(\mathbf{a}) \quad \text{and} \quad \langle i\rangle \nabla M \simeq \widetilde{\mathscr{S}}(\mathbf{b}),$$

where $\widetilde{\mathscr{S}}(\mathbf{a})$ is defined in (2.4). Since $\langle i\rangle$ is not a convolution factor of $M \nabla \langle i\rangle$ and $\langle i\rangle \nabla M$ by Lemma 2.19 (ii), we have $a_{k_0} = b_{k_0} = 0$. Define $\mathbf{c} := (c_k)_{k\in\widetilde{\mathsf{J}}} \in \mathbb{Z}_{\geqslant 0}^{\oplus\widetilde{\mathsf{J}}}$ by $c_k = \min(a_k, b_k)$. Then $[\langle i\rangle]\cdot[M]$ is divisible by $[\widetilde{\mathscr{S}}(\mathbf{c})]$ in $K(\mathscr{C}_w)|_{q=1}$, which means that $[M]$ is divisible by $[\widetilde{\mathscr{S}}(\mathbf{c})]$. Hence [24, Theorem 4.1] implies that $[\widetilde{\mathscr{S}}(\mathbf{c})]$ is a convolution factor of M, i.e., there exists a simple $X \in \mathscr{C}_w$ such that $M \simeq X \circ \mathscr{S}(\mathbf{c})$. Hence, (3.1) implies that $X \in \mathscr{C}_{w,v}$ and

$$X \nabla \langle i \rangle \simeq \widetilde{\mathscr{S}}(\mathbf{a}-\mathbf{c}) \quad \text{and} \quad \langle i \rangle \nabla X \simeq \widetilde{\mathscr{S}}(\mathbf{b}-\mathbf{c}).$$

Replacing M with X from the beginning, we may assume that

$$M \nabla \langle i \rangle \simeq \widetilde{\mathscr{S}}(\mathbf{a}) \quad \text{and} \quad \langle i \rangle \nabla M \simeq \widetilde{\mathscr{S}}(\mathbf{b})$$

with $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{\oplus \widetilde{\mathsf{J}}}$ and $\mathbf{b} \in \mathbb{Z}_{\geqslant 0}^{\oplus \widetilde{\mathsf{J}}}$ such that

$$a_{k_0} = b_{k_0} = 0 \quad \text{and} \quad a_k b_k = 0 \quad \text{for any } k \in \widetilde{\mathsf{J}}. \tag{6.5}$$

Moreover M commutes with all $\widetilde{M}_k$ ($k \in \mathsf{J}$). Thus we have an exact sequence

$$0 \longrightarrow \widetilde{\mathscr{S}}(\mathbf{a}) \longrightarrow \langle i \rangle \circ M \longrightarrow \widetilde{\mathscr{S}}(\mathbf{b}) \longrightarrow 0. \tag{6.6}$$

We shall show that M commutes with $\widetilde{M}_k$ for $k \in \mathsf{K}$. Applying $\widetilde{M}_k \circ -$ to the above exact sequence (6.6), we obtain

$$0 \longrightarrow \widetilde{M}_k \circ \widetilde{\mathscr{S}}(\mathbf{a}) \longrightarrow \widetilde{M}_k \circ \langle i \rangle \circ M \longrightarrow \widetilde{M}_k \circ \widetilde{\mathscr{S}}(\mathbf{b}) \longrightarrow 0.$$

Note that $\widetilde{M}_k \circ \widetilde{\mathscr{S}}(\mathbf{a})$ and $\widetilde{M}_k \circ \widetilde{\mathscr{S}}(\mathbf{b})$ are simple since $\widetilde{\mathscr{T}}$ is a seed in $\mathscr{C}_w$. Hence $\langle i \rangle \circ \widetilde{M}_k \circ M \simeq \widetilde{M}_k \circ \langle i \rangle \circ M$ has length 2. If $\widetilde{M}_k$ and M do not commute, then

$$[\langle i \rangle \circ \widetilde{M}_k \circ M] = [\langle i \rangle \circ (\widetilde{M}_k \nabla M)] + [\langle i \rangle \circ (M \nabla \widetilde{M}_k)],$$

and $\langle i \rangle \circ (\widetilde{M}_k \nabla M)$, $\langle i \rangle \circ (M \nabla \widetilde{M}_k)$ are simple. Since $\langle i \rangle \circ \widetilde{M}_k \circ M$ has simple head, we have

$$\langle i \rangle \circ (\widetilde{M}_k \nabla M) \simeq \widetilde{M}_k \nabla (\langle i \rangle \nabla M) \simeq \widetilde{M}_k \nabla \widetilde{\mathscr{S}}(\mathbf{b}) \simeq \widetilde{M}_k \circ \widetilde{\mathscr{S}}(\mathbf{b}).$$

By (6.5), $\widetilde{M}_k \circ \widetilde{\mathscr{S}}(\mathbf{b})$ cannot have $\langle i \rangle$ as a convolution factor, which is a contradiction. Thus M commutes with $\widetilde{M}_k$ for any $k \in \mathsf{K}$, which means that M is a mutation of $\widetilde{\mathscr{T}}$ at k_0 by Lemma 5.1 (ii).

It follows from (6.6) that $\widetilde{b}_{k,k_0} = 0$ for any $k \in \widetilde{\mathsf{K}} \setminus \widetilde{\mathsf{J}}$, which implies that

$$\widetilde{B}|_{(\widetilde{\mathsf{K}} \setminus \widetilde{\mathsf{J}}) \times \{k_0\}} = 0.$$

This completes the proof. □

6.2. **Seed construction for $\mathscr{C}_{w,v}$.** Recall the seed construction for $\mathscr{C}_w$ given in Section 5.2. Let $w, v \in \mathsf{W}$ with $v \leqslant w$ and take a reduced sequence of w

$$\underline{w} = (i_1, i_2, \ldots, i_r).$$

For $k \in [1, r+1]$, we set $w_{\geqslant k} := s_{i_k} s_{i_{k+1}} \cdots s_{i_r}$ and define $v^{\underline{w}}_{\geqslant k}$ inductively by

$$v^{\underline{w}}_{\geqslant 1} = v,$$
$$v^{\underline{w}}_{\geqslant k} = \begin{cases} v^{\underline{w}}_{\geqslant k-1} & \text{if } s_{i_{k-1}} v^{\underline{w}}_{\geqslant k-1} > v^{\underline{w}}_{\geqslant k-1}, \\ s_{i_{k-1}} v^{\underline{w}}_{\geqslant k-1} & \text{if } s_{i_{k-1}} v^{\underline{w}}_{\geqslant k-1} < v^{\underline{w}}_{\geqslant k-1}, \end{cases} \qquad \text{for } k \in [2, r+1].$$

Note that $w_{\geqslant r+1} = v^{\underline{w}}_{\geqslant r+1} = \mathsf{id}$. We set

$$w_{\leqslant k} := w(w_{\geqslant k+1})^{-1} = s_{i_1} \cdots s_{i_k} \quad \text{and} \quad v_{\leqslant k} := v(v_{\geqslant k+1})^{-1} \quad \text{for } k \in [1, r].$$

We set

$$[1, r]^{\underline{w}}_{v} := \left\{ k \in [1, r] \mid v^{\underline{w}}_{\geqslant k} = v^{\underline{w}}_{\geqslant k+1} \right\}. \tag{6.7}$$

If no confusion arises, we simply write $v_{\geqslant k}$ and $[1, r]_v$ instead of $v^{\underline{w}}_{\geqslant k}$ and $[1, r]^{\underline{w}}_{v}$, respectively. We then consider the seed of $\mathscr{C}_w$

$$\mathscr{T} := \mathscr{T}_{\underline{w}, \mathsf{c}} = (M_k)_{k \in [1, r]},$$

where $\mathsf{c} = (c_1, \ldots, c_r) \in \{\mathsf{k}, \mathsf{f}\}^r$ given by

$$c_t := \begin{cases} \mathsf{k} & \text{if } t \in [1, r]_v, \\ \mathsf{f} & \text{otherwise} \end{cases} \tag{6.8}$$

(see (5.4) for $\mathscr{T}_{\underline{w}, \mathsf{c}}$). This sequence c is called the *KF sequence associated with $\underline{w}$ and v*.

