On Strong and Weak Logics for Paraconsistent Computability

Zach Weber
Philosophy Programme, University of Otago, Dunedin, New Zealand
<zach.weber@otago.ac.nz>

Fernando Cano-Jorge*
Philosophy Programme, University of Otago, Dunedin, New Zealand
Department of Philosophy, University of Canterbury, Christchurch, New Zealand
Facultad de Filosofía, Universidad Panamericana, Ciudad de México, México
<fernando.cano-jorge@otago.ac.nz>

Abstract

One tradition in relevant and paraconsistent logics has been to develop systems intended for applications to arithmetic and computability theory. The aspiration, as in Meyer [38] and others, is to recover enough working mathematics for real computation, but without the limitative results of Turing, Gödel, etc.; or more cautiously, as in Dunn [22], to respect *relevance* and with that be insulated against the possibility of a genuine inconsistency. We distill these goals into GUIDING QUESTIONS, and study the options for logics within a range of relevant systems. We focus on strong truth functional logics **RM3** and **PAC** [6] and their expansions, with application to inconsistent arithmetics [62, 63]. We argue that this approach, while having many virtues, does not fully answer our guiding questions. This points to weak relevant logics like Routley/Sylvan's **DKQ** [54], Brady's **MCQ** [14], and Logan and Boccuni's $\mathbf{DL2Q}^{t,fc}$ [31]. The recurring theme is that paraconsistent computability struggles with *functionality* [17, 41, 43]. A method for advancing on the 'function problem' is sketched with Kleene's theorem as a worked example.

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1 Introduction: Logics Strong and Weak

For a long time, research in paraconsistency has sought a logic (or logics) for *computability*. Motivations range from wanting to compute better with inconsistent information, as in Belnap [7], or "wagering" carefully about the possibility of inconsistency as in Dunn [22], to the more speculative hopes of surpassing the limitative results of Turing, Gödel, etc., as in Meyer [38], Brady [11], and others.[1] This would be to recover enough working mathematics for 'real' computation, and to go beyond the classical Church–Turing barrier—and maybe, as in Routley/Sylvan and Priest [50], compute better with inconsistent truths.

Whatever the motivations, insofar as computation is about operations from numbers to numbers, in practice this project means looking for paraconsistent *arithmetic* (or arithmetics) that can support calculating but evade classical collapse. And this in turn means considering what happens with arithmetic under different paraconsistent logics.[2] Here, we consider a spectrum of suitable relevant and closely related logics, ranging from logics so strong they aren't relevant at all, like **RM3** and **PAC**, to logics so relevant that they are extremely weak, like **DK** and **DL**, and corresponding arithmetics. In the background, we hear Hilbert's quixotic question [28]: Is *ignorabimus* inevitable? Or, per impossible, could everything be *decidable*? Of course, this has been definitively answered classically in the negative, but the non-classicist may still wonder.[3]

Less dreamily, more operationally, around every corner lurks the *function problem for (inconsistent) paraconsistency*, most recently raised in [17], which is as follows. Computation is often glossed in terms of recursive functions, which do not sit well with inconsistency. For a recursive relation A one usually expects a computable function f such that

$$f_A(n) = \begin{cases} 1 & \text{if } A(n) \\ 0 & \text{if } \neg A(n). \end{cases}$$

[1]Asenjo explicitly connects his pioneering work to avoiding Gödel [4].

[2]One may wonder why we take this low-level approach to computability, i.e., looking at computability via arithmetic, instead of a high-level approach using some modal logic for computation, like dynamic logics. The latter assumes some form of imperative programming language in the background and some properties of algorithms which we do not take for granted in a paraconsistent setting. Since the different paraconsistent arithmetics we examine here are not as well known as classical or intuitionistic arithmetics, and since all vary in their properties, we think it is best at this stage of research to look at the most fundamental theory underlying computation, i.e., arithmetic, in order to examine the prospects of paraconsistent computability. For some other approaches in related areas, see [27, 18, 29].

[3]For discussion of paraconsistent computability, see also [32, 60, 61].

