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Reasons and Grounds: A Proof-Theoretical Investigation

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Abstract
The key idea of this paper is that grounds are a special kind of reasons, so their logic is part of the logic of reasons. We outline a natural deduction calculus that provides a basic formal characterization of reasons and enables us to obtain some distinctive and relatively uncontentious principles about grounds. Then we show that the calculus outlined is consistent and decidable, which we take to be an interesting result in its own right.

Keywords Ground · Epistemic reason · Proof theory · Translation · Normalisation · Cut-elimination · Decidability · Consistency

1 Overview

Let us start with some preliminary clarifications about our use of the terms 'reason' and 'ground'. In what follows, 'reason' is understood as 'epistemic reason', that is, reason for belief. For two propositions p and q, to say that p is a reason for q in the sense that matters here is to say that assuming p provides a justification for believing q. In other words, p is a reason for q when p supports q, or equivalently when q can be inferred from p. We use the symbol $\triangleright$ to represent this relation, so $p \triangleright q$ is to be read as 'p is a reason for q'.

For example, the following sentences are plausibly understood as sentences of the form $p \triangleright q$:

(1) Tweetie's being a bird implies that it has wings
(2) If Tweetie is a bird, it has wings
(3) Tweetie's being a bird suggests that it can fly
(4) If Tweetie is a bird, it can fly

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(1) and (2) convey in different ways the claim that the proposition that Tweetie is a bird supports the proposition that Tweetie has wings. In other words, ‘Tweetie is a bird’ is a reason for ‘Tweetie has wings’. Similarly, (3) and (4) convey in different ways the claim that the proposition that Tweetie is a bird supports the proposition that Tweetie can fly. In other words, ‘Tweetie is a bird’ is a reason for ‘Tweetie can fly’.

The relation of support stated in (1) and (2) is stronger than the relation of support stated in (3) and (4). ‘Tweetie is a bird’ is a *conclusive* reason for ‘Tweetie has wings’, for birds have wings by definition, so Tweety cannot be a bird and lack wings. By contrast, ‘Tweetie is a bird’ is only a *defeasible* reason for ‘Tweetie can fly’ for Tweetie might be one of the atypical birds which cannot fly. Yet there is a clear sense in which (1)-(4) are all alike from a formal point of view. Our use of the symbol $\triangleright$ is neutral as to the distinction between conclusive and defeasible reasons because is intended to model reasons in the most general sense.[1]

As far as ‘ground’ is concerned, in what follows we restrict consideration to the canonical notion of ground, the notion that has been most studied by philosophers and logicians in the last decade. For two propositions p and q, to say that p grounds q in the sense that matters here is to say that p makes it the case that q, where ‘makes it the case’ can be spelled out at least in two ways: one is that p necessitates q, that is, if p is the case, q must be the case; the other is that p is more fundamental than q in some explanatory sense. We use the symbol $<$ to express this relation, so $p < q$ is to be read as ‘p grounds q’.

For example, the following sentences are plausibly understood as sentences of the form $p < q$:

(5) The sky is coloured because it is blue
(6) The sky is coloured in virtue of being blue
(7) John is a bachelor because he is an adult, unmarried male
(8) John is a bachelor in virtue of being an adult, unmarried male

(5) and (6) convey in different ways the claim that the sky being blue grounds its being coloured. In other words, ‘The sky is blue’ grounds ‘The sky is coloured’. Similarly, (7) and (8) convey in different ways the claim that John being an adult, unmarried male grounds his being a bachelor. In other words, ‘John is an adult, unmarried male’ grounds ‘John is a bachelor’.

It must be noted that (5)-(6) differ from (1)-(4) in at least two respects. First, the relation stated in (5)-(6) is naturally understood as factive, that is, as implying that the propositions that feature as its terms are both true. In the case (1)-(4), instead, factivity is less obvious, especially in the case of (2) and (4). Second, while in (1)-(4) the relation stated is one-to-one, in that it is taken to obtain between two propositions, in (7) and (8) the relation stated may naturally be read as many-to-one, in the sense that different facts — John’s being adult, John’s being unmarried, and John’s being male — are taken to ground a single fact, John’s being a bachelor.

However, neither of these two differences matters for our purposes. As to the first, although ground is generally understood as factive, nothing in the main formal accounts of ground forces a factive interpretation of $<$, and it turns out that interpreting $<$ as

[1] As customary in the literature on defeasible reasoning, we understand ‘defeasible’ as ‘non-monotonic’.

non-factive is a natural option.[2] So it will do no harm to assume that both $\triangleright$ and $<$ express relations between propositions without requiring that these propositions are true. As to the second, it is certainly plausible to model grounds in terms of many-to-one relations, but the same goes for reasons. For example, the following sentence may be read as expressing the claim that John being adult, John being unmarried, and John being male together provide a reason for believing that John is a bachelor:

(9) If John is an adult, unmarried male, he is a bachelor

More generally, for any finite set of propositions $\{p_1, \ldots, p_n\}$, it is plausible to treat the claim that $\{p_1, \ldots, p_n\}$ is a reason for q as equivalent to the claim that the conjunction of $p_1, \ldots, p_n$ is a reason for q. So it can be assumed that statements about reasons and statements about grounds are structurally similar.

Given this structural similarity, it is natural to ask whether there is some interesting logical relation between reasons and grounds. At least two points may be taken as uncontentious, or so we assume. The first is that whenever p grounds q, it is also the case that p is a reason for q. For example, just as it seems correct to say that the sky being blue grounds its being coloured, as in (5) and (6), it seems equally correct to say that the sky being blue is a reason for thinking that the sky is coloured. The second is that the converse does not hold: p can be a reason for q without grounding q. This is due, among other things, to the fact that ground is essentially asymmetric, which means that it is both irreflexive and antisymmetric. As the expressions 'because' and 'in virtue of' imply, a single proposition cannot ground itself, and two distinct propositions cannot ground each other. For example, it would make no sense to say that John is a bachelor because he is a bachelor, or that John is an adult, unmarried male because he is a bachelor, which is the converse of (7). By contrast, there seems to be nothing conceptually wrong in the claim that a single proposition supports itself, even though it may be a trivial claim, and it is certainly possible that two distinct propositions support each other. For example, consider (9) and (10):

(10) If John is a bachelor, he is an adult, unmarried male

Just as John's being an adult, unmarried male is a reason for thinking that he is a bachelor, John's being a bachelor is a reason for thinking that he is an adult, unmarried male.

This paper outlines a formal account of the relation between reasons and grounds that accords with these two points. Its key idea is that grounds are a special kind of reasons, so their logic is part of the logic of reasons. In order to develop this idea we set out two systems of natural deduction named **R** and **G**. The system **R**, which is the main focus of the paper, is a basic logic for $\triangleright$ inspired by the formal treatment of reasons recently suggested by Crupi and Iacona.[3] The system **G**, instead, is a minimal logic for $<$ that includes some canonical principles discussed by Fine, Correia, Litland and others.[4] As will be shown, for any claim of the form $\{\alpha_1, \ldots, \alpha_n\} < \beta$ provable in **G**, the corresponding claim $\{\alpha_1, \ldots, \alpha_n\} \triangleright \beta$ is provable in **R**, so there is a sense in

[2] See, for instance, [19, page 58], on the notion of potential explanation.

[3] [5, 6, 18].

[4] See [26], [8–10], [1, 2], [20, 21]. Here we mainly follow [4].

which the main principles concerning grounds are also principles concerning reasons. On the other hand, some principles about reasons provable in **R** do not correspond to principles about grounds provable in **G**. Once it is established that the relation expressed by $<$ is a subset of the relation expressed by $\triangleright$, we prove that **R** is consistent and decidable, which we take to be an interesting result in its own right.

The paper is structured as follows. Section 2 provides some preliminary definitions. Section 3 sets out the system **R**. Section 4 sets out the system **G**. Section 5 presents a cut-elimination result for **G**. Section 6 shows how $<$, as characterised by **G**, is a subset of $\triangleright$, as characterised by **R**. Section 7 establishes a normalisation result for **R**. In Sections 8 and 9 this result is used to prove that **R** is consistent and decidable. Finally, Section 10 provides some concluding remarks.

2 Preliminary Definitions

Definition 2.1 $\mathcal{L}$ is a language whose vocabulary is constituted by a set of sentence letters $p, q, r, \ldots$, the symbol $\bot$, and the connectives $\wedge$ and $\supset$. The formulas of $\mathcal{L}$ are defined as follows:

- if α is one of the sentence letters $p, q, r, \ldots$ or $\bot$, then $\alpha \in \mathcal{L}$;
- if $\alpha, \beta \in \mathcal{L}$, then $\alpha \wedge \beta \in \mathcal{L}$ and $\alpha \supset \beta \in \mathcal{L}$.

Definition 2.2 $\mathcal{L}_{\triangleright}$ is a language with the vocabulary of $\mathcal{L}$ plus the symbol $\triangleright$. The formulas of $\mathcal{L}_{\triangleright}$ are defined as follows:

- if $\alpha \in \mathcal{L}$, then $\alpha \in \mathcal{L}_{\triangleright}$;
- if $\alpha_1, \ldots, \alpha_n, \beta \in \mathcal{L}$, then $\{\alpha_1, \ldots, \alpha_n\} \in \mathcal{L}_{\triangleright}$ and $\{\alpha_1, \ldots, \alpha_n\} \triangleright \beta \in \mathcal{L}_{\triangleright}$;
- if $\alpha, \beta \in \mathcal{L}_{\triangleright}$, $\alpha \wedge \beta \in \mathcal{L}_{\triangleright}$ and $\alpha \supset \beta \in \mathcal{L}_{\triangleright}$.

Definition 2.3 $\mathcal{L}_{<}$ is a language with the vocabulary of $\mathcal{L}$ plus the symbol $<$. The formulas of $\mathcal{L}_{<}$ are defined as follows:

- if $\alpha \in \mathcal{L}$, then $\alpha \in \mathcal{L}_{<}$;
- if $\alpha_1, \ldots, \alpha_n, \beta \in \mathcal{L}$, then $\{\alpha_1, \ldots, \alpha_n\} < \beta \in \mathcal{L}_{<}$.

In Definitions 2.2 and 2.3, the second clause covers the possibility that $n = 1$, in which case $\{\alpha\}$ will be denoted as α, and consequently $\{\alpha\} \triangleright \beta$ and $\{\alpha\} < \beta$ will be abbreviated respectively as $\alpha \triangleright \beta$ and $\alpha < \beta$. Notice, moreover, that these definitions rule out the possibility of nesting $<$ and $\triangleright$ inside other occurrences of these connectives.

Independently of which language is adopted, we use Greek lowercase letters $\alpha, \beta, \gamma, \ldots$ to denote its formulas. Finite sets of formulas are denoted by capital Greek letters $\Gamma, \Delta \ldots$. Given any formula α and any finite set of formulas Γ, the notation Γ, α, or α, Γ, refers to the set obtained by adding α to Γ. Moreover, for any formula α, we stipulate that $\sim\alpha$ abbreviates $\alpha \supset \bot$.

Definition 2.4 For any formula α and any binary connective $\star$, the *depth* of α, that is, $|\alpha|$ is inductively defined as follows:

- $|\bot| = |p| = 0$

- $|\beta{\star}\gamma| = \max(|\beta|, |\gamma|) + 1$

A final clarification concerns the notation employed for derivations throughout the paper. We explicitly indicate the hypotheses on which the conclusion of a derivation depends. That is, a derivation with conclusion $\Gamma \Rightarrow \alpha$ is a derivation of α that depends on the formulas contained in Γ. Moreover, we adopt the abbreviation $\alpha \Leftrightarrow \beta$ to indicate the pair of premises $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. As a consequence, by the notation

$$\begin{array}{c}\pi\\ \alpha \Leftrightarrow \beta\end{array}$$

we indicate a pair of derivations, one for $\alpha \Rightarrow \beta$ and one for $\beta \Rightarrow \alpha$. We denote these two derivations by by $\pi^{\Rightarrow}$ and $\pi^{\Leftarrow}$ in such a way that the following are well-defined derivations:

$$\begin{array}{c}\pi^{\Rightarrow}\\ \alpha \Rightarrow \beta\end{array} \qquad \begin{array}{c}\pi^{\Leftarrow}\\ \beta \Rightarrow \alpha\end{array}$$

Sometimes, instead of using the notation below on the left for a pair of premises, we use the notation below on the right in order to display the derivations $\pi^{\Rightarrow}$ and $\pi^{\Leftarrow}$ of the pair of premises α—depending on β—and β—depending on α:

$$\begin{array}{c}\pi^{\Rightarrow}\\ \alpha \Rightarrow \beta\end{array} \quad \begin{array}{c}\pi^{\Leftarrow}\\ \beta \Rightarrow \alpha\end{array} \qquad\qquad \begin{array}{c}\left(\begin{array}{c}\pi^{\Rightarrow}\\ \alpha \Rightarrow \beta\end{array} \;\; \begin{array}{c}\pi^{\Leftarrow}\\ \beta \Rightarrow \alpha\end{array}\right)\\ \alpha \Leftrightarrow \beta\end{array}$$

3 The System R

In Fig. 1, we present the set of classical rules that constitutes the deductive basis for **R**.

Here i stands for 'introduction' and e stands for 'elimination'. In general, for any connective $\star$, we call $\star i$ the *introduction rule* for $\star$ and $\star e$ the *elimination rule* for $\star$. The *major premise* of an elimination rule for a connective $\star$ is the premise of the form $\alpha{\star}\beta$. We call *minor premise* any other premise of an elimination rule. Note that here EFQ and DNE are defined by using the sentence letter p, which means that they are restricted to atomic formulas.

R is the system in $\mathcal{L}_{\triangleright}$ obtained by adding the rules for $\triangleright$ in Fig. 2 to the rules in Fig. 1.

The rule C expresses the principle of *Contraposition*, which Crupi and Iacona regard as the cornerstone of the logic of reasons. According to them, this principle plausibly holds independently of the distinction between conclusive and defeasible reasons.[5]

The rule $\equiv$ guarantees that the relation expressed by $\triangleright$ preserves substitution of logically equivalent formulas. The major premise of this rule is the premise of the form $\alpha \triangleright \beta$, while the other premises are minor premises.

The rule $\triangleright i$ is the introduction rule for $\triangleright$. This rule expresses the principle of *Supraclassicality*, according to which $\triangleright$ holds whenever β logically follows from α.[6] Indeed, the restriction on the context of the premise of $\triangleright i$—that is, only α can occur

[5] This point is discussed in [6] and formally developed in [5].

[6] This label is adopted in [18] and in [7].

$$\frac{}{\Gamma, \alpha \Rightarrow \alpha}\ \text{Id} \qquad \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow p}\ \text{EFQ} \qquad \frac{\Gamma \Rightarrow \sim\sim p}{\Gamma \Rightarrow p}\ \text{DNE}$$

$$\frac{\Gamma, \gamma \Rightarrow \delta}{\Gamma \Rightarrow \gamma \supset \delta} \supset i \qquad \frac{\Gamma \Rightarrow \delta}{\Gamma \Rightarrow \gamma \supset \delta} \supset i \qquad \frac{\Gamma \Rightarrow \alpha \supset \beta \quad \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \beta} \supset e$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \wedge \beta} \wedge i \qquad \frac{\Gamma \Rightarrow \alpha \wedge \beta}{\Gamma \Rightarrow \alpha} \wedge e \qquad \frac{\Gamma \Rightarrow \alpha \wedge \beta}{\Gamma \Rightarrow \beta} \wedge e$$

Fig. 1 Classical Rules

to the left of $\Rightarrow$—implies that we can only infer the conclusion $\alpha \triangleright \beta$ when β can be derived from the sole assumption α. The rule $\triangleright e$ is the elimination rule for $\triangleright$. The plausibility of this rule is obvious, since it expresses the principle of *Modus Ponens*. Note that $\triangleright e$ does not force us to interpret α in $\alpha \triangleright \beta$ as a conclusive reason. Regardless of whether the relation of support represented by $\triangleright$ is understood as conclusive or defeasible, it seems correct to say that if one has a justification for α and α is a reason for β, then one has thereby a justification for β. Finally, Expl is a rule of *Explosion* that specifically concern formulas of the form $\alpha \triangleright \beta$, so it complements the restricted rule EFQ presented in Fig. 1.[7]

A derivation in **R** is defined in the usual way.

In order to properly compare **R** with **G**, we introduced in $\mathcal{L}_{\triangleright}$ the notation $\{\,\}$ for sets of formulas. The rules governing $\{\,\}$ are presented in Fig. 3. Let $\mathbf{R} + \{\,\}$ be the system defined by adding the rules $\{\,\}i$ and $\{\,\}e$ to **R**.

In $\mathbf{R} + \{\,\}$, sets of formulas indicated by $\{\,\}$ behave like conjunctions, which enables us to treat all formulas of the form $\{\alpha_1, \ldots, \alpha_n\}$ in terms of $\wedge$. Indeed, we can prove that there is a translation from $\mathcal{L}_{\triangleright}$ to the $\{\,\}$-free fragment of $\mathcal{L}_{\triangleright}$ that preserves derivability. This enables us to restrict consideration to **R** leaving aside the rules for $\{\,\}$. The translation from derivations in $\mathbf{R} + \{\,\}$ to derivations in **R** also guarantees that the results for **R** presented in the rest of the paper also hold for $\mathbf{R} + \{\,\}$, as we could treat the rules for $\{\,\}$ as we treat the rules for $\wedge$.

Definition 3.1 The translation $t_{\{\}}$ from formulas of $\mathcal{L}_{\triangleright}$ to the $\{\,\}$-free fragment of $\mathcal{L}_{\triangleright}$ is defined as follows.

- for any atomic formula α, $t_{\{\}}(\alpha) = \alpha$,
- $t_{\{\}}(\alpha \wedge \beta) = t_{\{\}}(\alpha) \wedge t_{\{\}}(\beta)$,
- $t_{\{\}}(\alpha \supset \beta) = t_{\{\}}(\alpha) \supset t_{\{\}}(\beta)$,
- $t_{\{\}}(\alpha \triangleright \beta) = t_{\{\}}(\alpha) \triangleright t_{\{\}}(\beta)$,
- $t_{\{\}}(\{\alpha_1, \ldots, \alpha_n\}) = t_{\{\}}(\alpha_1) \wedge \ldots \wedge t_{\{\}}(\alpha_n)$,

For any set of formulas Γ, $t_{\{\}}(\Gamma) = \{t_{\{\}}(\gamma) \mid \gamma \in \Gamma\}$.