Define

$$\mathscr{S}_{\underline{w}, v} := (M_k)_{k \in [1, r]^{\underline{w}}_{v}} \subset \mathscr{T}. \tag{6.9}$$

It follows from (5.5) that

$$M_k = \mathscr{G}_1 \cdots \mathscr{G}_{k-1}(\langle i_k \rangle) \quad \text{for any } k \in [1, r]_v,$$

where $\mathscr{G}_t = \mathscr{K}_{i_t}$ if $c_t = \mathsf{k}$ and $\mathscr{G}_t = \mathscr{F}_{i_t}$ if $c_t = \mathsf{f}$.

Proposition 6.5. *Let $k \in [1, r]$ such that $c_k = \mathsf{f}$. Then we have*

$$M_k = \mathrm{M}(w\Lambda_{i_k}, v_{\leqslant k-1}\Lambda_{i_k}),$$

where $v_{\leqslant k-1} := v(v_{\geqslant k})^{-1}$. In particular, $M_k \notin \mathscr{C}_{w,v}$ and M_k commutes with any simple in $\mathscr{C}_{w,v}$.

Proof. Note that

$$M_k = \mathscr{G}_1 \cdots \mathscr{G}_{k-1}(\mathrm{M}(w_{\geqslant k}\Lambda_{i_k}, \Lambda_{i_k})).$$

For $1 \leqslant a \leqslant b \leqslant r$, we set $v_{[a,b]} := v_{\leqslant a-1}^{-1} v_{\leqslant b} = v_{\geqslant a} v_{\geqslant b+1}^{-1}$. We shall use descending induction on $t \in [1,k]$ to show

$$\mathscr{G}_t \mathscr{G}_{t+1} \cdots \mathscr{G}_{k-1} \mathrm{M}(w_{\geqslant k}\Lambda_{i_k}, \Lambda_{i_k}) \simeq \mathrm{M}(w_{\geqslant t}\Lambda_{i_k}, v_{[t,k-1]}\Lambda_{i_k}).$$

It is obvious at $t = k$, and for $t < k$, we have

$$\mathscr{G}_t \cdots \mathscr{G}_{k-1} \mathrm{M}(w_{\geqslant k}\Lambda_{i_k}, \Lambda_{i_k}) \simeq \mathscr{G}_t \mathrm{M}(w_{\geqslant t+1}\Lambda_{i_k}, v_{[t+1,k-1]}\Lambda_{i_k}) \simeq \mathrm{M}(w_{\geqslant t}\Lambda_{i_k}, v_{[t,k-1]}\Lambda_{i_k}).$$

The last statement follows from the fact that $v_{\leqslant k-1} \lneqq v$ and M_k commutes with any simple in $\mathscr{C}_{w, v_{\leqslant k-1}} \supset \mathscr{C}_{w,v}$. □

We set

$$\mathsf{J}_{\underline{w},v} := [1,r]^{\underline{w}}_{v}, \quad \mathsf{J}^{\mathrm{fr}}_{\underline{w},v} := \{k \in \mathsf{J}_{\underline{w},v} \mid s_{i_k} v^{\underline{w}}_{\geqslant k} \not\leqslant w^{\underline{w}}_{\geqslant k+1}\}, \quad \mathsf{J}^{\mathrm{ex}}_{\underline{w},v} := \mathsf{J}_{\underline{w},v} \setminus \mathsf{J}^{\mathrm{fr}}_{\underline{w},v}. \tag{6.10}$$

Lemma 6.6. $\mathsf{J}^{\mathrm{fr}}_{\underline{w},v} = \{k \in \mathsf{J}_{\underline{w},v} \mid M_k$ *satisfies* (3.4) (b)$\}$.

Proof. We set $\mathsf{J} := \mathsf{J}^{\mathrm{fr}}_{\underline{w},v}$ and $\mathsf{J}' := \{k \in \mathsf{J}_{\underline{w},v} \mid M_k$ satisfies (3.4) (b)$\}$. By Theorem 3.3 and Proposition 3.6, we have $\mathsf{J} \subset \mathsf{J}'$.

Let $k \in \mathsf{J}'$ and assume that $s_{i_k} v_{\geqslant k} \leqslant w_{\geqslant k+1}$. Then Theorem 3.3 says that there is a simple module $X \in \mathscr{C}_{w_{\geqslant k}, v_{\geqslant k}}$ such that $\mathfrak{d}(\langle i_k \rangle, X) > 0$. By Lemma 2.22 (iii) and Proposition 3.1 (ii), we have

$$\mathfrak{d}(M_k, \mathscr{G}_1 \cdots \mathscr{G}_{k-1}(X)) = \mathfrak{d}(\langle i_k \rangle, X) > 0,$$

where $\mathscr{G}_t = \mathscr{K}_{i_t}$ if $c_t = \mathsf{k}$ and $\mathscr{G}_t = \mathscr{F}_{i_t}$ if $c_t = \mathsf{f}$. Since $\mathscr{G}_1 \cdots \mathscr{G}_{k-1}(X)$ belongs to $\mathscr{C}_{w,v}$, this is a contradiction to $k \in \mathsf{J}'$. Hence we have $\mathsf{J}' \subset \mathsf{J}$, which completes the proof. □

Remark 6.7. We define non-negative integers $d_{w,v}$ $(v \leqslant w)$ inductively as follows:

- $d_{w,w} = 0$ for any $w \in \mathsf{W}$,
- for any $v \leqslant w$ and $i \in I$ such that $s_i w < w$,

$$d_{w,v} = \begin{cases} d_{s_i w, s_i v} & \text{if } s_i v < v, \\ d_{s_i w, v} + 1 & \text{if } v < s_i v \text{ and } s_i v \not\leqslant s_i w, \\ d_{s_i w, v} & \text{if } v < s_i v \text{ and } s_i v \leqslant s_i w. \end{cases}$$

Then we have $d_{w,v} = |\mathsf{J}^{\mathrm{fr}}_{\underline{w},v}|$. This $d_{w,v}$ coincides with the formula for the coefficient of the second largest power in *Kazhdan-Lusztig R-polynomials* in [9, (3.1)]. We refer the readers to [9, Section 3.4] for details on Kazhdan-Lusztig R-polynomials and the number of frozen variables in open Richardson varieties.