But if some A is inconsistent,

$$A(n) \wedge \neg A(n)$$

then this leads by transitivity to the absurdity $0 = f_A(n) = 1$, which is bad—cf. [41, 43].[4]

Since inconsistency does not agree well with functionality, it is tempting to consider doing without functions, if it were somehow possible (and see §5.4). But one (presumably) needs some working notion of function in order to calculate—that is, to compute. As Priest outlines the situation,

> First-order arithmetic could be formulated just as well with no function symbols, but ... the inconsistent models would have no interesting structure, as far as I can see. ... At the other extreme, we could formulate arithmetic with many more function symbols. This would make [finding any inconsistent models] much more difficult [48, p. 1528].

We look for ways this problem might be addressed, measured against standard results like the Rabin–Scott theorem about non-deterministic finite automata [52], and a related limiting result by Agudelo and Carnielli [1] for strong paraconsistent logics. At the end we will suggest a way that the apparent impasse can be turned to an advantage, namely, the problem can be used to differentiate computable *functions* from properly paraconsistent *relations*.

It turns out that the choice of logic, *per se*, in most cases makes little difference to the function problem. Rather it is the difference between approaches—cautiously *paraconsistent*, and ultimately conservative, or *inconsistent* and so ultimately revisionary—that marks the fork in the path.

2 Paraconsistent Arithmetic: Some Guiding Questions

The quest for a paraconsistent computability theory, specially one which is immune to some classical limitative results, goes hand in hand with the quest for an appropriate theory of arithmetic. For if arithmetic itself is seriously limited, it is only natural that this in turn restricts the class of computable relations. We should then elaborate what an appropriate theory of arithmetic means in this context.

Gödel's incompleteness theorems stand as the most important limitative results about arithmetical theories. But these come with various assumptions, most notably the negation consistency of the theory. It is then natural to ask, as e.g., Meyer [35]

[4] A referee suggests defining $f_A(n) = 0$ when $A(n)$ is untrue, rather than false (thus distinguishing a true negation (falsity) from not being true). This points to larger questions, about the logic of the metatheory and relatedly the *just true* problem for paraconsistent logics—see [44], and further discussion in §4.4 below.

and Priest [47] do, whether a paraconsistent logic, in being able to accommodate possible inconsistencies, would sidestep Gödel's results. For, indeed, we have that:

G1. If a theory of arithmetic is negation consistent and can represent all computable relations, then that theory is incomplete.

G2. Moreover, such a theory cannot prove its own (absolute) consistency.

While a paraconsistent theory of computability will certainly demand a theory of arithmetic that can represent all computable relations, it will not in general assume that arithmetic is negation consistent.[5] The whole idea behind going paraconsistent in the underlying logic of our theory is to be prepared to handle inconsistencies without triviality. So natural GUIDING QUESTIONS to ask, reasoning in the other direction, are:

Q1. Is there a theory of arithmetic T that is negation inconsistent, non-trivial, and capable of representing all computable relations, such that T is complete?

Q2. Can such T prove its own absolute consistency, i.e., its non-triviality?

While Gödel's theorems decisively show that Hilbert's dream of a consistent and complete system of mathematics, expressively rich enough to show its own consistency through finitary means, has no hope whatsoever, one may still ask whether features like completeness and non-triviality can be had by going inconsistent. Notably, **G1** says that if a theory of arithmetic is complete and can represent all computable relations, then the theory is inconsistent.

And a similar line of thought can be followed regarding decidability—another salient goal of Hilbert's programme. Leaning on Turing's work, we have:

T1. If a theory of arithmetic is negation consistent, and can represent all computable relations, and is decidable, then the halting problem is effectively solvable.

T2. The halting problem is effectively unsolvable.