We can now show that any derivation in $\mathbf{R} + \{\,\}$ can be translated into a derivation in **R**. This enables us to treat any formula of the form $\{\alpha_1, \ldots, \alpha_n\}$ in **R** as an abbreviation for $\alpha_1 \wedge \ldots \wedge \alpha_n$, and to assume, from now on, that no derivation in **R** contains $\{\,\}$.

[7] Crupi et al., [5] presents a system of "minimal evidential logic" defined in terms of C, $\equiv$, and Expl. Although **R** is slightly stronger, as it includes $\triangleright i$ and $\triangleright e$, the underlying ideas are essentially the same.

$$\frac{\Gamma \Rightarrow \alpha \rhd \beta}{\Gamma \Rightarrow \sim\beta \rhd \sim\alpha}\ \mathrm{C} \qquad \frac{\Gamma \Rightarrow \alpha \rhd \beta \quad \alpha \Leftrightarrow \gamma \quad \beta \Leftrightarrow \delta}{\Gamma \Rightarrow \gamma \rhd \delta}\ \equiv$$

$$\frac{\alpha \Rightarrow \beta}{\Rightarrow \alpha \rhd \beta}\ \rhd i \qquad \frac{\Gamma \Rightarrow \alpha \rhd \beta \quad \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \beta}\ \rhd e$$

$$\frac{\Gamma \Rightarrow \bot \quad \Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rhd \beta}\ \mathrm{Expl}$$

Fig. 2 Rules for $\rhd$

Theorem 3.1 *For any finite set of formulas Γ and formulas $\alpha_1, \ldots, \alpha_n, \beta \in \mathcal{L}_{\rhd}$, if β is derivable from $\alpha_1, \ldots, \alpha_n$ by the rules $\{\}i$ and $\{\}e$ given Γ, then $t_{\{\}}(\beta)$ is derivable from $t_{\{\}}(\alpha)$ given $t_{\{\}}(\Gamma)$.*

Proof If the rule employed is $\{\}i$, then $\beta = \{\alpha_1, \ldots, \alpha_n\}$ and the $\{\}$-rule application is of the form

$$\frac{\Gamma_1 \Rightarrow \alpha_1 \quad \ldots \quad \Gamma_n \Rightarrow \alpha_n}{\Gamma_1, \ldots, \Gamma_n \Rightarrow \{\alpha_1, \ldots, \alpha_n\}}\ \{\}i$$

We simulate this by the derivation

$$\dfrac{\dfrac{\dfrac{t_{\{\}}(\Gamma_1) \Rightarrow t_{\{\}}(\alpha_1) \quad t_{\{\}}(\Gamma_2) \Rightarrow t_{\{\}}(\alpha_2)}{t_{\{\}}(\Gamma_1, \Gamma_2) \Rightarrow t_{\{\}}(\alpha_1) \wedge t_{\{\}}(\alpha_2)}\ \wedge i \quad \vdots \quad}{\dfrac{t_{\{\}}(\Gamma_1, \ldots, \Gamma_{n-2}) \Rightarrow t_{\{\}}(\alpha_1) \wedge \ldots \wedge t_{\{\}}(\alpha_{n-2}) \quad t_{\{\}}(\Gamma_{n-1}) \Rightarrow t_{\{\}}(\alpha_{n-1})}{t_{\{\}}(\Gamma_1, \ldots, \Gamma_{n-1}) \Rightarrow t_{\{\}}(\alpha_1) \wedge \ldots \wedge t_{\{\}}(\alpha_{n-1})}\ \wedge i \quad t_{\{\}}(\Gamma_n) \Rightarrow t_{\{\}}(\alpha_n)}}{t_{\{\}}(\Gamma_1, \ldots, \Gamma_n) \Rightarrow t_{\{\}}(\alpha_1) \wedge \ldots \wedge t_{\{\}}(\alpha_n)}\ \wedge i$$

If the rule employed is $\{\}e$, then $\alpha = \{\alpha_1, \ldots, \alpha_n\}$, $\beta = \alpha_j$ for $j \in \{1, \ldots, n\}$, and the $\{\}$ rule application is of the form

$$\frac{\Gamma \Rightarrow \{\alpha_1, \ldots, \alpha_n\}}{\Gamma \Rightarrow \alpha_j}\ \{\}e$$

Since $t_{\{\}}(\{\alpha_1, \ldots, \alpha_n\}) = t_{\{\}}(\alpha_1) \wedge \ldots \wedge t_{\{\}}(\alpha_n)$, we simulate this by the derivation

$$\dfrac{\dfrac{t_{\{\}}(\Gamma) \Rightarrow t_{\{\}}(\alpha_1) \wedge \ldots \wedge t_{\{\}}(\alpha_n)}{\vdots}\ \wedge e}{t_{\{\}}(\Gamma) \Rightarrow t_{\{\}}(\alpha_j)}\ \wedge e$$

□

Corollary 3.1.1 *Any derivation in* $\mathbf{R} + \{\}$ *can be translated into a derivation in* $\mathbf{R}$.

$$\frac{\Gamma_1 \Rightarrow \alpha_1 \quad \ldots \quad \Gamma_n \Rightarrow \alpha_n}{\Gamma_1, \ldots, \Gamma_n \Rightarrow \{\alpha_1, \ldots, \alpha_n\}}\ \{\}i \qquad \frac{\Gamma \Rightarrow \{\alpha_1, \ldots, \alpha_n\}}{\Gamma \Rightarrow \alpha_j}\ \{\}e$$

where $\alpha_1, \ldots, \alpha_n$ do not contain $\{\}$ and $j \in \{1, \ldots, n\}$

Fig. 3 Rules for sets of formulas

Proof Since $t_{\{\}}$ leaves unchanged the connectives of $\mathcal{L}$, the result follows from Theorem 3.1 by a simple inductive argument on the number of rules applied in the $\mathbf{R} + \{\}$ derivation. □

We establish two further facts concerning the rules EFQ and DNE. Although these rules are restricted to atomic formulas in Fig. 1, the fully generalized version of EFQ and the generalization of DNE to the whole classical fragment of our language are derivable in **R**.

Theorem 3.2 *The rule* $\dfrac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \alpha}$ *is admissible in* **R**.

Proof We reason by induction on the depth of α to show that, if there is a derivation $\Gamma \overset{\pi}{\Rightarrow} \bot$, then there is a derivation with conclusion α depending on Γ. In the base case, α is either of the form p or of the form $\bot$. If $\alpha = p$, then

$$\dfrac{\overset{\pi}{\Gamma \Rightarrow \bot}}{\Gamma \Rightarrow p}\ \text{EFQ}$$

If $\alpha = \bot$, then the derivation π is already there.

Suppose now that the statement holds for any formula of depth less than n. We show that it holds for any α of depth n. We distinguish cases depending on the form of α.

- $\alpha = \beta \wedge \gamma$. By inductive hypothesis, two derivations $\Gamma \overset{\pi_1}{\Rightarrow} \beta$ and $\Gamma \overset{\pi_2}{\Rightarrow} \gamma$ exist. So, we obtain the desired derivation as follows:

$$\dfrac{\overset{\pi_1}{\Gamma \Rightarrow \beta} \quad \overset{\pi_2}{\Gamma \Rightarrow \gamma}}{\Gamma \Rightarrow \beta \wedge \gamma}\ \wedge i$$

- $\alpha = \beta \supset \gamma$. By inductive hypothesis, the derivation $\Gamma \overset{\pi_1}{\Rightarrow} \gamma$ exists. So, we obtain the desired derivation as follows:

$$\dfrac{\overset{\pi_1}{\Gamma \Rightarrow \gamma}}{\Gamma \Rightarrow \beta \supset \gamma}\ \supset i$$

- $\alpha = \beta \rhd \gamma$. By inductive hypothesis, the derivations $\Gamma \overset{\pi_1}{\Rightarrow} \beta$ and $\Gamma \overset{\pi_2}{\Rightarrow} \gamma$ exist. Moreover, by the hypothesis in the statement, there exists a derivation $\Gamma \overset{\pi}{\Rightarrow} \bot$. Hence, we obtain the desired derivation as follows:

$$\dfrac{\overset{\pi}{\Gamma \Rightarrow \bot} \quad \overset{\pi_1}{\Gamma \Rightarrow \beta} \quad \overset{\pi_2}{\Gamma \Rightarrow \gamma}}{\Gamma \Rightarrow \beta \rhd \gamma}\ \text{Expl}$$

□

Theorem 3.3 *For any formula* $\alpha \in \mathcal{L}$*, the rule* $\dfrac{\Gamma \Rightarrow {\sim}{\sim}\alpha}{\Gamma \Rightarrow \alpha}$ *is admissible in* **R**.

Proof Consider any application of the rule $\dfrac{\overset{\pi}{\Gamma \Rightarrow {\sim}{\sim}\gamma}}{\Gamma \Rightarrow \gamma}$ occurring inside a derivation. Let us reason by induction on the depth of γ.

If $\gamma = p$, then the rule application is just an application of DNE. If $\gamma = \bot$, we can replace $\dfrac{\overset{\pi}{\Gamma \Rightarrow {\sim}{\sim}\bot}}{\Gamma \Rightarrow \bot}$ by

$$\dfrac{\dfrac{\dfrac{}{\bot \Rightarrow \bot}\,\text{Id}}{\Rightarrow {\sim}\bot}\supset i \qquad \overset{\pi}{\Gamma \Rightarrow {\sim}{\sim}\bot}}{\Gamma \Rightarrow \bot}\supset e$$

Assume that the statement holds for any formula of depth less than n. We show that it holds also for a formula γ of depth n.

- $\gamma = \alpha \supset \beta$. By inductive hypothesis, the statement holds for α and β, and hence we can show that it also holds for $\alpha \supset \beta$ as follows:

$$\begin{array}{c}\dfrac{\dfrac{\overset{\pi}{\Gamma \Rightarrow {\sim}{\sim}(\alpha \supset \beta)} \qquad \dfrac{\dfrac{\dfrac{}{{\sim}\beta \Rightarrow {\sim}\beta} \quad \dfrac{\dfrac{}{\alpha \supset \beta \Rightarrow \alpha \supset \beta} \quad \dfrac{}{\alpha \Rightarrow \alpha}}{\alpha \supset \beta, \alpha \Rightarrow \beta}}{{\sim}\beta, \alpha \supset \beta, \alpha \Rightarrow \bot}}{{\sim}\beta, \alpha \Rightarrow {\sim}(\alpha \supset \beta)}}{\Gamma, {\sim}\beta, \alpha \Rightarrow \bot}}{\Gamma, \alpha \Rightarrow {\sim}{\sim}\beta} \\ \vdots\ \text{i.h.} \\ \dfrac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \supset \beta}\end{array}$$

- $\gamma = \alpha \wedge \beta$. By inductive hypothesis, the statement holds for α and β, and hence we can show that it also holds for $\alpha \wedge \beta$ as follows:

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{\overset{\pi}{\Gamma \Rightarrow {\sim}{\sim}(\alpha \wedge \beta)} \qquad \dfrac{\dfrac{\dfrac{}{{\sim}\alpha \Rightarrow {\sim}\alpha} \quad \dfrac{\dfrac{}{\alpha \wedge \beta \Rightarrow \alpha \wedge \beta}}{\alpha \wedge \beta \Rightarrow \alpha}}{{\sim}\alpha, \alpha \wedge \beta \Rightarrow \bot}}{{\sim}\alpha \Rightarrow {\sim}(\alpha \wedge \beta)}}{\Gamma, {\sim}\alpha \Rightarrow \bot}}{\Gamma \Rightarrow {\sim}{\sim}\alpha} \\ \vdots\ \text{i.h.} \\ \Gamma \Rightarrow \alpha\end{array} \qquad \begin{array}{c}\dfrac{\dfrac{\overset{\pi}{\Gamma \Rightarrow {\sim}{\sim}(\alpha \wedge \beta)}}{\Gamma \Rightarrow \bot} \qquad \dfrac{\dfrac{\dfrac{}{{\sim}\beta \Rightarrow {\sim}\beta} \quad \dfrac{\dfrac{}{\alpha \wedge \beta \Rightarrow \alpha \wedge \beta}}{\alpha \wedge \beta \Rightarrow \beta}}{{\sim}\beta, \alpha \wedge \beta \Rightarrow \bot}}{{\sim}\beta \Rightarrow {\sim}(\alpha \wedge \beta)}}{\Gamma \Rightarrow {\sim}{\sim}\beta} \\ \vdots\ \text{i.h.} \\ \Gamma \Rightarrow \beta\end{array}}{\Gamma \Rightarrow \alpha \wedge \beta}$$

□

Since **R** contains all standard classical introduction and elimination rules for the connectives of classical logic, Theorems 3.2 and 3.3 guarantee that **R** is complete with respect to classical logic.

4 The System G

The system **G** is defined by the rules in Figs. 4 and 5. The first three 0-premise rules in Fig. 4 are characterized by the fact that the set to the left of $<$ contains the immediate

$$\frac{}{\Rightarrow \{\alpha, \beta\} < \alpha \wedge \beta} \qquad \frac{}{\Rightarrow \{\alpha\} < \alpha \vee \beta} \qquad \frac{}{\Rightarrow \{\beta\} < \alpha \vee \beta}$$

$$\frac{}{\Rightarrow \{{\sim}\alpha\} < {\sim}(\alpha \wedge \beta)} \qquad \frac{}{\Rightarrow \{{\sim}\beta\} < {\sim}(\alpha \wedge \beta)}$$

$$\frac{}{\Rightarrow \{{\sim}\alpha, {\sim}\beta\} < {\sim}(\alpha \vee \beta)} \qquad \frac{}{\Rightarrow \{\alpha\} < {\sim}{\sim}\alpha}$$

$$\frac{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha \quad \Rightarrow \{\beta_1, \ldots, \beta_m\} < \beta}{\Rightarrow \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\} < \alpha \wedge \beta}$$

$$\frac{\Rightarrow \{\alpha_1, \ldots, \beta \wedge \gamma, \ldots, \alpha_n\} < \delta}{\Rightarrow \{\alpha_1, \ldots, \beta, \gamma, \ldots, \alpha_n\} < \delta}$$

$$\frac{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha}{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha \vee \beta} \qquad \frac{\Rightarrow \{\beta_1, \ldots, \beta_n\} < \beta}{\Rightarrow \{\beta_1, \ldots, \beta_n\} < \alpha \vee \beta}$$

$$\frac{\Rightarrow \{\alpha_1, \ldots, \beta \vee \gamma, \ldots, \alpha_n\} < \delta}{\Rightarrow \{\alpha_1, \ldots, \beta, \ldots, \alpha_n\} < \delta} \qquad \frac{\Rightarrow \{\alpha_1, \ldots, \beta \vee \gamma, \ldots, \alpha_n\} < \delta}{\Rightarrow \{\alpha_1, \ldots, \gamma, \ldots, \alpha_n\} < \delta}$$

$$\frac{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < {\sim}\alpha}{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < {\sim}(\alpha \wedge \beta)} \qquad \frac{\Rightarrow \{\beta_1, \ldots, \beta_n\} < {\sim}\beta}{\Rightarrow \{\beta_1, \ldots, \beta_n\} < {\sim}(\alpha \wedge \beta)}$$

$$\frac{\Rightarrow \{\alpha_1, \ldots, {\sim}(\beta \wedge \gamma), \ldots, \alpha_n\} < \delta}{\Rightarrow \{\alpha_1, \ldots, {\sim}\beta, \ldots, \alpha_n\} < \delta} \qquad \frac{\Rightarrow \{\alpha_1, \ldots, {\sim}(\beta \wedge \gamma), \ldots, \alpha_n\} < \delta}{\Rightarrow \{\alpha_1, \ldots, {\sim}\gamma, \ldots, \alpha_n\} < \delta}$$

$$\frac{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < {\sim}\alpha \quad \Delta \Rightarrow \{\beta_1, \ldots, \beta_m\} < {\sim}\beta}{\Rightarrow \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\} < {\sim}(\alpha \vee \beta)}$$

$$\frac{\Rightarrow \{\alpha_1, \ldots, {\sim}(\beta \vee \gamma), \ldots, \alpha_n\} < \delta}{\Rightarrow \{\alpha_1, \ldots, {\sim}\beta, {\sim}\gamma, \ldots, \alpha_n\} < \delta}$$

$$\frac{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha}{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < {\sim}{\sim}\alpha} \qquad \frac{\Rightarrow \{\alpha_1, \ldots, {\sim}{\sim}\beta, \ldots, \alpha_n\} < \gamma}{\Rightarrow \{\alpha_1, \ldots, \beta, \ldots, \alpha_n\} < \gamma}$$

Fig. 4 Rules for connectives

subformulas of the formula to the right of $<$. Quite simply, they say that $\{\alpha, \beta\}$ grounds the conjunction $\alpha \wedge \beta$ and either element of the set $\{\alpha, \beta\}$ alone grounds the disjunction $\alpha \vee \beta$. The rest of the 0-premise rules in Fig. 4 are justified by similar principles but also involve negation. A single negation, nevertheless, cannot be handled just as an occurrence of any other connective. Indeed, rather obviously, there is no way to ground the truth of a formula ${\sim}\alpha$ in the truth of its immediate subformula α. Therefore, the rules for ground need to be so defined as to handle an occurrence of negation in combination with an occurrence of some other connective. To ground the truth of a negated conjunction ${\sim}(\alpha \wedge \beta)$, we can either use ${\sim}\alpha$ or ${\sim}\beta$; to ground the truth of a negated disjunction ${\sim}(\alpha \vee \beta)$, we must use both elements of the set $\{{\sim}\alpha, {\sim}\beta\}$. The interaction of negation with itself is rather simple: the ground of ${\sim}{\sim}\alpha$ is simply α. The latter, indeed, is the largest subformula of ${\sim}{\sim}\alpha$ that can justify ${\sim}{\sim}\alpha$. The 1-premise and 2-premise rules in Fig. 4 can be divided into *introduction* and *elimination* rules. The former enable us to introduce an occurrence of a connective in the formula to the

$$\frac{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha}{\Rightarrow \{\beta_1, \ldots, \beta_m\} < \alpha} \; \{\}c$$

where, as sets, $\{\beta_1, \ldots, \beta_m\} = \{\alpha_1, \ldots, \alpha_n\}$

$$\frac{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha \quad \Rightarrow \{\beta_1, \ldots, \beta_m\} < \alpha}{\Rightarrow \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\} < \alpha} \; \text{Am}$$

$$\frac{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha \quad \Rightarrow \{\beta_1, \ldots, \beta_{i-1}, \alpha, \beta_{i+1}, \ldots, \beta_m\} < \beta}{\Rightarrow \{\beta_1, \ldots, \beta_{i-1}, \alpha_1, \ldots, \alpha_n, \beta_{i+1}, \ldots, \beta_m\} < \beta} \; \text{Cut}$$

Fig. 5 Structural rules

right of $<$, while the latter enable us to eliminate an occurrence of a connective in the formulas to the left of $<$.