Theorem 6.8. *The set $\mathscr{S}_{\underline{w},v}$ is a Laurent family in $\mathscr{C}_{w,v}$. Moreover, $\mathscr{S}_{\underline{w},v}$ with $\mathsf{J}_{\underline{w},v} = \mathsf{J}^{\mathrm{ex}}_{\underline{w},v} \sqcup \mathsf{J}^{\mathrm{fr}}_{\underline{w},v}$ is a completely admissible seed of $\mathscr{C}_{w,v}$.*

Proof. For $k \in [1, r+1]$, we define $\mathscr{S}^{(k)} \subset \mathscr{C}_{w_{\geqslant k}, v_{\geqslant k}}$ inductively by

$$\begin{aligned} &\mathscr{S}^{(r+1)} := \varnothing, \\ &\mathscr{S}^{(k)} := \begin{cases} \mathscr{F}_{i_k}(\mathscr{S}^{(k+1)}) & \text{if } c_k = \mathsf{f}, \\ \{\langle i_k \rangle\} \sqcup \mathscr{K}_{i_k}(\mathscr{S}^{(k+1)}) & \text{if } c_k = \mathsf{k}, \end{cases} \quad \text{for } k \in [1, r]. \end{aligned} \tag{6.11}$$

By the construction, we have $\mathscr{S}_{\underline{w},v} = \mathscr{S}^{(1)}$. Proposition 3.5 and 3.6 say that $\mathscr{S}^{(k)}$ is a Laurent family of $\mathscr{C}_{w_{\geqslant k}, v_{\geqslant k}}$ and satisfies (3.4) for each $k \in [1, r+1]$. As $\mathscr{F}_i$ preserves the completely admissibility, the assertion follows by applying Theorem 6.4 to the above construction (6.11). $\square$

Example 6.9. We continue with Example 5.8. Recall the reduced expression $\underline{w} = (1, 2, 1, 3, 2, 1)$, and take $v = s_2 s_3$. Then $w_{\geqslant k}$ and $v_{\geqslant k}$ are computed as

k	1	2	3	4	5	6
$w_{\geqslant k}$	$s_1 s_2 s_1 s_3 s_2 s_1$	$s_2 s_1 s_3 s_2 s_1$	$s_1 s_3 s_2 s_1$	$s_3 s_2 s_1$	$s_2 s_1$	s_1
$v_{\geqslant k}$	$s_2 s_3$	$s_2 s_3$	s_3	s_3	id	id

Hence, we have $[1, 6]_v = \{1, 3, 5, 6\}$, the associated KF sequence

$$\mathsf{c} = (c_1, \ldots, c_6) = (\mathsf{k}, \mathsf{f}, \mathsf{k}, \mathsf{f}, \mathsf{k}, \mathsf{k}),$$

and the decomposition $\mathsf{J}_{\underline{w},v} = [1, 6]_v = \mathsf{J}^{\mathrm{ex}}_{\underline{w},v} \sqcup \mathsf{J}^{\mathrm{fr}}_{\underline{w},v}$ given by

$$\mathsf{J}^{\mathrm{ex}}_{\underline{w},v} = \{3\} \quad \text{and} \quad \mathsf{J}^{\mathrm{fr}}_{\underline{w},v} = \{1, 5, 6\}.$$

Since the KF sequence c coincides with the KF sequence in Example 5.8, we have

$$\mathscr{S}_{\underline{w},v} = \{M_1, M_3, M_5, M_6\},$$

where M_k are given in Example 5.8. The seed $\mathscr{S}_{\underline{w},v}$ can be depicted as follows:

$$\begin{tikzcd} M_3 \arrow[r] & \boxed{M_5} & \boxed{M_1} \\ \boxed{M_6} \arrow[u] & & \end{tikzcd}$$

where the boxed ones are frozen. Note that the above quiver can be obtained from the quiver in Example 5.8 by deleting M_2, M_4 and freezing M_1, M_5. The mutation of M_3 is $\langle 3 \rangle$.

The next proposition gives the relation between our seed of $\mathscr{C}_{w,v}$ and the one of Leclerc. Note that the the isomorphism class of M_k ($k \in [1,r]^{w}_{v}$) in the proposition gives a prime element of the Grothendieck ring $K(\mathscr{C}_w)|_{q=1}$ ([14]).

Proposition 6.10. *Let $v \leqslant w \in \mathsf{W}$ and let $\underline{w}$ be a reduced expression of w. Let $\mathscr{S}_{\underline{w},v} = (M_k)_{k\in[1,r]^{w}_{v}}$ be the seed of $\mathscr{C}_{w,v}$ associated with $\underline{w}$ and v.*

(i) *Then there exists a family $\{n_{j,k}\}_{j,k\in[1,r]^{w}_{v}}$ of non-negative integers such that*

$$M(w_{\leqslant k}\Lambda_{i_k}, v_{\leqslant k}\Lambda_{i_k}) \simeq \mathop{\circ}_{j\in[1,r]^{w}_{v}} M_j^{\circ n_{j,k}} \quad \textit{for any } k \in [1,r]^{w}_{v}$$

and $n_{j,k} \in \mathbb{Z}_{\geqslant 0}$ satisfy the unitriangular condition: $n_{j,k} = 0$ for any $j > k$ and $n_{k,k} = 1$.

(ii) *In particular, we have*

$$\begin{aligned}\mathscr{S}_{\underline{w},v} &= \{M_k \mid k \in [1,r]^{w}_{v}\}\\ &= \{\textit{prime convolution factors of } \mathrm{M}(w_{\leqslant k}\Lambda_{i_k}, v_{\leqslant k}\Lambda_{i_k}) \mid k \in [1,r]^{w}_{v}\}.\end{aligned}$$

Note that the right hand side is the seed of Leclerc in [42, Corollary 4.4] *when $\mathfrak{g}$ is of finite type ADE.*

Proof. Note that (ii) immediately follows from (i) and [24, Theorem 4]. We prove (i) by induction on $\ell(w)$. Assume that $\underline{w} = (i_1, i_2, \dots, i_r)$. Set $i = i_1$ and $\underline{w}' = (i'_1, i'_2, \dots, i'_{r-1}) = (i_2, i_3, \dots, i_r)$. Then $\underline{w}'$ is a reduced expression of $w' := s_i w < w$. Set $v' = v^{w}_{\geqslant 2}$. Let $\mathsf{c} = (c_k)_{k\in[1,r]}$ be the KF-sequence for $\underline{w}$ and v, and let $\mathsf{c}' = (c'_k)_{k\in[1,r-1]}$ be the KF-sequence for $\underline{w}'$ and v'. Then $c_{k+1} = c'_k$ for $k \in [1, r-1]$. Let

$$\mathscr{S}_{\underline{w}',v'} := (M'_k)_{k\in[1,r-1]^{w'}_{v'}} \quad \text{and} \quad \mathscr{S}_{\underline{w},v} := (M_k)_{k\in[1,r]^{w}_{v}}.$$

Case 1 : Assume that $s_i v < v$. Then $s_i v < s_i w$, $c_1 = \mathsf{f}$, $w_{\leqslant k+1} = s_i w'_{\leqslant k}$ and $v_{\leqslant k+1} = s_i v'_{\leqslant k}$. and

$$[1,r]^{w}_{v} = \left\{k+1 \mid k \in [1,r-1]^{w'}_{v'}\right\} \quad \text{and} \quad M_{k+1} = \mathscr{F}_i(M'_k).$$

Since

$$\mathrm{M}(w_{\leqslant k+1}\Lambda_{i_{k+1}}, v_{\leqslant k+1}\Lambda_{i_{k+1}}) \simeq \mathscr{F}_i\big(\mathrm{M}(w'_{\leqslant k}\Lambda_{i'_k}, v'_{\leqslant k}\Lambda_{i'_k})\big),$$

we have the desired result.