But the argument for **T2** famously rests on the inadmissibility of contradictions in the underlying logic of our theory. So again, reasoning dually, we ask:

[5] According to tradition, if T is a consistent theory in the language of arithmetic, we say that a k-place function f is *representable* in T if there is a formula $A(x_1, \ldots, x_k, y)$ such that, whenever $f(a_1, \ldots, a_k) = b$, then $\vdash_T \forall y(A(\overline{a_1}, \ldots, \overline{a_k}, y) \leftrightarrow y = \overline{b})$—where the $\overline{a_i}$ and $\overline{b}$ are numerals standing for the a_i and b, and $\leftrightarrow$ is a biconditional to be specified by choice of logic. Similarly, a k-place relation R is representable in T if there is a formula $A(x_1, \ldots, x_k)$ such that, whenever $R(a_1, \ldots, a_k)$ holds, $\vdash_T A(\overline{a_1}, \ldots, \overline{a_k})$. See [58, Ch. 12].

Q3. Is there a theory T as in Q1–Q2 where it can be shown that the halting function is computable? And if so, is T decidable then?

Certainly, these considerations would require that the provability relation of arithmetic is itself a representable, computable relation in the theory. We know that self-reference by Gödelian constructions gives arithmetic the possibility of talking about itself. But this is related to another important limitative result, due to Löb (see §5.1 below):

L1. If a theory of arithmetic can prove its own soundness, then it is trivial.

A theory T answering to the GUIDING QUESTIONS we have been considering so far should indeed be a sound theory and, moreover, be capable of expressing such a fact within itself through finitary means [49, Ch. 3, 17]. While the limitative results of Gödel and Turing directly depend on the negation consistency of a theory of arithmetic, Löb's theorem, like Curry's paradox, crucially depends (given other assumptions) on a different feature of the logic underlying our theory of arithmetic, namely, that Contraction,

$$\vdash A \to (A \to B) \;\therefore\; \vdash A \to B$$

is logically valid. So we ask:

Q4. Is there a theory T as in Q1–Q3 whose underlying logic is not only paraconsistent but also Contraction-free? If so, can T prove its own soundness without this leading to triviality?

These GUIDING QUESTIONS make it look like we are after a non-trivial theory of arithmetic that represents all computable relations and which is negation inconsistent, complete, decidable and capable of proving its own soundness. But such a theory might only be a mythological creature, and indeed the arithmetics surveyed in this paper show that no investigation has yet found it. The lesson of limitative results is that *something* has to give; we will see that there are pros and cons for different candidate paraconsistent arithmetics.

These pros and cons hinge largely on the logic that underlies a given arithmetic. Indeed, this logic must be paraconsistent and perhaps even Contraction-free, but at the same time it must also be strong enough to carry out valid reasoning in arithmetical proofs and describe the structure of the number line; the arithmetic must be, in some sense, recognizable as *arithmetic*.[6] This might make it impossible to have all these features in a single system. This is why we will analyze systems of logic which we deem strong enough to fulfill some or all of the properties involved in Q1–Q4. This is to ask a final, more practical GUIDING QUESTION:

Q5. Is there a theory T as in Q1–Q4 that is 'sufficiently rich' to carry out calculation and mathematical reasoning?

Much as constructive mathematics differs from classical in the allowable methods, and leads to differences in which proofs (and so theorems) are valid, so here we note where e.g., reductio is available or not, and the effects this may have on what is deemed uncomputable.

In looking at applications to arithmetic and computability, it would be too quick to think that these goals are achievable as a result of the choice of logic alone. More is needed. When we consider the requisite that all computable relations are representable in the theory, we must have some answer to the question 'What is a computable relation in a paraconsistent setting?', and the answer to this question will necessarily depart from the standard, classical response insofar as contradictions must be accommodated in the theory—as shown by the function problem. To illustrate this point, by the end of the paper we will sketch a start at redesigning concepts and methods from computability theory so as to escape the function problem and glimpse how a paraconsistent computability theory might take shape.

3 Relevant Arithmetics

3.1 $\mathbf{R}^\sharp$ and Some of its Extensions

A nice place to begin our expedition is first-order relevant logic **RQ**. Not only is **RQ** one of the strong systems in the spectrum of relevant logics—which are paraconsistent—but it is also considered to be THE logic of relevance by Anderson and Belnap. Moreover, Meyer developed relevant arithmetic $\mathbf{R}^\sharp$ [37] using this system and made enough progress in his investigations to give some answers to our QUESTIONS Q1 and Q2. The axiomatization of **RQ** and a number of relevant logics that will be used throughout this paper (like **RM** and **RM3**) can be found in Appendix 1; cf. §3.3, 4.2, Appendix 2.