The rules in Fig. 5 are basic structural rules. $\{\}c$ enables us to handle sets in such a way that the order and number of occurrences of their elements do not matter: here the list $\beta_1, \ldots, \beta_m$ contains the same elements as $\alpha_1, \ldots, \alpha_n$ but without repetitions and, possibly, in a different order. Am—for *Amalgamation*—enables us to merge possibly different grounds $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_m\}$ of the same formula α in case they are indeed grounds of α. Cut, on the other hand, enables us to replace a formula α—the *cut formula*—by one of its grounds—that is, $\alpha_1, \ldots, \alpha_n$—in case α occurs inside a ground $\{\beta_1, \ldots, \beta_{i-1}, \alpha, \beta_{i+1}, \ldots, \beta_m\}$ of a formula β. This rule, rather obviously, looks very similar to the infamous rule of cut often employed in sequent systems.[8] Even though the validity of Amalgamation and Cut is a controversial issue in the literature on ground, we include them here.[9] It is important to note, though, that the results presented below would hold *a fortiori* if one or both of these rules were to be removed from the calculus.

Derivations in **G** are defined just as derivations in **R** but with reference to the rules in Figs. 4 and 5. Even though the rules of **G** do not use contexts to express the dependencies of the derived expressions, we keep the notation $\Rightarrow \alpha$ in order to stress that, technically, we handle $<$ as a connective of the language and not as a relation on the language.[10]

5 Cut Elimination for G

In order to show that **G** constitutes a proof-theoretically adequate set of rules for ground, we prove that it enjoys cut-elimination. This is a novel result, as far as we

8 See [17].

9 On the amalgamation rule, see, for instance, [3, 9, 11]. On the cut rule and, more generally, transitivity, see, for instance, [12, 20, 24, 25].

10 Nevertheless, $<$ exactly captures the derivability relation induced by calculi such as—if we exclude the Am rule—the one in [3]. Indeed, the conclusions of the 0-premise rules of **G** exactly capture individual applications of the rules in Correia's calculus, introduction rules in **G** have the same effect as Correia's rules, and elimination rules in **G** have the same effect as applying these rules to derive a hypothesis of an existing derivation.

know. Even though deductive systems are often employed to characterize the formal behaviour of ground, not many formal results concerning the proof-theoretical properties of these systems can be found in the literature.[11]

As a first step we show that any derivation in **G** containing only one application of Cut can be transformed into a derivation in **G** where Cut does not occur.

Lemma 5.1 *For any derivation containing only one* Cut *application r, we can transform it into an equivalent derivation—that is, one with the same conclusion and hypothesis—that does not contain any* Cut *application.*

Proof The proof is by induction on the number of rules applied above the leftmost premise of r.

The base case—no rules applied above the leftmost premise of r—is impossible since a Cut application must have at least one 0-premise rule application above its leftmost premise.

Now suppose that the statement holds for any suitable Cut application with less than h rule applications above its leftmost premise. We prove that applications of Cut holds also for any suitable Cut application r with exactly h rule applications occurring above its leftmost premise. We reason by cases on the form of the last rule applied in the derivation of the leftmost premise of r.

If a 0-premise rule application occurs immediately above the leftmost premise of r, then we can remove it from the derivation. In particular, if the derivation is of the following form

$$\dfrac{\dfrac{}{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha} \quad \overset{\ldots}{\Rightarrow \{\ldots, \alpha, \ldots\} < \beta}}{\Rightarrow \{\ldots, \alpha_1, \ldots, \alpha_n, \ldots\} < \beta} \text{ Cut}$$

we can replace it by the following one:

$$\dfrac{\overset{\ldots}{\Rightarrow \{\ldots, \alpha, \ldots\} < \beta}}{\Rightarrow \{\ldots, \alpha_1, \ldots, \alpha_n, \ldots\} < \beta} *$$

where $*$ is a suitable elimination rule. Indeed, it is clear by inspection of the rules in Fig. 4 that, if $\overline{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha}$ is a 0-premise rule application, then some elimination rule enables us to replace the occurrence of α by $\alpha_1, \ldots, \alpha_n$ inside the ground $\{\ldots, \alpha, \ldots\}$.

Suppose that an application of an elimination or $\{\,\}c$ rule occurs immediately above the leftmost premise of r:

$$\dfrac{\dfrac{\overset{\ldots}{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha}}{\Rightarrow \{\alpha'_1, \ldots, \alpha'_m\} < \alpha} * \quad \overset{\ldots}{\Rightarrow \{\ldots, \alpha, \ldots\} < \beta}}{\Rightarrow \{\ldots, \alpha'_1, \ldots, \alpha'_m, \ldots\} < \beta} \text{ Cut}$$

where $*$ is either the rule $\{\,\}c$ or any elimination rule. Then, we can construct the following derivation:

$$\dfrac{\dfrac{\overset{\ldots}{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha} \quad \overset{\ldots}{\Rightarrow \{\ldots, \alpha, \ldots\} < \beta}}{\Rightarrow \{\ldots, \alpha_1, \ldots, \alpha_n, \ldots\} < \beta} \text{ Cut}}{\Rightarrow \{\ldots, \alpha'_1, \ldots, \alpha'_m, \ldots\} < \beta} *$$

[11] A notable exception are the results presented in [13] for the notion of ground defined in [22, 23].

Since less than h rule applications occur now above the leftmost premise of r, by inductive hypothesis, we obtain a derivation of $\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \dots\} < \beta$ that does not contain any Cut application.

If, instead, an application of Am occurs immediately above the leftmost premise of r:

$$\dfrac{\dfrac{\begin{array}{c}\cdots\\ \Rightarrow \{\alpha_1, \dots, \alpha_n\} < \alpha\end{array} \quad \begin{array}{c}\cdots\\ \Rightarrow \{\beta_1, \dots, \beta_m\} < \alpha\end{array}}{\Rightarrow \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\} < \alpha}\,\text{Am} \quad \begin{array}{c}\cdots\\ \Rightarrow \{\dots, \alpha, \dots\} < \beta\end{array}}{\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m, \dots\} < \beta}\,\text{Cut}$$

we can construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\cdots\\ \Rightarrow \{\alpha_1, \dots, \alpha_n\} < \alpha\end{array} \quad \begin{array}{c}\cdots\\ \Rightarrow \{\dots, \alpha, \dots\} < \beta\end{array}}{\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \dots\} < \beta}\,\text{Cut} \quad \dfrac{\begin{array}{c}\cdots\\ \Rightarrow \{\beta_1, \dots, \beta_m\} < \alpha\end{array} \quad \begin{array}{c}\cdots\\ \Rightarrow \{\dots, \alpha, \dots\} < \beta\end{array}}{\Rightarrow \{\dots, \beta_1, \dots, \beta_m, \dots\} < \beta}\,\text{Cut}}{\cdots}\,\text{Am}}{\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m, \dots\} < \beta}\,\{\}c$$

Since less than h rule applications occur now above each of the leftmost premises of the two applications of Cut, by inductive hypothesis, we can obtain derivations of $\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \dots\} < \beta$ and $\Rightarrow \{\dots, \beta_1, \dots, \beta_m, \dots\} < \beta$ that do not contain any Cut rule application.

Let us now consider introduction rules. If an application of a 1-premise introduction rule $*_i$ occurs immediately above the leftmost premise of r:

$$\dfrac{\dfrac{\begin{array}{c}\cdots\\ \Rightarrow \{\alpha_1, \dots, \alpha_n\} < \alpha\end{array}}{\Rightarrow \{\alpha_1, \dots, \alpha_n\} < \alpha'}\,*_i \quad \begin{array}{c}\cdots\\ \Rightarrow \{\dots, \alpha', \dots\} < \beta\end{array}}{\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \dots\} < \beta}\,\text{Cut}$$

we can construct the following derivation:

$$\dfrac{\begin{array}{c}\cdots\\ \Rightarrow \{\alpha_1, \dots, \alpha_n\} < \alpha\end{array} \quad \dfrac{\begin{array}{c}\cdots\\ \Rightarrow \{\dots, \alpha', \dots\} < \beta\end{array}}{\Rightarrow \{\dots, \alpha, \dots\} < \beta}\,*_e}{\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \dots\} < \beta}\,\text{Cut}$$

where $*_e$ is the elimination rule that corresponds to the introduction rule $*_i$ used in the derivation above—that is, the elimination rule that eliminates the same connective introduced by $*_i$. Since less than h rule applications occur now above the leftmost premise of the application of Cut, by inductive hypothesis, we can obtain a derivation of $\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \dots\} < \beta$ that does not contain any Cut application.

We now consider 2-premise introduction rules. We only show the case of conjunction introduction since the other one is completely analogous.

$$\dfrac{\dfrac{\Rightarrow \{\alpha_1, \dots, \alpha_n\} < \alpha \quad \Rightarrow \{\beta_1, \dots, \beta_m\} < \beta}{\Rightarrow \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\} < \alpha \wedge \beta} \quad \begin{array}{c}\cdots\\ \Rightarrow \{\dots, \alpha \wedge \beta, \dots\} < \gamma\end{array}}{\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m, \dots\} < \gamma}\,\text{Cut}$$

where the rule above the leftmost premise of r is the rule that introduces a conjunction to the right of $<$. We can construct the following derivation:

$$\dfrac{\begin{array}{c}\cdots\\ \Rightarrow \{\beta_1, \dots, \beta_m\} < \beta\end{array} \quad \dfrac{\begin{array}{c}\cdots\\ \Rightarrow \{\alpha_1, \dots, \alpha_n\} < \alpha\end{array} \quad \dfrac{\begin{array}{c}\cdots\\ \Rightarrow \{\dots, \alpha \wedge \beta, \dots\} < \gamma\end{array}}{\Rightarrow \{\dots, \alpha, \beta, \dots\} < \gamma}}{\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \beta, \dots\} < \gamma}\,\text{Cut}}{\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m, \dots\} < \gamma}\,\text{Cut}$$

Since less than h rule applications occur now above the leftmost premise of the uppermost application of Cut, by inductive hypothesis, we can obtain a derivation

$$\dfrac{\overset{\dots}{\Rightarrow \{\beta_1, \dots, \beta_m\} < \beta} \quad \overset{\dots}{\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \beta, \dots\} < \gamma}}{\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m, \dots\} < \gamma} \text{ Cut}$$

where the derivation of $\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \beta, \dots\} < \gamma$ does not contain any Cut rule application and the derivation of $\Rightarrow \{\beta_1, \dots, \beta_m\} < \beta$ did not change. Afterwards, since less than h rule applications occur now above the leftmost premise of the application of Cut and since no application of Cut occurs above any of its premises, by inductive hypothesis, we can obtain a derivation of $\Rightarrow \{\dots, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m, \dots\} < \gamma$ that does not contain any Cut rule application. □

We can now prove the proper cut-elimination result.

Theorem 5.2 *All* Cut *applications are eliminable from any derivation in* **G**.

Proof The proof is by induction on the number of applications of Cut in π.

In the base case, π does not contain any application of Cut. Thus, we can just take $\pi' = \pi$.

Suppose now that the statement holds for all derivations containing less than n applications of Cut. Lemma 5.1 guarantees that we can remove an uppermost application r of Cut without changing any rule application in π that does not occur above r. By Lemma 5.1, we obtain a derivation π'' containing $n - 1$ applications of Cut. Thus, by inductive hypothesis, we can transform π'' into a Cut-free derivation π'. □

6 Subsumption of G Under R

In this section we show that $<$, as characterised by **G**, is a fragment of $\rhd$, as characterised by **R**. That is, for any statement about grounds provable in **G** there is a corresponding statement about reasons provable in **R**. Moreover, we show that it is possible to characterise, independently of the calculus **G**, the fragment of $\rhd$ corresponding to the provable instances of $<$.

The translation that we employ to associate formulas of $\mathcal{L}_<$ with formulas of $\mathcal{L}_\rhd$ is the following.

Definition 6.1 For any $\alpha \in \mathcal{L}_<$, we define

$$\overline{\alpha} = t_{\{\}}(\alpha[\rhd / <])$$

where $\alpha[\rhd / <]$ is obtained by replacing each occurrence of $<$ in α by an occurrence of $\rhd$, and $t_{\{\}}$ is the translation from $\mathcal{L}_\rhd$ to $\{\}$-free $\mathcal{L}_\rhd$ of Definition 3.1.

Notice that the translation is essentially an identical translation: we are simply rewriting $<$ into $\rhd$. Indeed, $t_{\{\}}$ simply enables us to treat the symbol $\{\}$ in **R** as an abbreviation for nested conjunctions. Moreover, the proof of Theorem 3.1 clearly shows that the rules for { } could be kept in the language without loosing any of the proof-theoretical properties of **R**.

Before proving that **G** can be subsumed under **R** it must be stressed that, for obvious reasons, the opposite direction of the subsumption cannot hold. For example, Contraposition is not—and should not be—a valid principle for ground: there is no reason to assume that, if $\alpha < \beta$ holds, $\sim\beta < \sim\alpha$ must hold as well. Actually, depending on the specific notion of ground under consideration, there might be very good reasons to assume that $\sim\beta < \sim\alpha$ should *not* hold. The fact that the rules of **G** never introduce single negations—if not in relation to a single negation already occurring in the premises—and never move formulas from one side of $<$ to the other side of $<$ shows that no version of C for $<$ is valid in **G**.

Theorem 6.1 *For any formula α in $\mathcal{L}_<$, if $\Rightarrow \alpha$ is derivable in* **G**, *then $\Rightarrow \overline{\alpha}$ is derivable in* **R**.

Proof We show that any derivation π of $\Rightarrow \alpha$ in **G** can be translated into a derivation of $\Rightarrow \overline{\alpha}$ in **R**. The proof is by induction on the number of rule applications in π.

In the base case, the only rule application in π must be an application of a 0-premise rule. Since all 0-premise rules have a conclusion of the form $\Rightarrow \{\gamma_1, \ldots, \gamma_n\} < \delta$, where δ classically follows from $\gamma_1, \ldots, \gamma_n$, we can easily show that $\gamma_1 \wedge \ldots \wedge \gamma_n \rhd \delta$ is provable in **R** by means of $\rhd i$, that is, in virtue of the fact that **R** is supraclassical. We only show one case since the others are either trivial or analogous. Let the rule in question be the following:

$$\dfrac{}{\Rightarrow \alpha < \alpha \vee \beta}$$

Since $\overline{\alpha} \vee \overline{\beta}$ is an abbreviation for $\sim(\sim\overline{\alpha} \wedge \sim\overline{\beta})$ and $\sim$ is an abbreviation for $\supset \bot$, the following derivation proves that the statement holds:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\sim\overline{\alpha} \wedge \sim\overline{\beta} \Rightarrow \sim\overline{\alpha} \wedge \sim\overline{\beta}}\,\text{Id}}{\sim\overline{\alpha} \wedge \sim\overline{\beta} \Rightarrow \sim\overline{\alpha}}\,\wedge e \qquad \dfrac{}{\overline{\alpha} \Rightarrow \overline{\alpha}}\,\text{Id}}{\sim\overline{\alpha} \wedge \sim\overline{\beta}, \overline{\alpha} \Rightarrow \bot}\,\supset e}{\overline{\alpha} \Rightarrow \sim(\sim\overline{\alpha} \wedge \sim\overline{\beta})}\,\supset i}{\Rightarrow \overline{\alpha} \rhd \sim(\sim\overline{\alpha} \wedge \sim\overline{\beta})}\,\rhd i$$

Suppose now that the statement holds for all derivations containing less than n rule applications, Let π a derivation that contains exactly n rule applications. We reason by cases on the last rule applied in π. We group the cases in which this is an application of an introduction rule, of an elimination rule and of a structural rule. Let us then begin with introduction rules. The main reason why these cases comply with the statement is that the formula to the right of $<$ in the conclusion of a **G** introduction rule—say, δ—classically follows from the formulas to the right of $<$ in the premises of such a rule—say, $\gamma_1, \ldots, \gamma_n$ for $1 \leq n \leq 2$. Therefore, we can use the inductive hypothesis on the premises of the introduction rule, apply $\rhd e$, classically derive δ from $\gamma_1, \ldots, \gamma_n$, and then apply $\rhd i$ to prove the desired statement about reasons. We show this in detail only for some exemplar cases since the other ones are analogous.