Case 2 : Assume that $s_i v > v$. Then $v' = v$, $\mathsf{c}_1 = \mathsf{k}$, and

$$[1,r]^{w}_{v} = \{1\} \sqcup \left\{k+1 \mid k \in [1,r-1]^{w'}_{v}\right\}.$$

Hence we have

$$M_1 = \langle i \rangle \quad \text{and} \quad M_{k+1} \simeq \mathscr{K}_i(M'_k) \quad \text{for } k \in [1,r-1]^{w'}_{v}$$

Set

$$L_k = \mathrm{M}(w_{\leqslant k}\Lambda_{i_k}, v^{\underline{w}}_{\leqslant k}\Lambda_{i_k}) \quad \text{and} \quad L'_k = \mathrm{M}(w'_{\leqslant k}\Lambda_{i'_k}, v'^{\underline{w}'}_{\leqslant k}\Lambda_{i'_k}).$$

Then we have $L_1 = M_1$, and $L_{k+1} \simeq \langle i\rangle^n \circ \mathscr{K}_i(L'_k)$ for some $n \in \mathbb{Z}_{\geqslant 0}$ by Lemma 3.4. Hence if $L'_k = \mathop{\circ}\limits_{j\in[1,r-1]^{\underline{w}'}_{v}} (M'_j)^{\circ n_{j,k}}$, then we have

$$L_{k+1} \simeq \langle i\rangle^{\circ n} \circ \mathscr{K}_i(L'_k) \simeq \langle i\rangle^{\circ n} \circ \mathop{\circ}\limits_{j\in[1,r-1]^{\underline{w}'}_{v'}} (\mathscr{K}_i M'_j)^{\circ n_{j,k}} \simeq M_1^{\circ n} \mathop{\circ}\limits_{j\in[1,r-1]^{\underline{w}'}_{v'}} (M_{j+1})^{\circ n_{j,k}}. \qquad \square$$

7. Mutation invariance of the seeds $\mathscr{S}_{\underline{w},v}$

In this section, we prove that the mutation class of $\mathscr{S}_{\underline{w},v}$ does not depend on the choice of reduced expressions $\underline{w}$ of $w \in \mathsf{W}$.

Let $w, v \in \mathsf{W}$ such that $v \leqslant w$, and choose and fix a reduced sequence

$$\underline{w} = (i_1, i_2, \ldots, i_r)$$

throughout this section. We consider the seed $\mathscr{T}_{\underline{w},\mathsf{c}}$ for $\mathscr{C}_w$ and the seed $\mathscr{S}_{\underline{w},v}$ for $\mathscr{C}_{w,v}$, where $\mathsf{c} = (c_1, \ldots, c_r) \in \{\mathsf{k}, \mathsf{f}\}^r$ is defined in (6.8). We keep all notations appearing in Section 6.2, and simply write

$$\mathscr{S}_{\underline{w}} := \mathscr{S}_{\underline{w},v}$$

in this section. We remark that the mutation class of $\mathscr{T}_{\underline{w},\mathsf{c}}$ does not depend on the choice of $\mathsf{c} \in \{\mathsf{k}, \mathsf{f}\}^r$ by Proposition 5.7. Recall that for a set A, the symmetric group $\mathfrak{S}_r = \langle \mathsf{r}_k \mid k \in [1, r-1]\rangle$ acts on A^r by the place permutation, i.e., for $\mathbf{a} = (a_1, a_2, \ldots, a_r) \in A^r$,

$$\mathsf{r}_k\mathbf{a} := (a_1, \ldots, a_{k-1}, a_{k+1}, a_k, a_{k+2}, \ldots, a_r),$$

where $\mathsf{r}_k = (k, k+1)$ is the k-th transposition in $\mathfrak{S}_r$.

7.1. **Commutation equivalence.**

In this subsection, we show that $\mathscr{S}_{\underline{w},v}$ and $\mathscr{S}_{\underline{w}',v}$ belong to the same mutation class when $\underline{w}'$ is obtained from $\underline{w}$ by a single commutation move.

Lemma 7.1. *Let $i, j \in I$ such that $(\alpha_i, \alpha_j) = 0$. For a simple module M such that $\alpha_i, \alpha_j \notin \mathrm{W}(M)$, we have*

(i) $\mathscr{K}_i\mathscr{K}_j(M) \simeq \mathscr{K}_j\mathscr{K}_i(M)$,
(ii) $\mathscr{K}_i\mathscr{F}_j(M) \simeq \mathscr{F}_j\mathscr{K}_i(M)$,
(iii) $\mathscr{F}_i\mathscr{F}_j(M) \simeq \mathscr{F}_j\mathscr{F}_i(M)$.

Proof. It is easy to see that $\mathscr{K}_i(\langle j\rangle \nabla M) \simeq \langle j\rangle \nabla \mathscr{K}_i(M)$.
(i) Since $\langle i\rangle \circ \langle j\rangle \simeq \langle j\rangle \circ \langle i\rangle$ is simple and $\mathfrak{d}_i(M) = \mathfrak{d}_i(\langle j\rangle^k \nabla M)$ for any $k \geqslant 0$, we have $\mathscr{K}_i\mathscr{K}_j(M) \simeq \mathscr{K}_j\mathscr{K}_i(M)$ by Definition 2.18.
(ii) By Proposition 2.3, we have $\mathscr{F}_j(\langle i\rangle^k \nabla M) \simeq \langle i\rangle^k \nabla \mathscr{F}_j(M)$ and

$$\mathfrak{d}_i(M) = \mathfrak{d}(\langle i\rangle, M) = \mathfrak{d}(\mathscr{F}_j(\langle i\rangle), \mathscr{F}_j(M)) = \mathfrak{d}(\langle i\rangle, \mathscr{F}_j(M)) = \mathfrak{d}_i(\mathscr{F}_j(M)),$$

which implies (ii).
(iii) follows from the fact that the functors $\mathscr{F}_i$ satisfy the braid relations ([36, 30]). □

Let $a \in [1, r-1]$ and assume that

$$(\alpha_{i_a}, \alpha_{i_{a+1}}) = 0.$$

We denote by $\mathsf{c} = (c_1, \ldots, c_r) \in \{\mathsf{k}, \mathsf{f}\}^r$ the KF sequence associated with $\underline{w}$ and v, and by c' the KF sequence associated with $\mathsf{r}_a\underline{w}$ and v (see (6.8)). We set $\mathcal{K} := [1, r]_v^{\underline{w}}$ and $\mathcal{K}' := [1, r]_v^{\mathsf{r}_a\underline{w}}$, and write

$$\mathscr{S}_{\underline{w}} := (M_k)_{k\in\mathcal{K}} \quad \text{and} \quad \mathscr{S}_{\mathsf{r}_a\underline{w}} := (M'_k)_{k\in\mathcal{K}'}.$$

Proposition 7.2. *We have*

$$\mathsf{c}' = \mathsf{r}_a\mathsf{c} \quad \textit{and} \quad \mathscr{S}_{\mathsf{r}_a\underline{w}} = \mathscr{S}_{\underline{w}} \quad \textit{as a set.}$$

Precisely, we have $M'_k = M_k$ for $k \in \mathcal{K} \setminus \{a, a+1\}$ and
 (i) *if $(c_a, c_{a+1}) = (\mathsf{k}, \mathsf{k})$, then $\mathcal{K}' = \mathcal{K}$ and $M'_b = M_c$ where $\{b, c\} = \{a, a+1\}$,*
 (ii) *if $(c_a, c_{a+1}) = (\mathsf{f}, \mathsf{k})$, then $\mathcal{K}' = (\mathcal{K} \setminus \{a+1\}) \cup \{a\}$ and $M'_a = M_{a+1}$,*
 (iii) *if $(c_a, c_{a+1}) = (\mathsf{f}, \mathsf{f})$, then $\mathcal{K}' = \mathcal{K}$.*

Proof. It follows by applying Lemma 7.1 to the definition of $\mathscr{S}_{\underline{w}}$. □

7.2. **Braid equivalence.** In this subsection, we show that $\mathscr{S}_{\underline{w},v}$ and $\mathscr{S}_{\underline{w}',v}$ belong to the same mutation class when $\underline{w}'$ is obtained from $\underline{w}$ by a single braid move. Recall that we assume that the Cartan matrix C is symmetric.