[6] In the context of this paper, this somewhat informal and imprecise condition needs to be understood without a classical bias. For example, if one asserts that anything worth calling arithmetic must be undecidable, this would clearly be begging the question. The constraint is rather that if *all* we wanted was decidability, a one-element trivial structure would suffice. What more *is* needed, is an open philosophical question; see [33].

The relevant arithmetics $\mathbf{R}^\sharp$ and $\mathbf{RM}^\sharp$, formulated in the standard arithmetical language $\mathscr{L}$, are obtained by adding the following arithmetical axioms and rules to **RQ** and **RMQ**, respectively:

(A1) $\forall x\,\forall y\,(x = y \leftrightarrow x' = y')$

(A2) $\forall x\,\forall y\,\forall z\,(x = y \rightarrow (x = z \rightarrow y = z))$

(A3) $\forall x\,(x' \neq 0)$

(A4) $\forall x\,(x + 0 = x)$

(A5) $\forall x\,\forall y\,(x + y' = (x + y)')$

(A6) $\forall x\,(x \times 0 = 0)$

(A7) $\forall x\,\forall y\,(x \times y' = (x \times y) + x)$

(AR) $\vdash P0, \vdash \forall x\,(Px \rightarrow Px') \ \therefore\ \vdash \forall x\,(Px)$

$\mathbf{R}^{\sharp\sharp}$ and $\mathbf{RM}^{\sharp\sharp}$ are obtained by respectively adding to $\mathbf{R}^\sharp$ and $\mathbf{RM}^\sharp$ the following infinitary rule:

(Ω) $\vdash P0, \vdash P0', \ldots, \vdash P0^{(n)} \ \therefore\ \vdash \forall x\,(Px)$

Meyer was able to show that, though $\mathbf{R}^\sharp$ is not negation complete, since e.g., neither $0 = 2 \rightarrow 0 = 1$ nor $\neg(0 = 2 \rightarrow 0 = 1)$ are provable in $\mathbf{R}^\sharp$, it is however absolutely consistent, i.e., non-trivial, since $0 = 1$ is not provable in $\mathbf{R}^\sharp$ and this can be shown inside $\mathbf{R}^\sharp$ itself through finitistic means using **RM3** matrices [38]. Here we have a paraconsistent arithmetic that escapes one of Gödel's results and which resembles classical Peano arithmetic quite closely insofar as $\mathbf{R}^\sharp$ can represent all computable relations and is also undecidable.

Regarding negation consistency, Meyer tried to show that rule γ, i.e., $\vdash A, \vdash \neg A \vee B \ \therefore\ \vdash B$, was admissible in $\mathbf{R}^\sharp$ —in analogy with proofs of the admissibility of Cut in Gentzen systems. He was able to show that there is an exact translation of the theorems of $\mathbf{P}^\sharp$ (classical Peano arithmetic) into $\mathbf{R}^\sharp$, assuming that γ is admissible in $\mathbf{R}^\sharp$. A finitary proof of the admissibility of γ not only would mean that $\mathbf{R}^\sharp$ was negation consistent, but also that so was $\mathbf{P}^\sharp$—but by Gödel's second incompleteness theorem, no such proof was to be found.

In fact, Meyer thought that $\mathbf{R}^\sharp$ was Peano complete, i.e., that $\mathbf{P}^\sharp \subseteq \mathbf{R}^\sharp$—see remarks in [40, p. 917], for instance. However, this is not the case. Friedman and Meyer [26] showed that there is a strictly positive (negation-free) theorem, the quadratic residue formula, which is not provable in $\mathbf{R}^\sharp$ though it is, of course, provable in $\mathbf{P}^\sharp$. This also means that γ is not admissible in $\mathbf{R}^\sharp$. Perhaps this is quite surprising, given the fact that both the logic **RQ** and the arithmetic $\mathbf{R}^{\sharp\sharp}$ admit γ [39], [36]. Hence, Friedman and Meyer invite us to find out whether there is such a thing as $\mathbf{R}^{\sharp 1/2}$, a system between $\mathbf{R}^\sharp$ and $\mathbf{R}^{\sharp\sharp}$—which is as of yet unknown; see [30] for a recent approach.