- $\dfrac{\Rightarrow \{\alpha_1, \dots, \alpha_n\} < \alpha \quad \Rightarrow \{\beta_1, \dots, \beta_m\} < \beta}{\Rightarrow \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\} < \alpha \wedge \beta}$

  By inductive hypothesis, there are two derivations without hypotheses π_1 and π_2 of, respectively, $\{\overline{\alpha}_1, \dots, \overline{\alpha}_n\} \rhd \overline{\alpha}$—which translates to $\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \rhd \overline{\alpha}$—and $\{\overline{\beta}_1, \dots, \overline{\beta}_m\} \rhd \overline{\beta}$—which translates to $\overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m \rhd \overline{\beta}$—in **R**. We can then construct our derivation without hypotheses of $\{\overline{\alpha}_1, \dots, \overline{\alpha}_n, \overline{\beta}_1, \dots, \overline{\beta}_m\} \rhd \overline{\alpha} \wedge \overline{\beta}$—which translates to $\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m \rhd \overline{\alpha} \wedge \overline{\beta}$—as follows:

$$\dfrac{\dfrac{\pi_1 \qquad \dfrac{\dfrac{\dfrac{}{\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m \Rightarrow \overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m}\,\mathrm{Id}}{\vdots}\,\wedge e}{\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m \Rightarrow \overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n}\,\wedge e}{\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m \Rightarrow \overline{\alpha}}\,\rhd e \qquad \pi_3}{\dfrac{\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m \Rightarrow \overline{\alpha} \wedge \overline{\beta}}{\Rightarrow \overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m \rhd \overline{\alpha} \wedge \overline{\beta}}\,\rhd i}\,\wedge i$$

  where π_1 has conclusion $\Rightarrow \overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \rhd \overline{\alpha}$ and π_3 is

$$\dfrac{\begin{matrix}\pi_2 \\ \Rightarrow \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m \rhd \overline{\beta}\end{matrix} \qquad \dfrac{\dfrac{\dfrac{}{\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m \Rightarrow \overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m}\,\mathrm{Id}}{\vdots}\,\wedge e}{\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m \Rightarrow \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m}\,\wedge e}{\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \dots \wedge \overline{\beta}_m \Rightarrow \overline{\beta}}\,\rhd e$$

- $\dfrac{\Rightarrow \{\alpha_1, \dots, \alpha_n\} < \alpha}{\Rightarrow \{\alpha_1, \dots, \alpha_n\} < \alpha \vee \beta}$

  By inductive hypothesis, there is a derivation π_1 without hypotheses of $\{\overline{\alpha}_1, \dots, \overline{\alpha}_n\} \rhd \overline{\alpha}$—which translates to $\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \rhd \overline{\alpha}$. We can then construct our derivation without hypotheses of $\Rightarrow \{\overline{\alpha}_1, \dots, \overline{\alpha}_n\} \rhd \overline{\alpha} \vee \overline{\beta}$—which translates to $\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \rhd {\sim}({\sim}\overline{\alpha} \wedge {\sim}\overline{\beta})$—as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{{\sim}\overline{\alpha} \wedge {\sim}\overline{\beta} \Rightarrow {\sim}\overline{\alpha} \wedge {\sim}\overline{\beta}}\,\mathrm{Id}}{{\sim}\overline{\alpha} \wedge {\sim}\overline{\beta} \Rightarrow {\sim}\overline{\alpha}}\,\wedge e \qquad \dfrac{\begin{matrix}\pi_1 \\ \Rightarrow \overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \rhd \overline{\alpha}\end{matrix} \qquad \dfrac{}{\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \Rightarrow \overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n}\,\mathrm{Id}}{\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \Rightarrow \overline{\alpha}}\,\rhd e}{\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n, {\sim}\overline{\alpha} \wedge {\sim}\overline{\beta} \Rightarrow \bot}\,\supset e}{\dfrac{\overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \Rightarrow {\sim}({\sim}\overline{\alpha} \wedge {\sim}\overline{\beta})}{\Rightarrow \overline{\alpha}_1 \wedge \dots \wedge \overline{\alpha}_n \rhd {\sim}({\sim}\overline{\alpha} \wedge {\sim}\overline{\beta})}\,\rhd i}\,\supset i$$

We now consider the cases in which the last rule applied in π is an elimination rule. All these cases essentially rely on the fact that the formulas obtained by eliminating the connective classically follow from the formula in which the eliminated connective occurs. We only show one interesting case in order to exemplify the method used to handle them. Consider the following rule application:

$$\dfrac{\Rightarrow \{\alpha_1, \dots, \beta \vee \gamma, \dots, \alpha_n\} < \delta}{\Rightarrow \{\alpha_1, \dots, \beta, \dots, \alpha_n\} < \delta}$$

By inductive hypothesis, there is a derivation π_1 without hypotheses of $\{\overline{\alpha}_1, \ldots, \overline{\beta} \vee \overline{\gamma}, \ldots, \overline{\alpha}_n\} \rhd \overline{\delta}$—which translates to $\overline{\alpha}_1 \wedge \ldots \wedge \sim(\sim\overline{\beta} \wedge \sim\overline{\gamma}) \wedge \ldots \wedge \overline{\alpha}_n \rhd \overline{\delta}$. We can then construct our derivation without hypotheses of $\{\overline{\alpha}_1, \ldots, \overline{\beta}, \ldots, \overline{\alpha}_n\} \rhd \overline{\delta}$—which translates to $\overline{\alpha}_1 \wedge \ldots \wedge \overline{\beta} \wedge \ldots \wedge \overline{\alpha}_n \rhd \overline{\delta}$—as follows:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Rightarrow \overline{\alpha}_1 \wedge \ldots \wedge \sim(\sim\overline{\beta} \wedge \sim\overline{\gamma}) \wedge \ldots \wedge \overline{\alpha}_n \rhd \overline{\delta}\end{matrix} \quad \pi_2}{\overline{\alpha}_1 \wedge \ldots \wedge \overline{\beta} \wedge \ldots \wedge \overline{\alpha}_n \Rightarrow \overline{\delta}}\,\rhd e}{\Rightarrow \overline{\alpha}_1 \wedge \ldots \wedge \overline{\beta} \wedge \ldots \wedge \overline{\alpha}_n \rhd \overline{\delta}}\,\rhd i$$

where the derivation π_2, with conclusion $\overline{\alpha}_1 \wedge \ldots \wedge \overline{\beta} \wedge \ldots \wedge \overline{\alpha}_n \Rightarrow \overline{\alpha}_1 \wedge \ldots \wedge \sim(\sim\overline{\beta} \wedge \sim\overline{\gamma}) \wedge \ldots \wedge \overline{\alpha}_n$, is obtained by, first, eliminating the conjunction $\overline{\alpha}_1 \wedge \ldots \wedge \overline{\beta} \wedge \ldots \wedge \overline{\alpha}_n$, second, by deriving $\sim(\sim\overline{\beta} \wedge \sim\overline{\gamma})$ from $\overline{\beta}$ as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\sim\overline{\beta} \wedge \sim\overline{\gamma} \Rightarrow \sim\overline{\beta} \wedge \sim\overline{\gamma}}\,\text{Id}}{\sim\overline{\beta} \wedge \sim\overline{\gamma} \Rightarrow \sim\overline{\beta}}\,\wedge e \qquad \dfrac{\begin{matrix}\dfrac{}{\overline{\alpha}_1 \wedge \ldots \wedge \overline{\beta} \wedge \ldots \wedge \overline{\alpha}_n \Rightarrow \overline{\alpha}_1 \wedge \ldots \wedge \overline{\beta} \wedge \ldots \wedge \overline{\alpha}_n}\,\text{Id}\\ \vdots\end{matrix}}{\overline{\alpha}_1 \wedge \ldots \wedge \overline{\beta} \wedge \ldots \wedge \overline{\alpha}_n \Rightarrow \overline{\beta}}\,\wedge e}{\overline{\alpha}_1 \wedge \ldots \wedge \overline{\beta} \wedge \ldots \wedge \overline{\alpha}_n, \sim\overline{\beta} \wedge \sim\overline{\gamma} \Rightarrow \bot}\,\supset e}{\overline{\alpha}_1 \wedge \ldots \wedge \overline{\beta} \wedge \ldots \wedge \overline{\alpha}_n \Rightarrow \sim(\sim\overline{\beta} \wedge \sim\overline{\gamma})}\,\supset i$$

and, lastly, by reintroducing the conjunction $\overline{\alpha}_1 \wedge \ldots \wedge \sim(\sim\overline{\beta} \wedge \sim\overline{\gamma}) \wedge \ldots \wedge \overline{\alpha}_n$ from $\sim(\sim\overline{\beta} \wedge \sim\overline{\gamma})$ and from the conjuncts $\overline{\alpha}_1, \ldots, \overline{\alpha}_n$ other than $\overline{\beta}$.

Lastly, we consider the cases in which the last rule applied in π is structural.

- $\dfrac{\Gamma \Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha}{\Gamma \Rightarrow \{\beta_1, \ldots, \beta_m\} < \alpha}\,\{\}c$
  By inductive hypothesis, there is a derivation π_1 without hypotheses of $\{\overline{\alpha}_1, \ldots, \overline{\alpha}_n\} \rhd \overline{\alpha}$—which translates to $\overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \rhd \overline{\alpha}$. We can then construct our derivation without hypotheses of $\{\overline{\beta}_1, \ldots, \overline{\beta}_m\} \rhd \overline{\alpha}$—which translates to $\overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m \rhd \overline{\alpha}$—as follows:

  $$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Rightarrow \overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \rhd \overline{\alpha}\end{matrix} \quad \begin{matrix}\ldots\\ \overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m \Rightarrow \overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n\end{matrix}}{\overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m \Rightarrow \overline{\alpha}}\,\rhd e}{\Rightarrow \overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m \rhd \overline{\alpha}}\,\rhd i$$

  where the rightmost premise of $\rhd e$ is derived by first eliminating all required conjunctions in $\overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n$ and then by reintroducing as many as needed and in the suitable order to derive the formula $\overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m$. We can always do this since all conjuncts occurring in $\overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m$ also occur in $\overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n$.
- $\dfrac{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha \quad \Rightarrow \{\beta_1, \ldots, \beta_m\} < \alpha}{\Rightarrow \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\} < \alpha}\,\text{Am}$
  By inductive hypothesis, there exist two derivations without hypotheses π_1 and π_2 of, respectively, $\{\overline{\alpha}_1, \ldots, \overline{\alpha}_n\} \rhd \overline{\alpha}$—which translates to $\overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \rhd \overline{\alpha}$—and $\{\overline{\beta}_1, \ldots, \overline{\beta}_m\} \rhd \overline{\alpha}$—which translates to $\overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m \rhd \overline{\alpha}$—in **R**. We can then

construct our derivation without hypotheses of $\{\overline{\alpha}_1, \ldots, \overline{\alpha}_n, \overline{\beta}_1, \ldots, \overline{\beta}_m\} \rhd \overline{\alpha}$—which translates to $\overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m \rhd \overline{\alpha}$—as follows:

$$\dfrac{\dfrac{\begin{matrix}\pi_1 \\ \Rightarrow \overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \rhd \overline{\alpha}\end{matrix} \qquad \dfrac{\begin{matrix}\dfrac{}{\overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m \Rightarrow \overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m}\,\text{Id} \\ \vdots \end{matrix}}{\overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m \Rightarrow \overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n}\,\wedge e}{\overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m \Rightarrow \overline{\alpha}}\,\rhd e}{\Rightarrow \overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \wedge \overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_m \rhd \overline{\alpha}}\,\rhd i$$

- $$\dfrac{\Rightarrow \{\alpha_1, \ldots, \alpha_n\} < \alpha \quad \Rightarrow \{\beta_1, \ldots, \beta_{i-1}, \alpha, \beta_{i+1}, \ldots, \beta_m\} < \beta}{\Rightarrow \{\beta_1, \ldots, \beta_{i-1}, \alpha_1, \ldots, \alpha_n, \beta_{i+1}, \ldots, \beta_m\} < \beta}\,\text{Cut}$$

  By inductive hypothesis, there exist two derivations without hypotheses π_1 and π_2 of, respectively, $\{\overline{\alpha}_1, \ldots, \overline{\alpha}_n\} \rhd \overline{\alpha}$—which translates to $\overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \rhd \overline{\alpha}$—and $\{\overline{\beta}_1, \ldots, \overline{\beta}_{i-1}, \overline{\alpha}, \overline{\beta}_{i+1}, \ldots, \overline{\beta}_m\} \rhd \overline{\beta}$—which translates to $\overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_{i-1} \wedge \overline{\alpha} \wedge \overline{\beta}_{i+1} \wedge \ldots \wedge \overline{\beta}_m \rhd \overline{\beta}$—in **R**. We can then construct our derivation without hypotheses of $\{\overline{\beta}_1, \ldots, \overline{\beta}_{i-1}, \overline{\alpha}_1, \ldots, \overline{\alpha}_n, \overline{\beta}_{i+1}, \ldots, \overline{\beta}_m\} \rhd \overline{\beta}$—which translates to $\overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_{i-1} \wedge \overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \wedge \overline{\beta}_{i+1} \wedge \ldots \wedge \overline{\beta}_m \rhd \overline{\beta}$—as follows:

$$\dfrac{\dfrac{\begin{matrix}\pi_2 \\ \Rightarrow \overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_{i-1} \wedge \overline{\alpha} \wedge \overline{\beta}_{i+1} \wedge \ldots \wedge \overline{\beta}_m \rhd \overline{\beta}\end{matrix} \quad \pi_3}{\overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_{i-1}, \overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n, \overline{\beta}_{i+1} \wedge \ldots \wedge \overline{\beta}_m \Rightarrow \overline{\beta}}\,\rhd e}{\Rightarrow \overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_{i-1}, \overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n, \overline{\beta}_{i+1} \wedge \ldots \wedge \overline{\beta}_m \rhd \overline{\beta}}\,\rhd i$$

  where the derivation π_3 is

$$\dfrac{\dfrac{\begin{matrix}\dfrac{}{\overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_{i-1} \Rightarrow \overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_{i-1}}\,\text{Id} \\ \vdots\ \wedge e\end{matrix} \quad \begin{matrix}\dfrac{\begin{matrix}\pi_1 \\ \Rightarrow \overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \rhd \overline{\alpha}\end{matrix}}{\overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n \Rightarrow \overline{\alpha}} \\ \vdots\ \wedge e\end{matrix} \quad \begin{matrix}\dfrac{}{\overline{\beta}_{i+1} \wedge \ldots \wedge \overline{\beta}_m \Rightarrow \overline{\beta}_{i+1} \wedge \ldots \wedge \overline{\beta}_m}\,\text{Id} \\ \vdots\ \wedge e\end{matrix}}{\ldots}\,\wedge i}{\overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_{i-1}, \overline{\alpha}_1 \wedge \ldots \wedge \overline{\alpha}_n, \overline{\beta}_{i+1} \wedge \ldots \wedge \overline{\beta}_m \Rightarrow \overline{\beta}_1 \wedge \ldots \wedge \overline{\beta}_{i-1} \wedge \overline{\alpha} \wedge \overline{\beta}_{i+1} \wedge \ldots \wedge \overline{\beta}_m}\,\wedge i$$

□

6.1 Characterisation of <

We can characterise the set of all formulas $\{\alpha_1, \ldots, \alpha_n\} < \alpha$ provable in **G** by a simple procedure on the syntactic tree of α. This implies, in particular, that it is possible to determine when a derivable formula $\{\alpha_1, \ldots, \alpha_n\} \rhd \alpha$ corresponds to a provable grounding formula.

Let us fix some terminological conventions and define some notions useful to prove this fact. We assume that the root of a tree is on top of it, and hence that a node a is below a node b if and only if the path from a to the root of the tree goes through b. Moreover, for any logical operator $\star$, we call $\star$-nodes all nodes of the syntactic tree of the formula α that correspond to a subformula of α with $\star$ as main connective. Lastly, we remind the reader that a *bar* of a tree T is a set B of nodes of T such that exactly one element of B occurs on each path from a leaf of T to its root. We call a bar *non trivial* if it does not contain the root of T.

Definition 6.2 Given any syntactic tree, we call *positive nodes* those that occur strictly below an even number of $\sim$-nodes and *negative nodes* those that occur strictly below an odd number of $\sim$-nodes.

Definition 6.3 Given any syntactic tree T, we call *feeble nodes* the positive $\vee$-nodes and the negative $\wedge$-nodes.

Definition 6.4 Given a formula α with syntactic tree T, a *selection tree* for α is any tree obtained by selecting exactly one child of each feeble node of T, and then by deleting from T all subtrees rooted at any of the selected nodes.

Definition 6.5 Given any formula α, a *grounding bar* G for α is any set that can be obtained as follows: pick a selection tree S for α; pick a bar B of S that is not trivial and that does not contain the only child of the root of S, if there is one such node; prefix all negative nodes in B by $\sim$.

Given the previous definitions, it is easy to show the following:

> A grounding claim $\{\alpha_1, \ldots, \alpha_n\} < \alpha$ is provable in **G** if, and only if, the set of formulas $\{\alpha_1, \ldots, \alpha_n\}$ is the union of some grounding bars for α.

Intuitively, constructing the selection tree corresponds to choosing one disjunct for each disjunction and one conjunct for each formula $\sim(\beta \wedge \gamma)$ that might be encountered as element of a ground along a derivation in **G**. Note that the definition of grounding bar guarantees that at least a negation is prefixed to each negative subformula of α in case it occurs as element of a ground. Lastly, the union of different grounding bars should be admitted since the Am rule is in the calculus and it precisely enables us to form a ground by taking the union of two grounds for the same formula.[12]

7 Normalisation of R

So far we have established one crucial relation between **R** and **G**, which rigorously substantiates the idea that the logic of grounds is part of the logic of reasons. The aim of the rest of the paper is to outline some key results about **R** itself. The first of these results, which is presented in this section, is the existence of a normalization procedure for the derivations in **R**.

We define a set of permutation and reduction rules by expressions of the form $\pi \mapsto \pi'$ indicating that any subderivation of the form π occurring in a derivation can be replaced by a subderivation of the form π'. We call any subderivation that has one of the forms displayed to the left of $\mapsto$ a *redex*, and the corresponding subderivation with the form indicated to the right of $\mapsto$ its *reductum*.