Lemma 7.3. *Let $i, j \in I$ such that $(\alpha_i, \alpha_j) = -1$. Let M be a simple module such that $\alpha_i, \alpha_j \notin \mathrm{W}(M)$, and set*

$$m := \mathfrak{d}_i(M) \quad \textit{and} \quad n := \mathfrak{d}_j(M).$$

Then we have
 (i) *$\mathfrak{d}_j(\langle i^c\rangle \nabla M) = c + \mathfrak{d}_j(M)$ for any $c \in \mathbb{Z}_{\geqslant 0}$. In particular, we have $\mathfrak{d}_j(\mathscr{K}_iM) = \mathfrak{d}_i(M) + \mathfrak{d}_j(M)$,*
 (ii) *$\mathscr{K}_j\mathscr{K}_iM \simeq \mathrm{hd}\big(\langle ji\rangle^{\circ m} \circ \langle j^n\rangle \circ M\big) \simeq \mathrm{hd}\big(\langle j^{m+n}\rangle \circ \langle i^m\rangle \circ M\big) \simeq \langle ji\rangle^{\circ m} \nabla \mathscr{K}_jM,$*

(iii) $\mathfrak{d}_i(\mathscr{K}_j\mathscr{K}_iM) = \mathfrak{d}_j(M)$,
(iv) $\mathrm{E}_i(\mathscr{K}_jM) \simeq 0$ *and* $\mathfrak{d}_i(\mathscr{F}_j\mathscr{K}_iM) = \mathfrak{d}(\langle ij\rangle, \mathscr{K}_iM) = \mathfrak{d}_j(M)$,
(v) $\mathrm{E}_i(\mathscr{F}_j\mathscr{K}_iM) \simeq 0$ *and* $\mathfrak{d}_j(\mathscr{F}_i\mathscr{F}_jM) = \mathfrak{d}_i(M)$.

Proof. (i) Since $(\langle j\rangle, M)$ and $(\langle i^c\rangle, \langle j\rangle)$ are unmixed pairs for $c \in \mathbb{Z}_{\geqslant 0}$, we have

$$\Lambda(\langle j\rangle, \langle i^c\rangle \nabla M) = \Lambda(\langle j\rangle, \langle i^c\rangle) + \Lambda(\langle j\rangle, M) \quad \text{and}$$
$$\Lambda(\langle i^c\rangle \nabla M, \langle j\rangle) = \Lambda(\langle i^c\rangle, \langle j\rangle) + \Lambda(M, \langle j\rangle)$$

by [29, Corollary 2.13] and [24, Lemma 2.7 and 2.8]. Hence $\mathfrak{d}_j(\langle i^c\rangle \nabla M) = \mathfrak{d}_j(\langle i^c\rangle) + \mathfrak{d}_j(M)$.

(ii) Since $(\langle j\rangle, M)$ is an unmixed pair, $\langle j^{m+n}\rangle \circ \langle i^m\rangle \circ M$ has a simple head. By (i), we have

$$\begin{aligned} \mathscr{K}_j\mathscr{K}_iM &\simeq \langle j^{m+n}\rangle \nabla (\langle i^m\rangle \nabla M) \simeq \mathrm{hd}\big(\langle j^{m+n}\rangle \circ \langle i^m\rangle \circ M\big) \\ &\simeq \mathrm{hd}\big(\langle ji\rangle^{\circ m} \circ \langle j^n\rangle \circ M\big) \\ &\simeq \langle ji\rangle^{\circ m} \nabla \mathscr{K}_jM. \end{aligned} \tag{7.1}$$

(iii) We have

$$\begin{aligned} \Lambda(\langle i\rangle, \mathscr{K}_j\mathscr{K}_iM) &= \Lambda\big(\langle i\rangle, \langle ji\rangle^{\circ m} \nabla (\langle j^n\rangle \nabla M)\big) \\ &= \Lambda\big(\langle i\rangle, \langle ji\rangle^{\circ m}\big) + \Lambda\big(\langle i\rangle, \langle j^n\rangle \nabla M\big) \\ &= \Lambda\big(\langle i\rangle, \langle ji\rangle^{\circ m}\big) + \Lambda\big(\langle i\rangle, \langle j^n\rangle\big) + \Lambda(\langle i\rangle, M) \\ &= -m + n + \Lambda(\langle i\rangle, M), \end{aligned}$$

where the first equality follows from (7.1), the second and third follow from the fact that $(\langle i\rangle, \langle j^n\rangle \nabla M)$ and $(\langle i\rangle, M)$ are unmixed, and the fourth follows from $\Lambda(\langle i\rangle, \langle ji\rangle) = -1$ and $\Lambda(\langle i\rangle, \langle j\rangle) = 1$. Similarly, we have

$$\begin{aligned} \Lambda(\mathscr{K}_j\mathscr{K}_iM, \langle i\rangle) &= \Lambda(\langle j^{m+n}\rangle \nabla \mathscr{K}_iM, \langle i\rangle) = \Lambda(\langle j^{m+n}\rangle, \langle i\rangle) + \Lambda(\mathscr{K}_iM, \langle i\rangle) \\ &= m + n - \Lambda(\langle i\rangle, \mathscr{K}_iM) \\ &= m + n - \Lambda(\langle i\rangle, M), \end{aligned}$$

where the first equality follows from (7.1), the second follows from the fact that $(\langle j\rangle, \langle i\rangle)$ is unmixed, the third follows from $\Lambda(\langle j\rangle, \langle i\rangle) = 1$ and $\mathfrak{d}(\langle i\rangle, \mathscr{K}_iM) = 0$, and the fourth follows from $\Lambda(\langle i\rangle, \mathscr{K}_iM) = \Lambda(\langle i\rangle, \langle i^m\rangle \nabla M) = \Lambda(\langle i\rangle, M)$. Hence, we have

$$2\,\mathfrak{d}_i(\mathscr{K}_j\mathscr{K}_iM) = \big(-m + n + \Lambda(\langle i\rangle, M)\big) + \big(m + n - \Lambda(\langle i\rangle, M)\big) = 2n.$$

(iv) Since $\alpha_i \notin \mathrm{W}(\langle j\rangle \circ M)$, we have $\mathrm{E}_i(\mathscr{K}_jM) \simeq 0$.

By the same manner as above, we have

$$\begin{aligned}\Lambda(\langle ij\rangle, \mathscr{K}_i M) &= \Lambda\big(\langle ij\rangle, \langle i\rangle^m \nabla M\big)\\ &= \Lambda\big(\langle ij\rangle, \langle i\rangle^m\big) + \Lambda\big(\langle ij\rangle, M\big)\\ &= -m + \Lambda(\langle i\rangle, M) + \Lambda(\langle j\rangle, M)\end{aligned}$$

and

$$\begin{aligned}\Lambda(\mathscr{K}_i M, \langle ij\rangle) &= \Lambda(\mathscr{K}_i M, \langle i\rangle) + \Lambda(\mathscr{K}_i M, \langle j\rangle)\\ &= -\Lambda(\langle i\rangle, \mathscr{K}_i M) + \Lambda(\langle i^m\rangle, \langle j\rangle) + \Lambda(M, \langle j\rangle)\\ &= -\Lambda(\langle i\rangle, M) + m + \Lambda(M, \langle j\rangle).\end{aligned}$$

Hence, by Proposition 2.3, we have

$$\begin{aligned}2\,\mathfrak{d}(\langle i\rangle, \mathscr{F}_j\mathscr{K}_i M) &= 2\,\mathfrak{d}\big(\mathscr{F}_j^{-1}\langle i\rangle, \mathscr{K}_i M\big) = 2\,\mathfrak{d}(\langle ij\rangle, \mathscr{K}_i M)\\ &= \big(-m + \Lambda(\langle i\rangle, M) + \Lambda(\langle j\rangle, M)\big) + \big(-\Lambda(\langle i\rangle, M) + m + \Lambda(M, \langle j\rangle)\big)\\ &= \Lambda(\langle j\rangle, M) + \Lambda(M, \langle j\rangle)\\ &= 2\,\mathfrak{d}_j(M).\end{aligned}$$