3.2 Relevant Robinson's Arithmetic

One of the difficulties in proving that γ is admissible in $\mathbf{R}^\sharp$ was the induction schema. This lead Dunn to explore Robinson arithmetic $\mathbf{Q_R}$ based on **RQ** in [22]. Recall that Robinson arithmetic is given by the Peano axioms without the induction schema (AR), plus the axiom

$$\forall x\,(x \neq 0 \to \exists y\,(x = y'))$$

(phrased this way rather than the more usual $\forall x\,(x = 0 \vee \exists y\,(x = y'))$ so as to avoid need of disjunctive syllogism).

Momentarily happy with his finding a proof of the admissibility of γ for $\mathbf{Q_R}$, Dunn soon realized that, for any formula A, $\vdash_{\mathbf{Q_R}} A$ iff $\vdash_{\mathbf{Q}} A$, i.e., that relevant Robinson arithmetic collapses with classical Robinson arithmetic —which makes the admissibility of γ a trivial fact, for γ, or material detachment, is surely admissible in **Q**. Hence, $\mathbf{Q_R}$ is a sound theory that can represent all recursive relations, whence it is incomplete, undecidable and cannot prove its own consistency.

Oddly enough, this collapse only happens when arithmetic includes zero. Relevant Robinson arithmetic on the *positive* numbers, denoted $\mathbf{Q_R}(1)$, is different from its classical counterpart $\mathbf{Q}(1)$ since $\nvdash_{\mathbf{Q_R}(1)} \forall x\,\forall y\,\forall z\,(x = y \to z = z)$ [22, §3]. Moreover, neither $\mathbf{Q_R}(1) \subseteq \mathbf{Q_R}$ nor $\mathbf{Q_R} \subseteq \mathbf{R}^\sharp$.

3.3 An Interlude on RMQ

The logic **RM** is **R** plus the Mingle axiom, $A \to (A \to A)$. This axiom was introduced by McCall in the form $(A \to B) \to ((A \to B) \to (A \to B))$; and it is equivalent to the converse of Contraction, $(A \to B) \to (A \to (A \to B))$.[7] **RM** is quasi-relevant: if $\vdash_{\mathbf{RM}} A \to B$ then either A and B share a variable, or else $\vdash_{\mathbf{RM}} \neg A$ and $\vdash_{\mathbf{RM}} B$; see [53].

But as Anderson and Belnap notice, the relevance of **RM** is highly dubious, since Linearity (a.k.a. the Chain Property), $(A \to B) \vee (B \to A)$, is provable in

[7] According to Dunn, McCall suggested the mingle axiom, $A \to (A \to A)$, to be added to **E**, and Anderson and Belnap misattribute their own conjecture about restricted mingle $(A \to B) \to ((A \to B) \to (A \to B))$ (giving it to McCall) in [2]. See [10] for a detailed discussion.

this logic (see [2, p. 429] and Appendix 2). Standefer [59] argues that, to the letter, this does not violate variable sharing and so is satisfactory from a purely relevantist standpoint. Perhaps so, but it remains highly counterintuitive and certainly not in the spirit of relevant implication.

Ideological objections aside, **RM** is paraconsistent and is not without its charms. In a 'consumer's report' style note, Dunn lists nice features like decidability [23].[8] The present note can be read as another report on related logics, specifically trialed on paraconsistent arithmetic and computability theory. The verdict is mixed.

One possible risk of using **RM** is Safety, $(A \wedge \neg A) \rightarrow (B \vee \neg B)$, which is derivable from the **RM**-theorem $\neg(A \rightarrow A) \rightarrow (B \rightarrow B)$. Tedder [64] warns against Safety, in the following terms. A *regular* theory has every theorem of **RM** as a member; a *funky* theory has the negation of a theorem of **RM** as a member. Because of Safety, all funky theories are regular, and all non-regular theories are consistent. Thus,

> **RM** dictates that a theory may be about something other than logic, or it may be inconsistent, but it may not be both. But why think that the theory of, say, an inconsistent model of arithmetic must also contain the complete theory of logic?