In defining some rules, we will employ derivations denoted by expressions of the form $\pi_{\bot\vdash\alpha}$ and $\pi_{\sim\sim\alpha\vdash\alpha}$. A derivation denoted by $\pi_{\bot\vdash\alpha}$ has conclusion α depending

[12] See [15] for a formal proof of this.

on Γ and is obtained by applying the inductive procedure defined in the proof of Theorem 3.2 on a derivation of $\Gamma \Rightarrow \bot$. A derivation denoted by $\pi_{\sim\sim\alpha\vdash\alpha}$, on the other hand, has conclusion α depending on Γ and is obtained by using the inductive procedure defined in the proof of Theorem 3.3 on a derivation of $\Gamma \Rightarrow {\sim\sim}\alpha$. Since derivations of this kind are exclusively used when α does not contain $\rhd$, they are always purely classical derivations. As a consequence, we can assume that they can be normalised before using them in a reductum.[13]

Definition 7.1 Permutations for C**:**

$$\dfrac{\dfrac{\overset{\pi_1}{\Rightarrow \alpha \rhd \beta} \quad \overset{\pi_2}{\alpha \Leftrightarrow \gamma} \quad \overset{\pi_3}{\beta \Leftrightarrow \delta}}{\Rightarrow \gamma \rhd \delta}\,\equiv}{\Rightarrow {\sim}\delta \rhd {\sim}\gamma}\,\mathrm{C} \quad \mapsto$$

$$\dfrac{\dfrac{\overset{\pi_1}{\Rightarrow \alpha \rhd \beta}}{\Rightarrow {\sim}\beta \rhd {\sim}\alpha}\,\mathrm{C} \quad \underset{{\sim}\beta \Leftrightarrow {\sim}\delta}{\left(\dfrac{\dfrac{\dfrac{}{{\sim}\delta \Rightarrow {\sim}\delta} \quad \overset{\pi_3^{\Rightarrow}}{\beta \Rightarrow \delta}}{{\sim}\delta, \beta \Rightarrow \bot}}{{\sim}\delta \Rightarrow {\sim}\beta} \quad \dfrac{\dfrac{\dfrac{}{{\sim}\beta \Rightarrow {\sim}\beta} \quad \overset{\pi_3^{\Leftarrow}}{\delta \Rightarrow \beta}}{{\sim}\beta, \delta \Rightarrow \bot}}{{\sim}\beta \Rightarrow {\sim}\delta}\right)} \quad \underset{{\sim}\alpha \Leftrightarrow {\sim}\gamma}{\left(\dfrac{\dfrac{\dfrac{}{{\sim}\gamma \Rightarrow {\sim}\gamma} \quad \overset{\pi_2^{\Rightarrow}}{\alpha \Rightarrow \gamma}}{{\sim}\gamma, \alpha \Rightarrow \bot}}{{\sim}\gamma \Rightarrow {\sim}\alpha} \quad \dfrac{\dfrac{\dfrac{}{{\sim}\alpha \Rightarrow {\sim}\alpha} \quad \overset{\pi_2^{\Leftarrow}}{\gamma \Rightarrow \alpha}}{{\sim}\alpha, \gamma \Rightarrow \bot}}{{\sim}\alpha \Rightarrow {\sim}\gamma}\right)}}{\Rightarrow {\sim}\delta \rhd {\sim}\gamma}\,\equiv$$

Reductions for C**.**

C-C-reduction:

$$\dfrac{\dfrac{\overset{\pi}{\Gamma \Rightarrow \alpha \rhd \beta}}{\Gamma \Rightarrow {\sim}\beta \rhd {\sim}\alpha}\,\mathrm{C}}{\Gamma \Rightarrow {\sim\sim}\alpha \rhd {\sim\sim}\beta}\,\mathrm{C} \quad \mapsto$$

$$\dfrac{\overset{\pi}{\Gamma \Rightarrow \alpha \rhd \beta} \quad \underset{\alpha \Leftrightarrow {\sim\sim}\alpha}{\left(\dfrac{\dfrac{\dfrac{}{\alpha \Rightarrow \alpha} \quad \dfrac{}{{\sim}\alpha \Rightarrow {\sim}\alpha}}{\alpha, {\sim}\alpha \Rightarrow \bot}}{\alpha \Rightarrow {\sim\sim}\alpha} \quad \begin{matrix}\dfrac{}{{\sim\sim}\alpha \Rightarrow {\sim\sim}\alpha}\\ \pi_{\sim\sim\alpha\vdash\alpha}\\ {\sim\sim}\alpha \Rightarrow \alpha\end{matrix}\right)} \quad \underset{\beta \Leftrightarrow {\sim\sim}\beta}{\left(\dfrac{\dfrac{\dfrac{}{\beta \Rightarrow \beta} \quad \dfrac{}{{\sim}\beta \Rightarrow {\sim}\beta}}{\beta, {\sim}\beta \Rightarrow \bot}}{\beta \Rightarrow {\sim\sim}\beta} \quad \begin{matrix}\dfrac{}{{\sim\sim}\beta \Rightarrow {\sim\sim}\beta}\\ \pi_{\sim\sim\beta\vdash\beta}\\ {\sim\sim}\beta \Rightarrow \beta\end{matrix}\right)}}{\Gamma \Rightarrow {\sim\sim}\alpha \rhd {\sim\sim}\beta}\,\equiv$$

where $\pi_{\sim\sim\alpha\vdash\alpha}$ and $\pi_{\sim\sim\beta\vdash\beta}$ are the normal forms of the purely classical derivations constructed according to Theorem 3.3.

C-$\equiv$-C-reduction:

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\Gamma \Rightarrow \alpha \rhd \beta}}{\Gamma \Rightarrow {\sim}\beta \rhd {\sim}\alpha}\,\mathrm{C} \quad \overset{\pi_2}{{\sim}\beta \Leftrightarrow \gamma} \quad \overset{\pi_3}{{\sim}\alpha \Leftrightarrow \delta}}{\Gamma \Rightarrow \gamma \rhd \delta}\,\equiv}{\Gamma \Rightarrow {\sim}\delta \rhd {\sim}\gamma}\,\mathrm{C} \quad \mapsto$$

[13] The classical rules of Fig. 1 are rather standard and, in any case, if one ignores all rules and reductions concerning $\rhd$, the proof that we are going to present is a normalisation proof for the purely classical fragment of **R**. Indeed, if we ignore the reductions involving rules for $\rhd$, no assumption on the normalisability of purely classical derivations is required. Therefore, the proof of Theorem 7.5 constitutes a perfectly legitimate normalisation proof for the calculus only featuring the classical rules in Fig. 1.

$$\dfrac{\overset{\pi_1}{\Gamma\Rightarrow\alpha\rhd\beta}\quad \begin{array}{c}\left(\dfrac{\dfrac{\overline{\alpha\Rightarrow\alpha}\quad \overset{\pi_3^{\Leftarrow}}{\delta\Rightarrow\sim\alpha}}{\alpha,\delta\Rightarrow\bot}}{\alpha\Rightarrow\sim\delta}\quad \begin{array}{c}\dfrac{\dfrac{\overline{\sim\delta\Rightarrow\sim\delta}\quad \overset{\pi_3^{\Rightarrow}}{\sim\alpha\Rightarrow\delta}}{\sim\alpha,\sim\delta\Rightarrow\bot}}{\sim\delta\Rightarrow\sim\sim\alpha}\\ \pi_{\sim\sim\alpha\vdash\alpha}\\ \sim\delta\Rightarrow\alpha\end{array}\right)\\ \alpha\Leftrightarrow\sim\delta\end{array}\quad \begin{array}{c}\left(\dfrac{\dfrac{\overline{\beta\Rightarrow\beta}\quad \overset{\pi_2^{\Leftarrow}}{\gamma\Rightarrow\sim\beta}}{\beta,\gamma\Rightarrow\bot}}{\beta\Rightarrow\sim\gamma}\quad \begin{array}{c}\dfrac{\dfrac{\overline{\sim\gamma\Rightarrow\sim\gamma}\quad \overset{\pi_2^{\Rightarrow}}{\sim\beta\Rightarrow\gamma}}{\sim\gamma,\sim\beta\Rightarrow\bot}}{\sim\gamma\Rightarrow\sim\sim\beta}\\ \pi_{\sim\sim\beta\vdash\beta}\\ \sim\gamma\Rightarrow\beta\end{array}\right)\\ \beta\Leftrightarrow\sim\gamma\end{array}}{\Gamma\Rightarrow\sim\delta\rhd\sim\gamma}\ \equiv$$

where $\pi_{\sim\sim\alpha\vdash\alpha}$ and $\pi_{\sim\sim\beta\vdash\beta}$ are the normal forms of the purely classical derivations constructed according to Theorem 3.3.

$\rhd i$-C-$\rhd e$-reduction:

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\alpha\Rightarrow\beta}}{\Rightarrow\alpha\rhd\beta}\,\rhd i}{\Rightarrow\sim\beta\rhd\sim\alpha}\,\mathrm{C}\quad \overset{\pi_2}{\Gamma\Rightarrow\sim\beta}}{\Gamma\Rightarrow\sim\alpha}\,\rhd e\quad\mapsto\quad \dfrac{\dfrac{\overset{\pi_2}{\Gamma\Rightarrow\sim\beta}\quad \overset{\pi_1}{\alpha\Rightarrow\beta}}{\Gamma,\alpha\Rightarrow\bot}\,\supset e}{\Gamma\Rightarrow\sim\alpha}\,\supset i$$

$\rhd i$-C-reduction:

$$\dfrac{\dfrac{\overset{\pi}{\alpha\Rightarrow\beta}}{\Rightarrow\alpha\rhd\beta}\,\rhd i}{\Rightarrow\sim\beta\rhd\sim\alpha}\,\mathrm{C}\quad\mapsto\quad \dfrac{\dfrac{\dfrac{\overline{\sim\beta\Rightarrow\sim\beta}\,\mathrm{Id}\quad \overset{\pi}{\alpha\Rightarrow\beta}}{\sim\beta,\alpha\Rightarrow\bot}\,\supset e}{\sim\beta\Rightarrow\sim\alpha}\,\supset i}{\Rightarrow\sim\beta\rhd\sim\alpha}\,\rhd i$$

Expl-C-reduction:

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma\Rightarrow\bot}\quad \overset{\pi_2}{\Gamma\Rightarrow\alpha}\quad \overset{\pi_3}{\Gamma\Rightarrow\beta}}{\Gamma\Rightarrow\alpha\rhd\beta}\,\mathrm{Expl}}{\Gamma\Rightarrow\sim\beta\rhd\sim\alpha}\,\mathrm{C}\quad\mapsto\quad \dfrac{\overset{\pi_1}{\Gamma\Rightarrow\bot}\quad \overset{\pi_{\bot\vdash\sim\beta}}{\Gamma\Rightarrow\sim\beta}\quad \overset{\pi_{\bot\vdash\sim\alpha}}{\Gamma\Rightarrow\sim\alpha}}{\Gamma\Rightarrow\sim\beta\rhd\sim\alpha}\,\mathrm{Expl}$$

where $\pi_{\bot\vdash\sim\beta}$ and $\pi_{\bot\vdash\sim\alpha}$ are the normal forms of the purely classical derivations constructed according to the inductive procedure defined in the proof of Theorem 3.2.

Reductions for $\equiv$.

$\equiv$-$\equiv$-reduction:

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma\Rightarrow\alpha\rhd\beta}\quad \overset{\pi_2}{\alpha\Leftrightarrow\gamma}\quad \overset{\pi_3}{\beta\Leftrightarrow\delta}}{\Gamma\Rightarrow\gamma\rhd\delta}\,\equiv\quad \overset{\pi_4}{\gamma\Leftrightarrow\zeta}\quad \overset{\pi_5}{\delta\Leftrightarrow\xi}}{\Gamma\Rightarrow\zeta\rhd\xi}\,\equiv\quad\mapsto\quad \dfrac{\overset{\pi_1}{\Gamma\Rightarrow\alpha\rhd\beta}\quad \begin{array}{c}\left(\begin{array}{cc}\overset{\pi_2^{\Rightarrow}}{\alpha\Rightarrow\gamma} & \overset{\pi_4^{\Leftarrow}}{\zeta\Rightarrow\gamma}\\ \overset{\pi_4^{\Rightarrow}}{\alpha\Rightarrow\zeta} & \overset{\pi_2^{\Leftarrow}}{\zeta\Rightarrow\alpha}\end{array}\right)\\ \alpha\Leftrightarrow\zeta\end{array}\quad \begin{array}{c}\left(\begin{array}{cc}\overset{\pi_3^{\Rightarrow}}{\beta\Rightarrow\delta} & \overset{\pi_5^{\Leftarrow}}{\xi\Rightarrow\delta}\\ \overset{\pi_5^{\Rightarrow}}{\beta\Rightarrow\xi} & \overset{\pi_3^{\Leftarrow}}{\xi\Rightarrow\beta}\end{array}\right)\\ \beta\Leftrightarrow\xi\end{array}}{\Gamma\Rightarrow\zeta\rhd\xi}\,\equiv$$

Expl-$\equiv$-$\rhd e$-reduction:

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\Gamma\Rightarrow\bot}\quad \overset{\pi_2}{\Gamma\Rightarrow\alpha}\quad \overset{\pi_3}{\Gamma\Rightarrow\beta}}{\Gamma\Rightarrow\alpha\rhd\beta}\,\mathrm{Expl}\quad \overset{\pi_4}{\alpha\Leftrightarrow\gamma}\quad \overset{\pi_5}{\beta\Leftrightarrow\delta}}{\Gamma\Rightarrow\gamma\rhd\delta}\,\equiv\quad \overset{\pi_6}{\Delta\Rightarrow\gamma}}{\Gamma,\Delta\Rightarrow\delta}\quad\mapsto\quad \overset{\pi_{\bot\vdash\delta}}{\Gamma\Rightarrow\delta}$$

where $\pi_{\bot\vdash\delta}$ is the normal derivation constructed according to the inductive procedure defined in the proof of Theorem 3.2.

Expl-≡-reduction:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma\Rightarrow\bot\end{matrix}\quad\begin{matrix}\pi_2\\ \Gamma\Rightarrow\alpha\end{matrix}\quad\begin{matrix}\pi_3\\ \Gamma\Rightarrow\beta\end{matrix}}{\Gamma\Rightarrow\alpha\rhd\beta}\,\text{Expl}\quad\begin{matrix}\pi_4\\ \alpha\Leftrightarrow\gamma\end{matrix}\quad\begin{matrix}\pi_5\\ \beta\Leftrightarrow\delta\end{matrix}}{\Gamma\Rightarrow\gamma\rhd\delta}\,\equiv\quad\mapsto\quad\dfrac{\begin{matrix}\pi_1\\ \Gamma\Rightarrow\bot\end{matrix}\quad\begin{matrix}\pi_{\bot\vdash\gamma}\\ \Gamma\Rightarrow\gamma\end{matrix}\quad\begin{matrix}\pi_{\bot\vdash\delta}\\ \Gamma\Rightarrow\delta\end{matrix}}{\Gamma\Rightarrow\gamma\rhd\delta}\,\text{Expl}$$

where $\pi_{\bot\vdash\gamma}$ and $\pi_{\bot\vdash\delta}$ are the normal forms of the purely classical derivations constructed according to the inductive procedure defined in the proof of Theorem 3.2.

$\rhd i$-≡-$\rhd e$-reduction:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma,\alpha\Rightarrow\beta\end{matrix}}{\Gamma\Rightarrow\alpha\rhd\beta}\,\rhd i\quad\begin{matrix}\pi_2\\ \alpha\Leftrightarrow\gamma\end{matrix}\quad\begin{matrix}\pi_3\\ \beta\Leftrightarrow\delta\end{matrix}}{\Gamma\Rightarrow\gamma\rhd\delta}\,\equiv\quad\begin{matrix}\pi_4\\ \Delta\Rightarrow\gamma\end{matrix}}{\Gamma,\Delta\Rightarrow\delta}\quad\mapsto\quad\begin{matrix}\pi_4\\ \Delta\Rightarrow\gamma\\ \pi_2^{\Leftarrow}\\ \Delta\Rightarrow\alpha\\ \pi_1\\ \Gamma,\Delta\Rightarrow\beta\\ \pi_3^{\Rightarrow}\\ \Gamma,\Delta\Rightarrow\delta\end{matrix}$$

$\rhd i$-≡-reduction:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \alpha\Rightarrow\beta\end{matrix}}{\Rightarrow\alpha\rhd\beta}\,\rhd i\quad\begin{matrix}\pi_2\\ \alpha\Leftrightarrow\gamma\end{matrix}\quad\begin{matrix}\pi_3\\ \beta\Leftrightarrow\delta\end{matrix}}{\Rightarrow\gamma\rhd\delta}\,\equiv\quad\mapsto\quad\dfrac{\begin{matrix}\pi_2^{\Leftarrow}\\ \gamma\Rightarrow\alpha\\ \pi_1\\ \gamma\Rightarrow\beta\\ \pi_3^{\Rightarrow}\\ \gamma\Rightarrow\delta\end{matrix}}{\Rightarrow\gamma\rhd\delta}$$

≡-$\rhd e$-reduction:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma\Rightarrow\alpha\rhd\beta\end{matrix}\quad\begin{matrix}\pi_2\\ \alpha\Leftrightarrow\gamma\end{matrix}\quad\begin{matrix}\pi_3\\ \beta\Leftrightarrow\delta\end{matrix}}{\Gamma\Rightarrow\gamma\rhd\delta}\,\equiv\quad\begin{matrix}\pi_4\\ \Delta\Rightarrow\gamma\end{matrix}}{\Gamma,\Delta\Rightarrow\delta}\,\rhd e\quad\mapsto\quad\begin{matrix}\dfrac{\begin{matrix}\pi_1\\ \Gamma\Rightarrow\alpha\rhd\beta\end{matrix}\quad\begin{matrix}\pi_4\\ \Delta\Rightarrow\gamma\\ \pi_2^{\Leftarrow}\\ \Delta\Rightarrow\alpha\end{matrix}}{\Gamma,\Delta\Rightarrow\beta}\,\rhd e\\ \pi_3^{\Rightarrow}\\ \Gamma,\Delta\Rightarrow\delta\end{matrix}$$

Detour eliminations for $\rhd$:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \alpha\Rightarrow\beta\end{matrix}}{\Rightarrow\alpha\rhd\beta}\,\rhd i\quad\begin{matrix}\pi_2\\ \Delta\Rightarrow\alpha\end{matrix}}{\Delta\Rightarrow\beta}\,\rhd e\quad\mapsto\quad\begin{matrix}\pi_2\\ \Delta\Rightarrow\alpha\\ \pi_1\\ \Delta\Rightarrow\beta\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma\Rightarrow\bot\end{matrix}\quad\begin{matrix}\pi_2\\ \Gamma\Rightarrow\alpha\end{matrix}\quad\begin{matrix}\pi_3\\ \Gamma\Rightarrow\beta\end{matrix}}{\Gamma\Rightarrow\alpha\rhd\beta}\,\text{Expl}\quad\begin{matrix}\pi_4\\ \Delta\Rightarrow\alpha\end{matrix}}{\Gamma,\Delta\Rightarrow\beta}\,\rhd e\quad\mapsto\quad\begin{matrix}\pi_3\\ \Gamma\Rightarrow\beta\end{matrix}$$

Detour eliminations for $\supset$:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma,\alpha\Rightarrow\beta\end{matrix}}{\Gamma\Rightarrow\alpha\supset\beta}\,\supset i\quad\begin{matrix}\pi_2\\ \Delta\Rightarrow\alpha\end{matrix}}{\Gamma,\Delta\Rightarrow\beta}\,\supset e\quad\mapsto\quad\begin{matrix}\pi_2\\ \Delta\Rightarrow\alpha\\ \pi_1\\ \Gamma,\Delta\Rightarrow\beta\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma\Rightarrow\beta\end{matrix}}{\Gamma\Rightarrow\alpha\supset\beta}\,\supset i\quad\begin{matrix}\pi_2\\ \Delta\Rightarrow\alpha\end{matrix}}{\Gamma,\Delta\Rightarrow\beta}\,\supset e\quad\mapsto\quad\begin{matrix}\pi_1\\ \Gamma\Rightarrow\beta\end{matrix}$$

Detour eliminations for conjunction:

$$\dfrac{\dfrac{\begin{matrix}\pi_1 \\ \Gamma \Rightarrow \alpha\end{matrix} \quad \begin{matrix}\pi_2 \\ \Delta \Rightarrow \beta\end{matrix}}{\Gamma, \Delta \Rightarrow \alpha \wedge \beta}\wedge i}{\Gamma, \Delta \Rightarrow \alpha}\wedge e \mapsto \begin{matrix}\pi_1 \\ \Gamma \Rightarrow \alpha\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\pi_1 \\ \Gamma \Rightarrow \alpha\end{matrix} \quad \begin{matrix}\pi_2 \\ \Delta \Rightarrow \beta\end{matrix}}{\Gamma, \Delta \Rightarrow \alpha \wedge \beta}\wedge i}{\Gamma, \Delta \Rightarrow \beta}\wedge e \mapsto \begin{matrix}\pi_2 \\ \Gamma \Rightarrow \beta\end{matrix}$$

In order to prove that the defined normalisation procedure terminates in a finite number of steps for each derivation, we associate a complexity value to each redex in the derivation.