(v) By [26, Corollary 3.8], we have

$$\begin{aligned}\varepsilon_i(\mathscr{F}_j\mathscr{K}_i M) &= \widetilde{\Lambda}(\langle i\rangle, \mathscr{F}_j\mathscr{K}_i M) = \widetilde{\Lambda}(\mathscr{F}_j^{-1}\langle i\rangle, \mathscr{K}_i M) = \widetilde{\Lambda}(\langle ij\rangle, \langle i^m\rangle \nabla M)\\ &= \widetilde{\Lambda}(\langle ij\rangle, \langle i^m\rangle) + \widetilde{\Lambda}(\langle ij\rangle, M)\\ &= 0,\end{aligned}$$

where the last equality follows from the fact that $(\langle ij\rangle, \langle i\rangle)$ and $(\langle ij\rangle, M)$ are unmixed. This tells us that $\mathrm{E}_i(\mathscr{F}_j\mathscr{K}_i M) \simeq 0$.

Since $\langle j\rangle = \mathscr{F}_i\mathscr{F}_j\langle i\rangle$, we have

$$\mathfrak{d}\big(\langle j\rangle, \mathscr{F}_i\mathscr{F}_j M\big) = \mathfrak{d}\big(\mathscr{F}_i\mathscr{F}_j\langle i\rangle, \mathscr{F}_i\mathscr{F}_j M\big) = \mathfrak{d}\big(\langle i\rangle, M\big). \qquad \square$$

Corollary 7.4. *Let $i, j \in I$ such that $(\alpha_i, \alpha_j) = -1$. Let M be a simple module such that $\alpha_i, \alpha_j \notin \mathrm{W}(M)$. Then, we have*

(i) $\mathscr{K}_i\mathscr{K}_j\mathscr{K}_i M \simeq \mathscr{K}_j\mathscr{K}_i\mathscr{K}_j M$,
(ii) $\mathscr{F}_i\mathscr{K}_j\mathscr{K}_i M \simeq \mathscr{K}_j\mathscr{F}_i\mathscr{K}_j M$,
(iii) $\mathscr{F}_i\mathscr{F}_j\mathscr{K}_i M \simeq \mathscr{K}_j\mathscr{F}_i\mathscr{F}_j M$.
(iv) $\mathscr{F}_i\mathscr{F}_j\mathscr{F}_i M \simeq \mathscr{F}_j\mathscr{F}_i\mathscr{F}_j M$.

Proof. Set $m := \mathfrak{d}_i(M)$ and $n := \mathfrak{d}_j(M)$.

(i) Since $\langle ji\rangle$ commutes with $\langle i\rangle$ and $\langle j\rangle$, Lemma 7.3 (ii) says that

$$\mathscr{K}_j\mathscr{K}_i M \simeq (\langle j^{m+n}\rangle \nabla \langle i^m\rangle) \nabla M \simeq (\langle j^n\rangle \circ \langle ji\rangle^{\circ m}) \nabla M,$$

which implies

$$\begin{aligned}\mathscr{K}_i\mathscr{K}_j\mathscr{K}_i M &\simeq \langle i^n\rangle \nabla \mathscr{K}_j\mathscr{K}_i M \\ &\simeq \langle i^n\rangle \nabla \big((\langle j^n\rangle \circ \langle ji\rangle^{\circ m}) \nabla M\big) \\ &\simeq \big(\langle i^n\rangle \nabla (\langle j^n\rangle \circ \langle ji\rangle^{\circ m})\big) \nabla M \\ &\simeq \big(\langle ij\rangle^{\circ n} \circ \langle ji\rangle^{\circ m})\big) \nabla M,\end{aligned}$$

where the first equality follows from Lemma 7.3 (iii). Exchanging the roles of i and j in the above, we also obtain

$$\mathscr{K}_j\mathscr{K}_i\mathscr{K}_j M \simeq \big(\langle ji\rangle^{\circ m} \circ \langle ij\rangle^{\circ n})\big) \nabla M.$$

The assertion follows from the fact that $\langle ij\rangle$ commutes with $\langle ji\rangle$.

(ii) By Lemma 7.3 (ii), we have

$$\begin{aligned}\mathscr{F}_i\mathscr{K}_j\mathscr{K}_i M &\simeq \mathscr{F}_i\big(\langle ji\rangle^{\circ m} \nabla \mathscr{K}_j M\big) \\ &\simeq \mathscr{F}_i(\langle ji\rangle^{\circ m}) \nabla (\mathscr{F}_i\mathscr{K}_j M) \\ &\simeq \langle j^m\rangle \nabla \mathscr{F}_i\mathscr{K}_j M \\ &\simeq \mathscr{K}_j\mathscr{F}_i\mathscr{K}_j M,\end{aligned}$$

where the last equality follows by exchanging the roles of i and j in Lemma 7.3 (iv).

(iii) By Lemma 7.3 (v), we have

$$\begin{aligned}\mathscr{F}_i\mathscr{F}_j\mathscr{K}_i M &\simeq \mathscr{F}_i\mathscr{F}_j\big(\langle i^m\rangle \nabla M\big) \\ &\simeq \mathscr{F}_i\mathscr{F}_j\langle i^m\rangle \nabla \mathscr{F}_i\mathscr{F}_j M \\ &\simeq \langle j^m\rangle \nabla \mathscr{F}_i\mathscr{F}_j M \\ &\simeq \mathscr{K}_j\mathscr{F}_i\mathscr{F}_j M.\end{aligned}$$

(iv) It follows from [36, 30]. □

Let $a \in [1, r-2]$ and assume that

$$i_a = i_{a+2} \quad \text{and} \quad (\alpha_{i_a}, \alpha_{i_{a+1}}) = -1. \tag{7.2}$$

We set $\underline{w}'$ to be the reduced expression obtained from $\underline{w}$ by applying a single braid move at the $a+1$-position, i.e.,

$$\underline{w}' = (i_1, \ldots, i_{a-1}, i_{a+1}, i_a, i_{a+1}, i_{a+3}, \ldots, i_r).$$

We denote by $\mathsf{c} = (c_1, c_2, \dots, c_r) \in \{\mathsf{k}, \mathsf{f}\}^r$ the KF sequence associated with $\underline{w}$ and v, and by $\mathsf{c}' = (c'_1, c'_2, \dots, c'_r)$ the KF sequence associated with $\underline{w}'$ and v. We set $\mathcal{K}{:=}[1,r]^{\underline{w}}_{v}$ and $\mathcal{K}' := [1,r]^{\underline{w}'}_{v}$, and write

$$\mathscr{S}_{\underline{w}} := (M_k)_{k\in\mathcal{K}} \quad \text{and} \quad \mathscr{S}_{\underline{w}'} := (M'_k)_{k\in\mathcal{K}'}.$$

Note that, by (7.2) and the definition of c (see (6.8)), the triple (c_a, c_{a+1}, c_{a+2}) can be neither $(\mathsf{k},\mathsf{k},\mathsf{f})$ nor $(\mathsf{f},\mathsf{k},\mathsf{f})$.

Proposition 7.5. *For $k \in [1,r] \setminus \{a, a+1, a+2\}$ and $t \in \mathcal{K} \setminus \{a, a+1, a+2\}$, we have*

$$c'_k = c_k \quad \textit{and} \quad M'_k = M_k.$$

Moreover, we have the following.