This is a philosophical objection to Safety, which perhaps could be brushed aside—what harm would there be in an inconsistent model also containing the complete theory of logic?—but does suggest that, at the same time as proving too much, these strong logics also are too constrained.

It is to the pros and cons of such logics for arithmetic that we now turn.

4 Arithmetic in Maximally Paraconsistent Logics

When it comes to strong logics for paraconsistent computability theory, there is nothing stronger than *maximally* paraconsistent logics. Avron and collaborators have advanced and refined this concept throughout several papers, e.g., [6, 3]. Most recently, maximal paraconsistency is taken to be the requirement from a paraconsistent logic **L** to retain as much of classical logic as possible, while still allowing non-trivial inconsistent theories. This in turn can be interpreted in two ways: (1) as *absolute* maximal paraconsistency, which means that any extension of **L** (without changing the language) is not paraconsistent; and (2) as maximal *relative to classical logic*, meaning that any further extension of **L** is exactly classical logic.

Some notable examples of maximally paraconsistent logics (in the absolute sense) are Sette's $\mathbf{P}^1$, Priest's **LP**, D'Ottaviano and da Costa's **J3**, all the **LFI** systems of

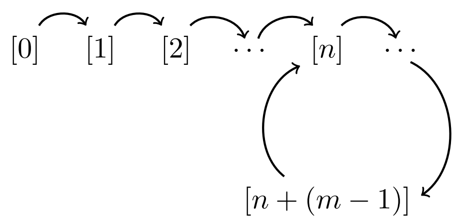
[8] Dunn [20] hoped that **R** would be the intersection of the **RM** systems. But **R** is not decidable [66], so this didn't work. Besides, the **RM** systems are not extensions of **RM**; rather they are intermediate logics between **R** and **RM**.

Carnielli and Marcos, Avron's **PAC** (also known as Baten's **CLuNs**), and Anderson and Belnap's **RM3**. Not all of these logics have been used as a basis for arithmetic. In what follows we will look at some attempts in this direction, with a particular attention to inconsistent models of arithmetic.

4.1 Inconsistent Models of Arithmetic

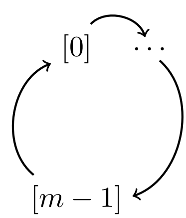
Paraconsistent arithmetics have been studied mostly in terms of their models. We assume the reader is familiar with these ideas and just mention a few memory joggers.

Dunn [21] initiated a general way of obtaining a three-valued structure from a two-valued classical one. Models for inconsistent arithmetic can be *heap* models, as in Priest [48],



In such a model, there is a consistent initial segment followed by a least inconsistent number, which then is and is not identical to all numbers after it. Such models are amenable to arithmetic based on the logic **LP** [47], as Priest has explored in detail.

Or models can be *cyclic*, as in Meyer and Mortensen [40],



This can be worked with as modular arithmetic—where being congruent modulo m is interpreted as identity. Such models are amenable to arithmetic based on **RM3**.

Both types of models are, notably, finite.[9] This makes properties like decidability much more tractable—Hilbert's call for a decision procedure by finitary methods taken very literally! Let us look at a couple of examples illustrating how inconsistent arithmetic looks in each kind of model.

[9] Though not necessarily; see [46]. The general structure of these models is now well understood. See [48, p. 1523], [46].

4.2 Arithmetic Based on RM3

As mentioned in Appendix 1, **RM3** can be obtained from **RM** by adding to it $A \vee (A \to B)$ as an axiom. In contrast with **RM** and its sublogics, **RM3** has a characteristic matrix:

$\neg$	
t	f
b	b
f	t

$\wedge$	t	b	f
t	t	b	f
b	b	b	f
f	f	f	f

$\vee$	t	b	f
t	t	t	t
b	t	b	b
f	t	b	f

$\to_{\mathbf{RM3}}$	t	b	f
t	t	f	f
b	t	b	f
f	t	t	t

The designated values are t and b, and logical consequence preserves designated values from premises to conclusion. Of course, having a characteristic matrix means that **RM3** is not a relevant logic. Nevertheless, **RM3** serves as a 'laboratory of relevance' when it comes to relevant arithmetic.