Definition 7.2 The complexity values of permutation and reduction rules are as follows:

- The redex complexity of C-permutation, C-C-reduction, C-$\equiv$-C-reduction, $\triangleright i$-C-reduction, Expl-C-reduction, $\triangleright i$-C-$\triangleright e$-reduction redexes is the depth of the conclusion of the uppermost C application displayed in the reduction rule.
- The redex complexity of $\equiv$-$\equiv$-reduction, $\triangleright i$-$\equiv$-$\triangleright e$-reduction, $\equiv$-$\triangleright e$-reduction and Expl-$\equiv$-$\triangleright e$-reduction redexes is the depth of the conclusion of the uppermost $\equiv$ rule application displayed in the reduction rule.
- The redex complexity of Expl-$\equiv$-reduction and $\triangleright i$-$\equiv$-reduction redexes is the depth of the major premise of the $\equiv$ rule application displayed in the reduction rule.
- The redex complexity of a detour elimination redex is the depth of the major premise of the bottommost elimination rule application displayed in the reduction rule.

We say that a rule application is *critical* if it is the instance of a schema displayed to the left of $\mapsto$ in any of the permutation and reduction rules of Definition 7.1.

Let us now introduce the notion of *block*. A block, roughly, is a sequence of C and $\equiv$ rule applications such that each rule application, except the lowermost, has as conclusion the major premise of the rule application below. If the block only contains one rule application or an application of C followed by one of $\equiv$, then we say that the block is *inert* since no redex occurs in the block. The complexity that we assign to a block enables us to predict the complexity of the new redexes that we will generate by reducing redexes occurring in that block.

Definition 7.3 A *block* is any sequence of formula occurrences $\alpha_1, \ldots, \alpha_n$ in a derivation such that:

- α_1 is the premise of a C or major premise of a $\equiv$ rule application but not the conclusion of a C or $\equiv$ rule application,
- for each $i \in \{1, \ldots, n-1\}$, α_i is the premise of a C or major premise of a $\equiv$ rule application r such that α_{i+1} is the conclusion of r,
- α_n is the conclusion of a C or $\equiv$ rule application but not the premise of a C or major premise of a $\equiv$ rule application.

Let us call a block *inert* if one of the following holds:

- the block only contains two formula occurrences,

- the block only contains three formula occurrences $\alpha_1, \alpha_2, \alpha_3$ such that α_1 is the premise of a C application and α_2 is the major premise of a $\equiv$ application.

Blocks that are not inert are called *active*.

The complexity of a block $\alpha_1, \ldots, \alpha_n$ is 0 if it is inert and, if it is active, the maximal depth of the formulas $\alpha_2, \ldots, \alpha_{n-1}$.

We now define a strategy for reducing derivations to their normal form. Intuitively, phase $\mathbb{BE}$ (*Block elimination*) aims at the reduction of all C and $\equiv$ redexes of maximal complexity. We usually only reduce C and $\equiv$ redexes that do not contain C and $\equiv$ redexes of maximal complexity. We make an exception for redexes with three critical rules—that is, $\triangleright i$-C-$\triangleright e$-reductions, $\triangleright i$-$\equiv$-$\triangleright e$-reductions and Expl-$\equiv$-$\triangleright e$-reductions—since these are reduced before the two-critical-rule redexes that share two of their three critical rules. During phase $\mathbb{DE}$ (*Detour Elimination*), we reduce all detour elimination redexes of maximal complexity that do not contain detour elimination redexes of maximal complexity. These two phases are interleaved until no redex occurs in the derivation. The division of the normalisation procedure in two phases is essential if we wish to show that the normalisation reduces the complexity of the derivation and thus terminates. Indeed, in general, the reduction of a detour redex can generate new redexes of greater complexity through the transformation of a block. If we make all blocks in the derivation inert, on the other hand, detour eliminations of complexity c can only generate new C or $\equiv$ redexes with complexity c or less. Since eliminating blocks can only produce detour elimination redexes of smaller complexity, it is possible to show that, by suitably interleaving $\mathbb{BE}$ and $\mathbb{DE}$, we can normalise any derivation in a finite number of steps.

Definition 7.4 The normalisation procedure for **R** derivations consists of the following phases which must be interleaved as long as possible:

$\mathbb{BE}$ We keep reducing, as long as possible,

- all redexes consisting in a C-permutation, C-C-reduction, C-$\equiv$-C-reduction, $\triangleright i$-C-reduction, Expl-C-reduction, $\equiv$-$\equiv$-reduction, Expl-$\equiv$-reduction, $\triangleright i$-$\equiv$-reduction, $\equiv$-$\triangleright e$-reduction if they
  - are of maximal complexity,
  - do not have critical rule applications that are part of a $\triangleright i$-C-$\triangleright e$-reduction, $\triangleright i$-$\equiv$-$\triangleright e$-reduction or Expl-$\equiv$-$\triangleright e$-reduction redex,
  - do not contain any other C or $\equiv$ redex of maximal complexity;
- all redexes consisting in a $\triangleright i$-C-$\triangleright e$-reduction, $\triangleright i$-$\equiv$-$\triangleright e$-reduction, Expl-$\equiv$-$\triangleright e$-reduction if
  - they are of maximal complexity,
  - all C and $\equiv$ redexes of maximal complexity that they contain share two of the three critical rules of the $\triangleright i$-C-$\triangleright e$-reduction, $\triangleright i$-$\equiv$-$\triangleright e$-reduction or Expl-$\equiv$-$\triangleright e$-reduction redex.

When no more reduction that complies with these conditions is possible, we proceed to $\mathbb{DE}$.

$\mathbb{DE}$ If possible, we reduce a detour elimination redex of maximal complexity that does not contain any other detour elimination redex of maximal complexity. If no detour elimination redex of maximal complexity that does not contain any other detour elimination redex of maximal complexity occurs in the derivation, we proceed to $\mathbb{BE}$.

Definition 7.5 We define a derivation in normal form as one in which no redex occurs.

We can finally show that the strategy of Definition 7.4 is terminating. In other words, we prove that, if we normalise any derivation by following this strategy, then the procedure terminates in a finite number of steps. We first establish that each individual iteration of phases $\mathbb{BE}$ and $\mathbb{DE}$ terminates by showing that some rather standard complexity measures strictly decrease; then we show that after one iteration of both phases, a less obvious complexity measure strictly decreases, and thus that the procedure terminates. The first iteration of $\mathbb{BE}$ and $\mathbb{DE}$, indeed, does not necessarily decrease the maximal complexity m of the redexes originally occurring in the derivation. This might happen simply because the first iteration of $\mathbb{BE}$ might leave in the derivation some detour of complexity m. Indeed, no detour is supposed to be eliminated during $\mathbb{BE}$. But this implies that the first iteration of $\mathbb{DE}$ might reactivate some block of complexity m. This might seem a serious problem, but it is not, because the second iteration of $\mathbb{BE}$ starts with a derivation that can contain some maximal block of complexity m but cannot contain any detour of complexity greater than $m - 1$. Since $\mathbb{BE}$ brings all blocks to complexity 0 and since block eliminations can only produce new detours of smaller complexity, the actual decrease of the complexity of the derivation finally starts. In simpler terms, the first iteration of each phase—one of $\mathbb{BE}$ and one of $\mathbb{DE}$—can be seen as a preliminary phase that levels the complexity of the derivation in such a way that the following iterations effectively decrease the complexity.

We first prove that each iteration of $\mathbb{BE}$ terminates.

Lemma 7.1 *For any derivation π in* **R**, *there is no infinite sequence $\pi \mapsto \pi_1 \mapsto \pi_2 \ldots$ of reductions following the strategy for phase $\mathbb{BE}$ specified in Definition 7.4.*

Proof Let the $\mathbb{BE}$-complexity of a derivation be (t, b) where

- t is the total number of critical applications of C and $\equiv$ which form redexes of maximal complexity, and
- b is the number of critical C applications that form redexes of maximal complexity and occur below at least one $\equiv$ application.

Now, each permutation for C in phase $\mathbb{BE}$ moves one critical C application which form a redex of maximal complexity above a $\equiv$ application. Moreover, it does not produce new $\equiv$ applications above any critical application of C and it does not duplicate any redex of maximal complexity. Hence, each permutation for C during $\mathbb{BE}$ decreases b without increasing t. Each reduction for C and $\equiv$ during $\mathbb{BE}$ eliminates one maximal C or $\equiv$ reduction redex without duplicating any other maximal redex—as usual, because we only reduce maximal redexes that do not contain any other maximal redex. Hence, reductions for C and $\equiv$ reduce t and the $\mathbb{BE}$-complexity strictly decreases at each reduction.The termination of each iteration of $\mathbb{BE}$ is thus guaranteed. □

We now show that each iteration of $\mathbb{DE}$ terminates in a finite number of steps.

Lemma 7.2 *For any derivation π in **R**, there is no infinite sequence $\pi \mapsto \pi_1 \mapsto \pi_2 \ldots$ of reductions following the strategy for phase $\mathbb{DE}$ specified in Definition 7.4.*

Proof The argument is rather standard since $\rhd$ detour eliminations are very similar to $\supset$ detour eliminations. Let the $\mathbb{DE}$-complexity of a derivation be (m, n) where m is the maximal redex complexity of the detours occurring in the derivation and n is the number of detours in the derivation that have redex complexity m. Each reduction during $\mathbb{DE}$ eliminates a detour with maximal redex complexity. Moreover, no redex of maximal complexity is duplicated during a reduction of $\mathbb{DE}$: we only reduce maximal detour redexes that do not contain other maximal detour redexes. Lastly, the new detour redexes possibly generated by the redex—those generated when we move the derivation of the minor premise α above the hypotheses α occurring above the major premise of $\rhd$ and $\supset$ detour eliminations—can only have smaller complexity than the one of the reduced redex. Hence, the $\mathbb{DE}$-complexity strictly decreases at each reduction of $\mathbb{DE}$. This guarantees the termination of each iteration of $\mathbb{DE}$. □

We now need to show that the whole normalisation procedure for **R** derivations terminates in a finite number of steps. Before doing it, we define the complexity measure that will be used in the proof and we establish Lemmas 7.4 and 7.3, which will be essential in the proof of Theorem 7.5.

Definition 7.6 For any derivation π, let the derivation complexity of π be the number $c = \max(b, r)$ where b is the maximal block complexity of the blocks in π and d is the maximal redex complexity of the redexes in π.

Lemma 7.3 *For any derivation π, the reduction steps of $\mathbb{BE}$ only produce new detours and* DNE *permutations of complexity smaller than the derivation complexity c of π.*

Proof Let us consider each kind of reduction separately. We refer to permutation and reduction rules as presented in Definition 7.1.

- Permutations for C. The complexity of C permutations is $|\gamma \rhd \delta|$ and no detours are generated since, after the permutation, the derivations $\pi_2^{\Rightarrow}, \pi_2^{\Leftarrow}, \pi_3^{\Rightarrow}$ and $\pi_3^{\Leftarrow}$ are used to derive the minor premises of $\supset e$ rule applications. Hence, if, after the permutation, an introduction occurs immediately above an elimination, then it already occurred immediately above an elimination before the permutation.
- C-$\equiv$-C-reductions. The complexity of C-$\equiv$-C-reductions is $|{\sim}\beta \rhd {\sim}\alpha|$. The only detours that can be generated have complexity smaller than $|{\sim}\beta \rhd {\sim}\alpha|$. These possible new redexes, indeed, are due to the new locations of $\pi_3^{\Leftarrow}$ and $\pi_2^{\Leftarrow}$ and have complexity, respectively, $|{\sim}\alpha|$ and $|{\sim}\beta|$. The relocation of $\pi_3^{\Rightarrow}$ and $\pi_2^{\Rightarrow}$ does not generate new detours since, after the reduction, these derivations are used to obtain the *minor* premises of $\supset e$ applications.
- C-C-reductions. The complexity of C-C-reductions is $|{\sim}\beta \rhd {\sim}\alpha|$ and no detour is generated since the derivations of the minor premises of $\equiv$ are in normal form and since no detour is generated by a $\equiv$ application.
- $\rhd i$-C-$\rhd e$-reductions. The complexity of $\rhd i$-C-$\rhd e$-reductions is $|{\sim}\beta \rhd {\sim}\alpha|$. The only new detours that can be generated are $\supset$ detours due to either the new location of derivation π_2 or to the fact that the formula ${\sim}\alpha$ is now introduced by $\supset i$. Both

possible detours—one involving $\sim\beta$ and one involving $\sim\alpha$ as major premise of the critical introduction application—will have redex complexity strictly smaller than the complexity of the redex just reduced by the $\triangleright i$-C-$\triangleright e$-reduction.

- $\triangleright i$-C-reductions. The complexity of $\triangleright i$-C-reductions is $|\sim\beta \triangleright \sim\alpha|$ and no detour is generated, for two reasons. First, the derivation π is used, after the permutation, to derive the minor premise of an $\supset e$ rule application. Hence, no new introduction occurs immediately above an elimination. Secondly, if the $\triangleright i$ application displayed in the result of the reduction occurs immediately above a $\triangleright e$ applications, then a $\triangleright i$-C-$\triangleright e$-reduction was already possible and $\triangleright i$-C-$\triangleright e$-reductions have precedence over $\triangleright i$-C-reductions.
- Expl-C-reductions. No new detour can be generated by this reduction.
- $\equiv$-$\equiv$-reductions. The complexity of $\equiv$-$\equiv$-reductions is $|\gamma \triangleright \delta|$. The possible new detours generated are due to the new locations of the derivations $\pi_2^{\Rightarrow}$, $\pi_4^{\Leftarrow}$, $\pi_3^{\Rightarrow}$ and $\pi_5^{\Leftarrow}$ and they all have redex complexity strictly smaller than the $\equiv$-$\equiv$-reduction just executed. Indeed, their complexity will either be $|\gamma|$ or $|\delta|$. No new detour can be generated by the $\equiv$ rule application displayed in the result of the reduction.
- Expl-$\equiv$-$\triangleright e$-reductions. The complexity of Expl-$\equiv$-$\triangleright e$-reductions is $|\gamma \triangleright \delta|$. The new detour possibly generated has complexity strictly smaller than that of the Expl-$\equiv$-$\triangleright e$-reduction just executed. Indeed, it can only be a detour with δ as conclusion of its critical introduction application.
- Expl-$\equiv$-reductions. The complexity of Expl-$\equiv$-reductions is $|\alpha \triangleright \beta|$. All new detours possibly generated have complexity strictly smaller than that of the Expl-$\equiv$-reduction just executed. Indeed, if the Expl application displayed in the result of the reduction is critical with respect to a detour and $\gamma \triangleright \delta$ does not contain less symbol occurrences than $\alpha \triangleright \beta$, then a Expl-$\equiv$-$\triangleright e$-reduction was possible and Expl-$\equiv$-$\triangleright e$-reductions have precedence over Expl-$\equiv$-reductions with the same, or smaller, redex complexity.
- $\triangleright i$-$\equiv$-$\triangleright e$-reductions. The complexity of $\triangleright i$-$\equiv$-$\triangleright e$-reductions is $|\gamma \triangleright \delta|$. All new detours possibly generated have complexity strictly smaller than that of the $\triangleright i$-$\equiv$-$\triangleright e$-reduction just executed. Indeed, a $\triangleright i$-$\equiv$-$\triangleright e$-reduction is triggered only if $\alpha \triangleright \beta$ and $\gamma \triangleright \delta$ have the same depth; if $\alpha \triangleright \beta$ contains more symbol occurrences than $\gamma \triangleright \delta$, then a $\triangleright i$-$\equiv$-reduction has precedence due to greater redex complexity; if $\alpha \triangleright \beta$ and $\gamma \triangleright \delta$ have the same depth, then the new detours generated by the new locations of π_4, $\pi_2^{\Leftarrow}$, π_1 and $\pi_3^{\Rightarrow}$ have complexity strictly smaller than the $\triangleright i$-$\equiv$-$\triangleright e$-reduction just executed since the depth of $\gamma.\alpha$, β and δ is strictly smaller than the depth of $\alpha \triangleright \beta$.
- $\triangleright i$-$\equiv$-reductions. The complexity of $\triangleright i$-$\equiv$-reductions is $|\alpha \triangleright \beta|$. All new detours possibly generated have complexity strictly smaller than that of the $\triangleright i$-$\equiv$-reduction just executed. Indeed, if the $\triangleright i$ application displayed in the result of the reduction is critical with respect to a detour and $\gamma \triangleright \delta$ does not contain less symbol occurrences than $\alpha \triangleright \beta$, then a $\triangleright i$-$\equiv$-$\triangleright e$-reduction was possible before our $\triangleright i$-$\equiv$-reduction, and $\triangleright i$-$\equiv$-$\triangleright e$-reductions have precedence over $\triangleright i$-$\equiv$-reductions of equal or smaller redex complexity.
- $\equiv$-$\triangleright e$-reductions. The complexity of $\equiv$-$\triangleright e$-reductions is $|\gamma \triangleright \delta|$. All new detours possibly generated have complexity strictly smaller than that of the $\equiv$-$\triangleright e$-reduction just executed. Indeed, any redex generated by the new location of π_4 has redex

complexity $|\gamma|$, any redex generated by the new location of $\pi_3^{\Rightarrow}$ has as redex complexity $|\delta|$. Lastly, if a detour is generated by the new location of π_1 and $\alpha \rhd \beta$ does not contain less symbol occurrences than $\gamma \rhd \delta$, then a $\rhd i$-$\equiv$-$\rhd e$-reduction or a Expl-$\equiv$-$\rhd e$-reduction was possible before our $\equiv$-$\rhd e$-reduction, and both $\rhd i$-$\equiv$-$\rhd e$-reductions and Expl-$\equiv$-$\rhd e$-reductions have precedence over $\equiv$-$\rhd e$-reductions with equal or smaller redex complexity.