(i) *If $(c_a, c_{a+1}, c_{a+2}) = (\mathsf{k},\mathsf{k},\mathsf{k})$, then $(c'_a, c'_{a+1}, c'_{a+2}) = (\mathsf{k},\mathsf{k},\mathsf{k})$, $\mathcal{K}' = \mathcal{K}$, and*

$$M'_a = \mu_a(M_a), \quad M'_{a+1} = M_{a+2}, \quad M'_{a+2} = M_{a+1},$$

which means $\mathsf{r}_{a+1}(\mathscr{S}_{\underline{w}'}) = \mu_a(\mathscr{S}_{\underline{w}})$.

(ii) *If $(c_a, c_{a+1}, c_{a+2}) = (\mathsf{f},\mathsf{k},\mathsf{k})$, then $(c'_a, c'_{a+1}, c'_{a+2}) = (\mathsf{k},\mathsf{f},\mathsf{k})$, $\mathcal{K}' = \mathcal{K} \cup \{a\} \setminus \{a+1\}$, and*

$$M'_a = M_{a+2} \quad \textit{and} \quad M'_{a+2} = M_{a+1},$$

which means $\mathscr{S}_{\underline{w}'} = \mathscr{S}_{\underline{w}}$ as a set.

(iii) *If $(c_a, c_{a+1}, c_{a+2}) = (\mathsf{f},\mathsf{f},\mathsf{k})$, then $(c'_a, c'_{a+1}, c'_{a+2}) = (\mathsf{k},\mathsf{f},\mathsf{f})$, $\mathcal{K}' = \mathcal{K} \cup \{a\} \setminus \{a+2\}$, and*

$$M'_a = M_{a+2},$$

which means $\mathscr{S}_{\underline{w}'} = \mathscr{S}_{\underline{w}}$ as a set.

(iv) *If $(c_a, c_{a+1}, c_{a+2}) = (\mathsf{f},\mathsf{f},\mathsf{f})$, then $(c'_a, c'_{a+1}, c'_{a+2}) = (\mathsf{f},\mathsf{f},\mathsf{f})$, $\mathcal{K}' = \mathcal{K}$, and $\mathscr{S}_{\underline{w}'} = \mathscr{S}_{\underline{w}}$.*

Proof. Set $\mathscr{T} := \mathscr{T}_{\underline{w},\mathsf{c}} = (M_k)_{k\in[1,r]}$ and $\mathscr{T}' := \mathscr{T}_{\underline{w}',\mathsf{c}'} = (M'_k)_{k\in[1,r]}$, and define

$$(N_k)_{k\in[1,r]} := \begin{cases} \mu_a(\mathscr{T}) & \text{in case (i) and (ii)}, \\ \mathscr{T} & \text{in case (iii)}. \end{cases}$$

As in Section 5.2, we write $S_k := \langle i_k \rangle$ if $c_k = \mathsf{k}$ and $S_k := \mathrm{M}(w_{\geqslant k}\Lambda_{i_k}, \Lambda_{i_k})$ if $c_k = \mathsf{f}$ so that

$$M_k = \mathscr{G}_1\mathscr{G}_2\cdots\mathscr{G}_{k-1}(S_k),$$

where $\mathscr{G}_k = \mathscr{K}_{i_k}$ if $c_k = \mathsf{k}$ and $\mathscr{G}_k = \mathscr{F}_{i_k}$ if $c_k = \mathsf{f}$.

We may assume that $a = 1$. Write

$$w' := s_{i_3}\cdots s_{i_r}, \quad i := i_1 \quad \text{and} \quad j := i_2,$$

so that $w = s_i s_j s_i w'$. It is obvious that $c'_k = c_k$ for $k > 3$. Since $M = \mathscr{G}_4 \cdots \mathscr{G}_{k-1}(S_k)$ satisfies $\mathrm{E}_i M = \mathrm{E}_j M = 0$, Corollary 7.4 implies $N_k = M_k = M'_k$ for $k > 3$.

(i) Assume $(c_1, c_2, c_3) = (\mathsf{k}, \mathsf{k}, \mathsf{k})$. In this case, it is clear that $(c'_1, c'_2, c'_3) = (\mathsf{k}, \mathsf{k}, \mathsf{k})$ and $\mathcal{K} = \mathcal{K}'$. By a direct computation, we have

$$M_1 = \langle i \rangle, \qquad M_2 = \mathscr{K}_i \langle j \rangle \simeq \langle ij \rangle, \qquad M_3 = \mathscr{K}_i \mathscr{K}_j \langle i \rangle \simeq \langle ji \rangle,$$
$$M'_1 = \langle j \rangle, \qquad M'_2 = \mathscr{K}_j \langle i \rangle \simeq \langle ji \rangle, \qquad M'_3 = \mathscr{K}_j \mathscr{K}_i \langle j \rangle \simeq \langle ij \rangle.$$

From the exact sequence $0 \to \langle ji \rangle \to \langle i \rangle \circ \langle j \rangle \to \langle ij \rangle \to 0$, we have $N_1 \simeq \langle j \rangle$, $N_2 \simeq \langle ij \rangle$ and $N_3 \simeq \langle ji \rangle$ by Lemma 5.1 (ii).

Thus we have

$$\mu_1(\mathscr{T}) = \mathsf{r}_2(\mathscr{T}'),$$

which implies the assertion.

(ii) Assume $(c_1, c_2, c_3) = (\mathsf{f}, \mathsf{k}, \mathsf{k})$. In this case, we have $s_i v < v$ and $s_j s_i v > s_i v$, which says $(c'_1, c'_2, c'_3) = (\mathsf{k}, \mathsf{f}, \mathsf{k})$ and $\mathcal{K}' = \mathcal{K} \cup \{a\} \setminus \{a+1\}$. By a direct computation, we have

$$\begin{aligned} &M_1 = \mathrm{M}(w\Lambda_i, \Lambda_i), && M_2 = \mathscr{F}_i \langle j \rangle \simeq \langle ij \rangle, \quad M_3 = \mathscr{F}_i \mathscr{K}_j \langle i \rangle \simeq \langle j \rangle, \text{ and} \\ &M'_1 = \langle j \rangle, && M'_2 = \mathscr{K}_j \mathrm{M}(s_i s_j w'\Lambda_i, \Lambda_i) \simeq \mathrm{M}(s_j s_i s_j w'\Lambda_i, \Lambda_i) \simeq \mathrm{M}(w\Lambda_i, \Lambda_i), \\ &M'_3 = \mathscr{K}_j \mathscr{F}_i \langle j \rangle \simeq \langle ij \rangle. \end{aligned}$$

Hence we have $\mathscr{T} = \mathscr{T}'$ as a set, which implies the assertion.

(iii) Assume $(c_1, c_2, c_3) = (\mathsf{f}, \mathsf{f}, \mathsf{k})$. In this case, we have $s_j s_i v < s_i v < v$ and $s_j s_i v < s_i s_j s_i v$, which says $(c'_1, c'_2, c'_3) = (\mathsf{k}, \mathsf{f}, \mathsf{f})$ and $\mathcal{K}' = \mathcal{K} \cup \{a\} \setminus \{a+2\}$. By a direct computation, we have

$$\begin{aligned} &M_1 = \mathrm{M}(w\Lambda_i, \Lambda_i), \qquad M_2 = \mathscr{F}_i \mathrm{M}(s_j s_i w'\Lambda_j, \Lambda_j) \simeq \mathrm{M}(w\Lambda_j, \Lambda_j), \\ &M_3 = \mathscr{F}_i \mathscr{F}_j \langle i \rangle \simeq \langle j \rangle, \\ &M'_1 = \langle j \rangle, \qquad M'_2 = \mathscr{K}_j \mathrm{M}(s_i s_j w'\Lambda_i, \Lambda_i) \simeq \mathrm{M}(s_j s_i s_j w'\Lambda_i, \Lambda_i) \simeq \mathrm{M}(w\Lambda_i, \Lambda_i), \\ &M'_3 = \mathscr{K}_j \mathscr{F}_i \mathrm{M}(s_j w'\Lambda_j, \Lambda_j) \simeq \mathrm{M}(s_j s_i s_j w'\Lambda_j, \Lambda_j) \simeq \mathrm{M}(w\Lambda_j, \Lambda_j) \end{aligned}$$

Hence we have $\mathscr{T} = \mathscr{T}'$ as a set, which implies the assertion.