□

Lemma 7.4 *For any derivation π, the reduction steps of $\mathbb{BE}$ do not increase the derivation complexity c of π.*

Proof We need to show that $c = \max(b, d)$—where b is the maximal complexity of blocks in π and d is the maximal complexity of detours in π—is never increased by a reduction of $\mathbb{BE}$. Since in Lemma 7.3 we already established that the new detours generated by $\mathbb{BE}$ reductions have redex complexity strictly smaller than c, we simply need to show that the reductions of $\mathbb{BE}$ do not increase the complexity of blocks. Now, the reduction of a permutation redex

$$\dfrac{\dfrac{\begin{matrix}\pi_1 \\ \Gamma \Rightarrow \alpha \rhd \beta\end{matrix} \quad \begin{matrix}\pi_2 \\ \alpha \Leftrightarrow \gamma\end{matrix} \quad \begin{matrix}\pi_3 \\ \beta \Leftrightarrow \delta\end{matrix}}{\Gamma \Rightarrow \gamma \rhd \delta}\,\equiv}{\Gamma \Rightarrow \sim\delta \rhd \sim\gamma}\,\mathrm{C} \quad \mapsto$$

$$\dfrac{\dfrac{\begin{matrix}\pi_1 \\ \Gamma \Rightarrow \alpha \rhd \beta\end{matrix}}{\Gamma \Rightarrow \sim\beta \rhd \sim\alpha}\,\mathrm{C} \quad \begin{matrix}\left(\dfrac{\dfrac{\dfrac{}{\sim\delta \Rightarrow \sim\delta} \quad \begin{matrix}\pi_3^{\Rightarrow} \\ \beta \Rightarrow \delta\end{matrix}}{\sim\delta, \beta \Rightarrow \bot}}{\sim\delta \Rightarrow \sim\beta} \quad \dfrac{\dfrac{\dfrac{}{\sim\beta \Rightarrow \sim\beta} \quad \begin{matrix}\pi_3^{\Leftarrow} \\ \delta \Rightarrow \beta\end{matrix}}{\sim\beta, \delta \Rightarrow \bot}}{\sim\beta \Rightarrow \sim\delta}\right) \\ \sim\beta \Leftrightarrow \sim\delta\end{matrix} \quad \begin{matrix}\left(\dfrac{\dfrac{\dfrac{}{\sim\gamma \Rightarrow \sim\gamma} \quad \begin{matrix}\pi_2^{\Rightarrow} \\ \alpha \Rightarrow \gamma\end{matrix}}{\sim\gamma, \alpha \Rightarrow \bot}}{\sim\gamma \Rightarrow \sim\alpha} \quad \dfrac{\dfrac{\dfrac{}{\sim\alpha \Rightarrow \sim\alpha} \quad \begin{matrix}\pi_2^{\Leftarrow} \\ \gamma \Rightarrow \alpha\end{matrix}}{\sim\alpha, \gamma \Rightarrow \bot}}{\sim\alpha \Rightarrow \sim\gamma}\right) \\ \sim\alpha \Leftrightarrow \sim\gamma\end{matrix}}{\Gamma \Rightarrow \sim\delta \rhd \sim\gamma}\,\equiv$$

for C does not increase the complexity of any block. Indeed, if $\alpha \rhd \beta$ is not the conclusion of a $\equiv$ or C application, then $\sim\beta \rhd \sim\alpha$ is not counted in the compute of the complexity of the block; if $|\gamma \rhd \delta| < |\alpha \rhd \beta|$ and $\alpha \rhd \beta$ is the conclusion of a $\equiv$ or C, then we do not reduce the permutation for C that we are considering since we have a redex of greater complexity inside this permutation redex; if $|\alpha \rhd \beta| \leq |\gamma \rhd \delta|$, then there is no increase in the complexity of the block since $|\sim\alpha \rhd \sim\beta| \leq |\sim\gamma \rhd \sim\delta|$.

By inspecting all other reduction rules, it easy to see that no reduction for C, $\equiv$, $\rhd i$, $\rhd e$, Expl increases the complexity of any block. Moreover, $\rhd i$-C-$\rhd e$-reductions, Expl-$\equiv$-$\rhd e$-reductions, Expl-$\equiv$-reductions, $\rhd i$-$\equiv$-$\rhd e$-reductions, $\rhd i$-$\equiv$-reductions, and $\equiv$-$\rhd e$-reductions remove a whole block. □

Theorem 7.5 *For any derivation π of α with hypotheses Γ, there is an effective method for obtaining a normal derivation ν with conclusion α and hypotheses Γ.*

Proof We show that, given any derivation π in $\mathbf{R}$, there is no infinite sequence $\pi \mapsto \pi_1 \mapsto \pi_2 \ldots$ of reductions that follows the normalisation strategy of Definition 7.4. Due to Lemmas 7.2 and 7.2, we only need to show that we cannot have an infinite number of consecutive iterations of $\mathbb{BE}$ and $\mathbb{DE}$. Let c be the derivation complexity of the derivation π. In order to conclude that the normalisation strategy of Definition 7.4 induces a terminating procedure, we will use the following statements.

1. The reduction steps of $\mathbb{BE}$ do not increase c.

2. The reduction steps of $\mathbb{BE}$ only produce new detours and DNE permutations of complexity smaller than c.
3. At the end of $\mathbb{BE}$, all blocks are inert and thus have complexity 0.
4. The reduction steps of $\mathbb{DE}$ do not increase c.
5. At the end of $\mathbb{DE}$, the maximal detour complexity has strictly decreased with respect to the beginning of the phase.

Since Points 1 and 2 have already been proved in Lemmas 7.4 and 7.3, respectively, we show that the other points hold as well.

We begin with Point 3. Since during $\mathbb{BE}$, according to the normalisation strategy of Definition 7.4, we can reduce any non-detour redex of maximal complexity that does not contain any other non-detour redex of maximal complexity and since $\mathbb{BE}$ terminates in a finite number of permutation and reduction steps, we have that, at the end of $\mathbb{BE}$, no more non-detour redexes are in the derivation. If this is the case, all blocks are inert and thus have complexity 0.

Consider now Point 4. The reduction of the redex can only generate new redexes the complexity of which is determined by the depth of a formula which has depth strictly smaller than the complexity of the detour just eliminated. Moreover, since after $\mathbb{BE}$ we only have inert blocks–that is, blocks consisting of a C application only, of a $\equiv$ application only, or of a C application immediately followed by a $\equiv$ application—the maximal block complexity generated by an elimination of a detour of complexity $|\alpha|$ is $|\alpha| + 1$. Indeed, a detour elimination can only generate an active block by 1. placing a derivation on top of an hypothesis α_1 which is the first formula of an inert block $\alpha_1, \alpha_2, \alpha_3$ or 2. by placing a derivation on top of an hypothesis α_2, thus generating an inert block $\alpha_1, \alpha_2, \alpha_3$. In case 1, the rule deriving α_2 from α_1 is C and the complexity of the active block is $|\alpha_2| = |\alpha| + 1$. In case 2, $\alpha = \alpha_2$ and the complexity of the active block is at most $|\alpha| = |\alpha_2|$. Hence, $\mathbb{DE}$ reductions do not increase c.

Lastly, we prove Point 5. Since during $\mathbb{DE}$, according to the normalisation strategy of Definition 7.4, we can reduce any detour redex of maximal complexity that does not contain any other detour redex of maximal complexity and since the reduction of detour redexes can only generate detour redexes of smaller complexity, we have that, at the end of $\mathbb{DE}$, the maximal detour complexity has strictly decreased with respect to the beginning of the phase.

We can now, finally, show that the complexity c of a derivation strictly decreases after two iterations of $\mathbb{BE}$ and $\mathbb{DE}$. Consider the first iteration $\mathbb{BE}1$, $\mathbb{DE}1$ of $\mathbb{BE}$ and $\mathbb{DE}$ on a generic derivation after the preliminary phase of reduction of DNE redexes has been executed. By 3, iteration $\mathbb{BE}1$ makes all blocks inert and, by 1, the complexity c of the derivation does not increase. Iteration $\mathbb{DE}1$, by 5, reduces all maximal detours and, by 4, only produces active blocks of complexity at most c. Therefore, we now have detours of complexity at most $c - 1$ and blocks of complexity at most c. From now on, any pair of iterations $\mathbb{BE}n$, $\mathbb{DE}n$ will strictly decrease the complexity c. We can prove, indeed, that, if we consider any pair $\mathbb{BE}n$, $\mathbb{DE}n$ for $n > 1$, we have that $\mathbb{BE}n$ starts with a derivation of complexity which is at most $c - (n - 2)$ and $\mathbb{DE}n$ ends with a derivation of complexity which is at most $c - (n - 1)$. Iteration $\mathbb{BE}n$, by 3, reduces all maximal non-detour redexes and makes all blocks inert, and, by 2, only produces detours of complexity at most $c - (n - 1)$. Since detour redexes now have maximal complexity $c - (n - 1)$, iteration $\mathbb{DE}n$, by 4, only produces active blocks of complexity at most $c - (n - 1)$ and reduces all maximal detour redexes. Detours have now maximal complexity $c - n$ and non-detour redexes have maximal complexity $c - (n - 1)$. The complexity of the proof is then now $c - (n - 1) < c - (n - 2)$. Hence, there is a strict decrease in complexity. This proves that the normalisation strategy is indeed terminating. □

8 Analyticity

In order to prove that **R** is consistent and decidable, we define a notion of analyticity and show that normal derivations are analytic. More specifically, our notion of analyticity is a weakening of the usual *subformula property*, which, nonetheless, is strong enough to prove that **R** is consistent and that the set of theorems of **R** is decidable.

Definition 8.1 For any formula α, the set $\mathrm{Subf}(\alpha)$ of subformulas of α is inductively defined as follows.

- if $\alpha = p$, then $\mathrm{Subf}(\alpha) = \{p\}$;
- if $\alpha = \alpha_1 \supset \alpha_2$, then $\mathrm{Subf}(\alpha) = \{\alpha_1 \supset \alpha_2\} \cup \mathrm{Subf}(\alpha_1) \cup \mathrm{Subf}(\alpha_2)$;
- if $\alpha = \alpha_1 \rhd \alpha_2$, then $\mathrm{Subf}(\alpha) = \{\alpha_1 \rhd \alpha_2\} \cup \mathrm{Subf}(\alpha_1) \cup \mathrm{Subf}(\alpha_2)$.
- if $\alpha = \alpha_1 \wedge \alpha_2$, then $\mathrm{Subf}(\alpha) = \{\alpha_1 \wedge \alpha_2\} \cup \mathrm{Subf}(\alpha_1) \cup \mathrm{Subf}(\alpha_2)$;
- if $\alpha = \alpha_1 \vee \alpha_2$, then $\mathrm{Subf}(\alpha) = \{\alpha_1 \vee \alpha_2\} \cup \mathrm{Subf}(\alpha_1) \cup \mathrm{Subf}(\alpha_2)$;

Given any set of formulas Γ, we define

$$\mathrm{Subf}(\Gamma) = \bigcup_{\gamma \in \Gamma} \mathrm{Subf}(\gamma)$$

In order to show that normal derivations are analytic, we first need to prove a preliminary result concerning sequences of rule applications only consisting of eliminations and C rule applications. This result will be used to solve the problematic case of eliminations. Indeed, for an introduction we can easily show that, if the derivations of its premises are analytic, then also the derivation of its conclusion is analytic. Eliminations, instead, do not have this property. Consider the derivation

$$\dfrac{\overset{\pi}{\Gamma, \alpha \Rightarrow \beta}}{\Gamma \Rightarrow \alpha \supset \beta} \supset i$$

If one knows that each formula occurring in π complies with the subformula property with respect to the conclusion β and the hypotheses Γ, α, one also knows that each formula occurring in $\overset{\pi}{\Gamma, \alpha \Rightarrow \beta}$ complies with the subformula property with respect to the conclusion $\alpha \supset \beta$ and the hypotheses Γ. Indeed, all formulas in $\Gamma, \alpha \Rightarrow \beta$ are subformulas of some formula in $\Gamma \Rightarrow \alpha \supset \beta$. On the other hand, if we consider the derivation

$$\dfrac{\overset{\pi_1}{\Gamma \Rightarrow \alpha \supset \beta} \quad \overset{\pi_2}{\Delta \Rightarrow \alpha}}{\Gamma, \Delta \Rightarrow \beta} \supset e$$

nothing guarantees that the analyticity of π_1 and π_2 implies the analyticity of the whole derivation. For instance, there is no evidence that α is a subformula of some formula in $\Gamma, \Delta \Rightarrow \beta$. More generally, if some formula occurring in π_1 or π_2 is a subformula of α, we have no direct way of showing that it is a subformula of some formula in $\Gamma, \Delta \Rightarrow \beta$ as well. Hence, we use the following lemma.

Lemma 8.1 *If π is a normal form derivation with conclusion γ and hypotheses Γ, and the last rule applied in π is an elimination rule, then one of the following holds:*

(*a*) $\gamma \in \text{Subf}(\Gamma)$,
(*b*) $\gamma = {\sim}\delta$ *and* $\delta \in \text{Subf}(\Gamma)$.

Proof In a normal derivation, no rule can be applied above the major premise of an elimination rule for $\star$, for $\star \in \{\wedge\ \supset, \rhd\}$, but another elimination rule, a Id application or, in case $\star = \rhd$, a C application. Indeed, the conclusion of a EFQ or DNE application cannot be the major premise of an elimination rule. Moreover, if an introduction occurs immediately above an elimination and if an application of $\equiv$ or of Expl occurs immediately above an elimination for $\rhd$, we have a redex, and thus the derivation is not normal. Hence, it is enough to prove the following statement:

> A normal derivation of $\Gamma \Rightarrow \gamma$ that only contains applications of Id, elimination rules, and C, and that has an elimination rule as last rule applied verifies one of the conditions (*a*) and (*b*).

We can prove it by induction on the number of rules applied in the derivation. The base case—the one in which the derivation consists in one application of Id—is impossible because of our assumption that the last rule applied in the derivation is an elimination rule. We then suppose that this statement holds for all derivations containing less than *n* rule applications and show that it also holds for those that contain *n* rule applications. We reason by cases on the last rule applied.

- $\dfrac{\Gamma \Rightarrow \alpha \rhd \beta \quad \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \beta}\, \rhd e$ The rule applied in order to obtain the major premise of this rule application must either be Id, an elimination rule or C. If it is Id, the statement holds since $\alpha \rhd \beta \in \Gamma$. If it is an elimination rule, by inductive hypothesis, the statement holds for the derivation of the premise. Since $\alpha \rhd \beta$ is not of the form ${\sim}\delta$, we have that $\alpha \rhd \beta \in \text{Subf}(\Gamma)$. But then $\beta \in \text{Subf}(\Gamma)$ and the statement holds for our derivation of $\Gamma, \Delta \Rightarrow \beta$ as well. Suppose then that the last rule applied above the premise $\alpha \rhd \beta$ is C. Now, the rule applied above this application of C cannot be C again since otherwise we would have a C-C-reduction redex and the derivation would not be normal. Hence, we must either have an application of Id or of an elimination rule. If we have

$$\dfrac{\dfrac{\dfrac{}{\Gamma \Rightarrow \beta_1 \rhd \alpha_1}\,\text{Id}}{\Gamma \Rightarrow {\sim}\alpha_1 \rhd {\sim}\beta_1}\,\text{C} \qquad \Delta \Rightarrow {\sim}\alpha_1}{\Gamma, \Delta \Rightarrow {\sim}\beta_1}\,\rhd e$$

  where $\alpha = {\sim}\alpha_1$ and $\beta = {\sim}\beta_1$, then $\beta_1 \rhd \alpha_1 \in \Gamma$ and hence our conclusion β is indeed of the form ${\sim}\beta_1$ where $\beta_1 \in \text{Subf}(\Gamma)$.

If we have

$$\dfrac{\dfrac{\dfrac{\dots}{\Gamma \Rightarrow \beta_1 \rhd \alpha_1}\,\star e}{\Gamma \Rightarrow {\sim}\alpha_1 \rhd {\sim}\beta_1}\,\mathrm{C} \qquad \Delta \Rightarrow {\sim}\alpha_1}{\Gamma, \Delta \Rightarrow {\sim}\beta_1}\,\rhd e$$

where $\alpha = {\sim}\alpha_1$, $\beta = {\sim}\beta_1$ and $\star e$ is an elimination for some connective $\star$, then the inductive hypothesis holds for the derivation of $\Gamma \Rightarrow \beta_1 \rhd \alpha_1$. Since $\beta_1 \rhd \alpha_1$ is not of the form ${\sim}\delta$, we have that $\beta_1 \rhd \alpha_1 \in \mathrm{Subf}(\Gamma)$. But then, also in this case, our conclusion β is indeed of the form ${\sim}\beta_1$ where $\beta_1 \in \mathrm{Subf}(\Gamma)$.

- $\dfrac{\Gamma \Rightarrow \alpha \supset \beta \quad \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \beta} \supset e$ The rule applied in order to obtain the major premise of this rule application must either be Id or an elimination rule. If it is Id, the statement holds since $\alpha \supset \beta \in \Gamma$. If it is an elimination rule, by inductive hypothesis, the statement holds for the derivation of the premise. Since $\alpha \supset \beta$ is not of the form ${\sim}\delta$, we have that $\alpha \supset \beta \in \mathrm{Subf}(\Gamma)$. But then $\beta \in \mathrm{Subf}(\Gamma)$ and the statement holds for our derivation of $\Gamma, \Delta \Rightarrow \beta$ as well.
- $\dfrac{\Gamma \Rightarrow \alpha \wedge \beta}{\Gamma \Rightarrow \alpha} \wedge e$ The rule applied in order to obtain the major premise of this rule application must either be Id or an elimination rule. If it is Id, the statement holds since $\alpha \wedge \beta \in \Gamma$. If it is an elimination rule, by inductive hypothesis, the statement holds for the derivation of the premise. Since $\alpha \wedge \beta$ is not of the form ${\sim}\delta$, we have that $\alpha \wedge \beta \in \mathrm{Subf}(\Gamma)$. But then $\alpha \in \mathrm{Subf}(\Gamma)$ and the statement holds for our derivation of $\Gamma \Rightarrow \alpha$ as well.
- $\dfrac{\Gamma \Rightarrow \alpha \wedge \beta}{\Gamma \Rightarrow \beta} \wedge e$ This case is analogous to the previous one.

□

We can finally show that normal derivations enjoy a version of the subformula property. As already mentioned, this version of the property is weaker than the usual one but still strong enough to prove the consistency and the decidability of the calculus. The usual formulation of the subformula property requires any formula β occurring in a normal derivation of $\Gamma \Rightarrow \alpha$ to be either $\bot$ or a subformula of the conclusion α or of some hypothesis in Γ; that is, $\beta \in \mathrm{Subf}(\Gamma \cup \{\alpha\}) \cup \{\bot\}$. The version of this property for classical logic usually admits also the possibility that β is the negation—or the negation of the negation—of a subformula of the conclusion or of some hypothesis; that is, the case in which $\beta = {\sim}\gamma$—or $\beta = {\sim\sim}\gamma$—and $\gamma \in \mathrm{Subf}(\Gamma \cup \{\alpha\})$ is admissible too. For the calculus **R** we also need to admit the possibility that β is a subformula of the contrapositive of a $\rhd$-statement which is, in turn, a subformula of the conclusion or of some hypothesis. In formal terms, we need to admit the possibility that $\beta \in \mathrm{Subf}({\sim}\delta \rhd {\sim}\gamma)$ for some $\gamma \rhd \delta \in \mathrm{Subf}(\Gamma \cup \{\alpha\})$.[14]

Theorem 8.2 *If π is a normal form derivation with conclusion α and hypotheses Γ, then, for any formula β that occurs in π, at least one of the following holds:*

- $\beta \in \mathrm{Subf}(\Gamma \cup \{\alpha\}) \cup \{\bot\}$,
- *there is $p \in \mathrm{Subf}(\Gamma \cup \{\alpha\})$ such that $\beta \in \mathrm{Subf}({\sim\sim}p)$,*

[14] The fact that we can restrict the clause of Theorem 8.2 on double negations to the case of propositional variables is due to the fact that the DNE rule can only act on propositional variables.

- *there is* $\gamma \rhd \delta \in \mathrm{Subf}(\Gamma \cup \{\alpha\})$ *such that* $\beta \in \mathrm{Subf}(\sim\delta \rhd \sim\gamma)$.

Proof The proof is by induction on the number of rules applied in the derivation of $\Gamma \Rightarrow \alpha$.

In the base case, the derivation is

$$\pi \quad = \quad \dfrac{}{\Gamma \Rightarrow \alpha}\ \mathrm{Id}$$

and $\alpha \in \Gamma$. If this holds, the statement is trivially true.

Suppose now that the statement holds for all derivations that contain less than n rule applications. We show that it holds also for any derivation that contains n rule applications. Let us reason by cases on the last rule applied in the derivation. All cases concerning introductions only require an application of the inductive hypothesis to the premises of the rule application. All those concerning eliminations require an application of Lemma 8.1 followed by such an application of the inductive hypothesis. Hence, we simply exemplify these arguments by presenting the cases of $\rhd i$ and $\rhd e$. We discuss in detail, instead, the cases concerning rules other than introductions and eliminations.

- $\dfrac{\Gamma \Rightarrow \alpha \rhd \beta}{\Gamma \Rightarrow \sim\beta \rhd \sim\alpha}\ \mathrm{C}$ Since the derivation is in normal form, above the C rule application we cannot have any application of $\equiv$, C, $\rhd i$ or Expl. Otherwise, we would have a redex. Moreover, no introduction rule other than $\rhd i$ can have $\alpha \rhd \beta$ as conclusion. Therefore, we must either have an application of Id or an elimination above the C application.
  Suppose we have an application of Id above the application of C. We then know that $\alpha \rhd \beta \in \Gamma$ and, by inductive hypothesis, that the statement holds for the derivation of $\Gamma \Rightarrow \alpha \rhd \beta$. Since $\alpha \rhd \beta$ is an element of Γ, the statement holds for our derivation of $\Gamma \Rightarrow \sim\beta \rhd \sim\alpha$ as well.
  Suppose now that an elimination occurs above our C application. By Lemma 8.1, since $\alpha \rhd \beta$ is not of the form $\sim\delta$, we know that $\alpha \rhd \beta \in \mathrm{Subf}(\Gamma)$. Since, moreover, by inductive hypothesis, the statement holds for all formulas occurring in the derivation of $\Gamma \Rightarrow \alpha \rhd \beta$, we can conclude that the statement holds for our derivation of $\Gamma \Rightarrow \sim\beta \rhd \sim\alpha$ as well.
- $\dfrac{\Gamma \Rightarrow \alpha \rhd \beta \quad \alpha \Leftrightarrow \gamma \quad \beta \Leftrightarrow \delta}{\Gamma \Rightarrow \gamma \rhd \delta}\ \equiv$ Since the derivation is in normal form, above this rule application we cannot have any application of $\equiv$, $\rhd i$ or Expl. Otherwise, we would have a redex. Moreover, no introduction rule other than $\rhd i$ can have $\alpha \rhd \beta$ as conclusion. Therefore, we must have an application of Id, another application of C or an elimination above the major premise of the displayed $\equiv$ application.
  Suppose we have an application of Id above the major premise of $\equiv$. We then know that $\alpha \rhd \beta \in \Gamma$ and, by inductive hypothesis, that the statement holds for the derivation of $\Gamma \Rightarrow \alpha \rhd \beta$. Since $\alpha \rhd \beta$ is an element of Γ and since all subformulas of γ and δ are subformulas of $\gamma \rhd \delta$, the statement holds for our derivation of $\Gamma \Rightarrow \gamma \rhd \delta$ as well.

Suppose we have an application of C above the major premise of $\equiv$ and that $\alpha \rhd \beta = {\sim}\zeta \rhd {\sim}\xi$. By the argument presented for the previous case, we know that the premise $\xi \rhd \zeta$ of the C application is either an element of Γ or an element of $\text{Subf}(\Gamma)$. Hence, the formula ${\sim}\zeta \rhd {\sim}\xi = \alpha \rhd \beta$ complies with the statement of the theorem with respect to the derivation of $\Gamma \Rightarrow \gamma \rhd \delta$, as do all formulas occurring in the derivation of $\Gamma \Rightarrow \alpha \rhd \beta$ by the inductive hypothesis. Since, moreover, the inductive hypothesis guarantees us that the statement holds for all formulas occurring in the derivations of the minor premises of the $\equiv$ rule application, all subformulas of γ and δ are also subformulas of $\gamma \rhd \delta$, and all subformulas of α and β are also subformulas of $\alpha \rhd \beta$; we can conclude that the statement holds also for our derivation of $\Gamma \Rightarrow \gamma \rhd \delta$.

Suppose now that an elimination occurs above the major premise of our $\equiv$ application. By Lemma 8.1, since $\alpha \rhd \beta$ is not of the form ${\sim}\delta$, we know that $\alpha \rhd \beta \in \text{Subf}(\Gamma)$. Hence, by the inductive hypothesis, all formulas occurring in the derivation of $\Gamma \Rightarrow \alpha \rhd \beta$ comply with the statement of the theorem with respect to the derivation of $\Gamma \Rightarrow \gamma \rhd \delta$ as well. Since, moreover, the inductive hypothesis guarantees us that the statement holds for all formulas occurring in the derivations of the minor premises of the $\equiv$ rule application, all subformulas of γ and δ are also subformulas of $\gamma \rhd \delta$, and all subformulas of α and β are also subformulas of $\alpha \rhd \beta$; we can conclude that the statement holds also for our derivation of $\Gamma \Rightarrow \gamma \rhd \delta$.

- $\dfrac{\alpha \Rightarrow \beta}{\Rightarrow \alpha \rhd \beta}\ \rhd i$
  By inductive hypothesis, the statement holds for all formulas occurring in the derivation of the premise. Since all subformulas of α and β are also subformulas of $\alpha \rhd \beta$ and since ${\sim}\alpha \in \text{Subf}({\sim}\beta \rhd {\sim}\alpha)$ and $\alpha \rhd \beta \in \text{Subf}(\alpha \rhd \beta)$, the statement holds for our derivation of $\Rightarrow \alpha \rhd \beta$ as well.
- $\dfrac{\Gamma \Rightarrow \alpha \rhd \beta \quad \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \beta}\ \rhd e$
  By Lemma 8.1, since $\alpha \rhd \beta$ is not of the form ${\sim}\delta$, $\alpha \rhd \beta \in \text{Subf}(\Gamma \cup \Delta)$. Since, by inductive hypothesis, the statement holds for all formulas occurring in the derivations of the premises of $\rhd e$ and since all subformulas of α are also subformulas of $\alpha \rhd \beta$, the statement holds for our derivation of $\Gamma, \Delta \Rightarrow \beta$ as well.
- $\dfrac{\Gamma \Rightarrow \bot \quad \Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rhd \beta}$ Expl
  Since, by inductive hypothesis, the statement holds for all formulas occurring in the derivations of the premises of Expl, since all subformulas of α and β are also subformulas of $\alpha \rhd \beta$ and since $\bot = \bot$; the statement holds for our derivation of $\Gamma \Rightarrow \alpha \rhd \beta$ as well.
- $\dfrac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow p}$ EFQ Since $\bot = \bot$ and since there are no proper subformulas of $\bot$, the statement holds for our derivation of $\Gamma \Rightarrow p$.
- $\dfrac{\Gamma \Rightarrow {\sim}{\sim}p}{\Gamma \Rightarrow p}$ DNE Since, by inductive hypothesis, the statement holds for all formulas occurring in the derivation of $\Gamma \Rightarrow {\sim}{\sim}p$ and since ${\sim}{\sim}p \in \text{Subf}({\sim}{\sim}p)$ for $p \in \text{Subf}(\Gamma \cup \{p\})$, the statement holds for our derivation of $\Gamma \Rightarrow p$ as well.□

9 Consistency and Decidability

The results established in the previous sections can be used to show that **R** is consistent and decidable.

Theorem 9.1 *In* **R** *there is no derivation of* $\Rightarrow \bot$.

Proof Suppose that there is a derivation π of $\Rightarrow \bot$. By Theorem 7.5, there is a normal derivation ν of $\Rightarrow \bot$. By Theorem 8.2, for any formula β that occurs in ν, at least one of the following holds:

- $\beta \in \text{Subf}(\bot)$,
- there exists $p \in \text{Subf}(\bot)$ and $\beta \in \text{Subf}({\sim}{\sim}p)$,
- there exists $\gamma \triangleright \delta \in \text{Subf}(\bot)$ and $\beta \in \text{Subf}({\sim}\delta \triangleright {\sim}\gamma)$.

Since $\text{Subf}(\bot) = \{\bot\}$, only the first condition can hold. Hence, each formula β occurring in ν must be $\bot$. Let us then reason on the last rule applied in ν. This rule cannot be Id since the conclusion of Id cannot have an empty set to the left of $\Rightarrow$. It cannot be C, $\equiv$, Expl or any introduction rule either since these rules must have a complex formula as conclusion. Similarly, the last rule applied in ν cannot be EFQ or DNE. The conclusion of these rules, indeed, must be a propositional variable p. But the fact that any formula occurring in ν must be $\bot$ excludes also all elimination rules. Indeed, at least one premise of each application of one of these rules must be a complex formula. Since no rule can be applied to obtain the conclusion $\bot$ of ν without hypotheses, ν does not exist. Therefore, no derivation π of $\Rightarrow \bot$ exists. □

We finally prove that there is an effective method for deciding whether a sequent is derivable or not in **R**. This implies, in turn, that the set of theorems of the logic characterised by **R** is decidable. The proof employs a version of the technique used by Gentzen.[15]

Theorem 9.2 *For any finite set of formulas* Γ *and formula* α*, it is possible to establish in a finite number of steps whether* $\Gamma \Rightarrow \alpha$ *is derivable in* **R**.

Proof If there is a derivation of $\Gamma \Rightarrow \alpha$, by Theorems 7.5 and 8.2 there is a normal derivation of that only contains formulas β such that

- $\beta \in \text{Subf}(\Gamma \cup \{\alpha\})$,
- there exists $p \in \text{Subf}(\Gamma \cup \{\alpha\})$ and $\beta \in \text{Subf}({\sim}{\sim}p)$,
- there exists $\gamma \triangleright \delta \in \text{Subf}(\Gamma \cup \{\alpha\})$ and $\beta \in \text{Subf}({\sim}\delta \triangleright {\sim}\gamma)$.

Let us then consider the set of formulas S defined as follows

$$\begin{aligned} \text{Subf}(\Gamma \cup \{\alpha\}) \cup \{\bot\} \cup \\ \{{\sim}p \mid p \in \text{Subf}(\Gamma \cup \{\alpha\})\} \cup \\ \{{\sim}{\sim}p \mid p \in \text{Subf}(\Gamma \cup \{\alpha\})\} \cup \\ \{{\sim}\delta \triangleright {\sim}\gamma \mid \gamma \triangleright \delta \in \text{Subf}(\Gamma \cup \{\alpha\})\} \cup \\ \{{\sim}\delta \mid \gamma \triangleright \delta \in \text{Subf}(\Gamma \cup \{\alpha\})\} \cup \\ \{{\sim}\gamma \mid \gamma \triangleright \delta \in \text{Subf}(\Gamma \cup \{\alpha\})\} \end{aligned}$$

15 [17].

S is a finite set. This can be clearly evinced from the fact that each set in the union is finite, which is in turn due to the fact that Γ is finite. We also have that S contains all formulas that can occur in any normal derivation of $\Gamma \Rightarrow \alpha$. Indeed, for any formula of the form $\sim\sim p$, we have

$$\mathrm{Subf}(\sim\sim p) = \{\sim\sim p, \sim p, p, \bot\}$$

and for any formula of the form $\sim\delta \rhd \sim\gamma$, we have

$$\mathrm{Subf}(\sim\delta \rhd \sim\gamma) \subset \{\sim\delta \rhd \sim\gamma, \sim\delta, \sim\gamma, \bot\} \cup \mathrm{Subf}(\gamma \rhd \delta)$$

Let us then consider the set J defined as follows

$$\{\Delta \Rightarrow \beta \mid \Delta \cup \{\beta\} \subseteq S\}$$

This set J is clearly finite too. Indeed the finite sets Δ only containing elements of the finite set S are in finite number—the number of these sets is exactly the cardinality of the set $\mathcal{P}(S)$ of the parts of S: the cardinality of S to the power of 2. Thus, also the judgements $\Delta \Rightarrow \alpha$ are in finite number—these judgements can be simply counted by counting the ordered pairs $\langle \Delta, \alpha \rangle$ where $\Delta \in \mathcal{P}(S)$ and $\alpha \in S$.

Let us consider now each element of J and check if it is derivable by Id. If it is, let us mark it as derivable. Afterwards, let us then proceed as follows:

1. Consider each individual element of J which has been marked as derivable and try to apply the rules C, $\rhd i$, EFQ, DNE, $\supset i$, $\wedge e$ to it. Whenever it is possible to apply a rule and the conclusion of the rule application is an element of J, mark the conclusion as derivable.
2. Consider each pair of elements of J which have been marked as derivable and try to apply the rules $\rhd e$, $\supset e$, $\wedge i$ to them. Whenever it is possible to apply a rule and the conclusion of the rule application is an element of J, mark the conclusion as derivable.
3. Consider each triple of elements of J which have been marked as derivable and try to apply the rule Expl to them. Whenever it is possible to apply a rule and the conclusion of the rule application is an element of J, mark the conclusion as derivable.
4. Consider each quintuple of elements of J which have been marked as derivable and try to apply the rule $\equiv$ to them. Whenever it is possible to apply a rule and the conclusion of the rule application is an element of J, mark the conclusion as derivable.
5. If $\Gamma \Rightarrow \alpha$ has been marked as derivable, stop; if no new element of J has been marked by the last iteration of the tentatives in 1, 2, 3, and 4, stop; otherwise, start over from 1.

Now, each of these steps require a finite number of operations. Indeed, elements, pairs, triples and quintuples of elements of the finite set J are in finite number, and the rules of **R** are in finite number. Moreover, the procedure must terminate, either because $\Gamma \Rightarrow \alpha$ has been marked as derivable or because the last iteration of 1–4 did not mark any new judgement as derivable. If $\Gamma \Rightarrow \alpha$ has been marked, we know that $\Gamma \Rightarrow \alpha$ is derivable, and a backward study of a record of the successful attempts to apply rules to elements of J can give us a derivation of $\Gamma \Rightarrow \alpha$. If the last iteration of 1, 2, 3 and 4 did not mark any new judgement as derivable and $\Gamma \Rightarrow \alpha$ has not been marked, then we know that $\Gamma \Rightarrow \alpha$ is not derivable since no normal derivation containing $\Gamma \Rightarrow \alpha$ exists. □

Note, moreover, that also the fragment of $\triangleright$ that corresponds to grounding claims provable in **G** is decidable. Indeed, the characterisation given in Section 6.1 implies that, in order to list all grounds of a formula α it is enough to construct its syntactic tree T; construct the set of S all selection trees of T, which is of finite cardinality; construct the set B of all grounding bars of elements of S, which is finite as well since there is a finite number of bars for each finite tree. Since each step produces a finite set of objects, it is always possible to execute all these construction steps in finite time and obtain all grounds of a formula.

10 Final Remarks

In this paper we have outlined a formal account of the relation between reasons and grounds by developing the idea that the logic of grounds is part of the logic of reasons. We have defined two systems of natural deduction **R** and **G** which capture two sets of relatively uncontentious principles concerning reasons and grounds. Then we have shown that **G** can be embedded in **R** in the sense that for every claim about grounds provable in **G** there is a corresponding claim about reasons provable in **R**. Lastly, we have established some crucial metatheoretical results about **R** that we take to be interesting in their own right.

Although the results presented in the foregoing sections are far from providing an exhaustive treatment of the topic, they open a new line of investigation that may lead to further results. In particular, at least two distinct routes might be pursued. One is to address the question of how **R** can be extended so as to obtain a stronger logic of reasons while preserving important properties such as consistency and decidability. The other is to explore different notions of ground. While we focused on a rather standard logic of ground, it would be interesting to develop a similar line of thought with different notions, such as that suggested by Poggiolesi.[16] Indeed, the existing proof-theoretical results on this notion and the studies of its connections with other logics constitute a rather promising starting point for an investigation of the connections with the notion of reason.[17]

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16 [22, 23]

17 See [13, 14, 16].

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