(iv) Assume $(c_1, c_2, c_3) = (\mathsf{f}, \mathsf{f}, \mathsf{f})$. In this case, it is obvious by the definition of $\mathscr{S}_{\underline{w}}$. □

We now have the main theorem of this section from Proposition 7.2 and 7.5 .

Theorem 7.6. *Let $w, v \in \mathsf{W}$ with $v \leqslant w$. Then the mutation class of the seed $\mathscr{S}_{\underline{w},v}$ of $\mathscr{C}_{w,v}$ does not depend on the choice of reduced expressions $\underline{w}$ of w.*

8. Cluster algebras structure for $\mathscr{C}_{w,v}$

In this section, we discuss the cluster algebra structure of $\mathscr{C}_{w,v}$.

Let $w, v \in \mathsf{W}$ with $v \leqslant w$. Let $\underline{w} = (i_1, i_2, \dots, i_r)$ be a reduced expression of w, and set $\mathsf{J} := [1, r]$. We consider the completely admissible seeds

$$\mathscr{T}_{\underline{w},\mathsf{c}} = \{M_k\}_{k\in\mathsf{J}} \text{ of } \mathscr{C}_w \text{ and } \mathscr{S}_{\underline{w},v} = \{M_k\}_{k\in\mathsf{J}_{\underline{w},v}} \text{ of } \mathscr{C}_{w,v}$$

given by (5.4), (6.8) and (6.9), respectively (see Propositions 5.7 and Theorem 6.8). Thanks to Proposition 5.7 and Theorem 7.6, the mutation classes of $\mathscr{T}_{\underline{w},\mathsf{c}}$ and $\mathscr{S}_{\underline{w},v}$ do not depend on the choice of $\underline{w}$.

Let $\mathsf{J}^{\rm ex}$ and $\mathsf{J}^{\rm fr}$ be the index sets for $\mathscr{T}_{\underline{w},\mathsf{c}}$ given in (5.6), and let $\mathsf{J}^{\rm ex}_{\underline{w},v}$ and $\mathsf{J}^{\rm fr}_{\underline{w},v}$ be the index sets for $\mathscr{S}_{\underline{w},v}$ given in (6.10). Let $\widetilde{B}_{\underline{w},\mathsf{c}} = (b_{i,j})_{i\in\mathsf{J},j\in\mathsf{J}^{\rm ex}}$ be the exchange matrix determined by $\mathscr{T}_{\underline{w},\mathsf{c}}$ with $\mathsf{J} = \mathsf{J}^{\rm ex} \sqcup \mathsf{J}^{\rm fr}$,and set

$$\widetilde{B} := \widetilde{B}_{\underline{w},\mathsf{c}}|_{\mathsf{J}_{\underline{w},v}\times\mathsf{J}^{\rm ex}_{\underline{w},v}}.$$

Note that $\widetilde{B}$ is the exchange matrix for $\mathscr{S}_{\underline{w},v}$. We set $\Lambda := \big(\Lambda(M_i, M_j)\big)_{i,j\in\mathsf{J}_{\underline{w},v}}$ and let

$$\Sigma_{\underline{w},v} := (\{q^{-(\mathrm{wt}(M_k),\mathrm{wt}(M_k))/4}[M_k]\}_{k\in\mathsf{J}_{\underline{w},v}}, -\Lambda, \widetilde{B})$$

be a quantum seed in $K(\mathscr{C}_{w,v})$. We denote by

$$\mathscr{A}_{\underline{w},v} := \mathscr{A}(\Sigma_{\underline{w},v}) \quad \text{and} \quad \mathscr{U}_{\underline{w},v} := \mathscr{U}(\Sigma_{\underline{w},v})$$

the quantum cluster algebra and the quantum upper cluster algebra with the initial seed $\Sigma_{\underline{w},v}$, respectively.

The following lemma follows from Theorem 6.8 and Proposition 2.17.

Lemma 8.1. *Every seed which is mutation equivalent to $\mathscr{S}_{\underline{w},v}$ is a Laurent family in $\mathscr{C}_{w,v}$.*

We now obtain the main theorem.

Theorem 8.2. *Let $w, v \in \mathsf{W}$ with $v \leqslant w$ and let $\underline{w}$ be a reduced expression of w. Then we have*

$$\mathscr{A}_{\underline{w},v} \subset K(\mathscr{C}_{w,v}) \subset \mathscr{U}_{\underline{w},v}.$$

Proof. Since $\{M_k\}_{k\in\mathsf{J}_{\underline{w},v}}$ is contained in $\mathscr{C}_{w,v}$, we have $\mathscr{A}_{\underline{w},v} \subset K(\mathscr{C}_{w,v})$.

For any seed $\mathscr{S}$ mutation-equivalent to $\mathscr{S}_{\underline{w},v}$, we have an injective homomorphism

$$\varphi_{\mathscr{S}} \colon K(\mathscr{C}_{w,v}) \rightarrowtail \mathcal{T}_{\mathscr{S}}$$

by Lemma 8.1 and (2.10). This implies $K(\mathscr{C}_{w,v}) \subset \mathscr{U}_{\underline{w},v}$. $\square$

The following corollary was proved in [3, Theorem 5.16] by a different method.

Corollary 8.3. *Let $\mathfrak{g}$ be of finite ADE type. For $v \leqslant w$, the category $\widetilde{\mathscr{C}}_{w,v}$ is a monoidal categorification of the coordinate ring $\mathbb{C}[\mathcal{R}_{w,v}]$ of open Richardson variety associated with w and v.*

Proof. By taking localizations by frozen variables, we have

$$\widetilde{\mathscr{A}_{\underline{w},v}} \subset K(\widetilde{\mathscr{C}}_{w,v}) \subset \widetilde{\mathscr{U}_{\underline{w},v}},$$

where $\widetilde{\mathscr{A}_{\underline{w},v}}$ and $\widetilde{\mathscr{U}_{\underline{w},v}}$ are cluster algebra and upper cluster algebra with invertible frozen variables, respectively.

By Proposition 6.10, $\mathscr{S}_{\underline{w},v}$ is the seed of Leclerc in [42, Corollary 4.4]. By [3, Theorem 5.14], it is the same with the seed of Ménard in [45]. By [7, Section 10], Ménard's seed is a seed (associated with a right inductive weave) of a braid variety. Since the cluster algebras of braid varieties are equal to their upper cluster algebras, we obtain

$$\widetilde{\mathscr{A}_{\underline{w},v}} = K(\widetilde{\mathscr{C}}_{w,v}) = \widetilde{\mathscr{U}_{\underline{w},v}}.$$

Now the assertion follows from Theorem 6.8. □

Conjecture 1. *For any $w, v \in \mathsf{W}$ with $v \leqslant w$, we have $\mathscr{A}_{\underline{w},v} = \mathscr{U}_{\underline{w},v}$. This means that $\mathscr{C}_{w,v}$ provides a monoidal categorification of $K(\mathscr{C}_{w,v})$.*

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