In [40] we find a general model-theoretic construction for several relevant arithmetics which are extensions of $\mathbf{R}^\sharp$ by taking modular arithmetics (of modulus $i \geqslant 2$) with **RM3** as their base logic. An *$RM3^i$-model* is an ordered pair $\langle D^i, I\rangle$, where D^i are the integers modulo i, and I is a function which assigns to the terms, operators and formulas of $\mathscr{L}$ the following values:

- $I(x) \in D^i$ for individual variables x;
- $I(0) = 0$;
- $I(+)$, $I(\times)$, $I(')$ are the operations $+$, $\times$, $'$ of arithmetic modulo i, respectively;
- $I(t_1 + t_2) = I(+)(I(t_1), I(t_2))$, $I(t_1 \times t_2) = I(\times)(I(t_1), I(t_2))$, and $I(t_1') = I(')(I(t_1))$ for any terms t_1 and t_2;
- I assigns the values t, b or f to open or closed formulas as follows:
  - For atomic formulas $t_1 = t_2$, where t_1 and t_2 are terms, $I(t_1 = t_2) = \mathsf{b}$ iff $I(t_1) = I(t_2) \pmod i$; otherwise $I(t_1 = t_2) = \mathsf{f}$.
  - $I(\neg A)$, $I(A \wedge B)$, $I(A \vee B)$ and $I(A \to B)$ are determined by the tables for **RM3** given above.
  - $I(\forall x\, A) = \mathrm{glb}\{y \colon I^*(A) = y\}$ for every x-variant I^* of I.
  - A sentence A is *$RM3^i$-true under interpretation I* iff $I(A)$ is designated; A is *true in the $RM3^i$-model* iff A is $RM3^i$-true under all interpretations I. The *arithmetic* $\mathbf{RM3}^i$ is the set of sentences true in the $RM3^i$-model.

Remark. The $\mathbf{RM3}^i$ can themselves be considered arithmetics, not just models of arithmetic, due to the following considerations. The $\mathbf{RM3}^i$ are axiomatized by adding to $\mathbf{RM}^\sharp$ the following axioms: $0 = 0^{(i)}$ and, for every j such that $2 \leqslant j < i$, $0 = 0^{(j)} \leftrightarrow 0 = 0'$. This arithmetic is called $\mathbf{RM3}^{i\sharp}$ and, as Meyer and Mortensen showed [40, Prop. 8], the theorems of $\mathbf{RM3}^{i\sharp}$ are exactly the truths of $\mathbf{RM3}^i$.

It follows immediately that the $\mathbf{RM3}^i$ are inconsistent, since, for any modulus $i \geqslant 2$ and any term t, $I(t = t) = \mathsf{b} = I(t \neq t)$. Moreover, the $\mathbf{RM3}^i$ are absolutely consistent [40, Prop. 2], given that $0 = 1$ is not provable in any of them—consider, for instance $i = 2$. And since **RQ** is contained in **RM3Q**, all $RM3^i$-models are models of $\mathbf{R}^\sharp$, from which the absolute consistency of $\mathbf{R}^\sharp$ also follows [40, Prop. 1]. In fact, $\mathbf{RM3}^i$ also contains $\mathbf{R}^{\sharp\sharp}$, $\mathbf{RM}^\sharp$ and $\mathbf{RM}^{\sharp\sharp}$ [40, Prop. 5].

With respect to our guiding questions, other interesting properties of the $\mathbf{RM3}^i$ are completeness and primeness [40, Prop. 6], decidability [40, Prop. 7], and ω-completeness and ω-inconsistency [40, Props. 2, 4]. Thus, $\mathbf{RM3}^i$ provides an example of an inconsistent arithmetic which answers positively to Q1 and Q2. Regarding Q3, it is unknown whether the halting function is a representable, computable relation, though $\mathbf{RM3}^i$ certainly represents all primitive recursive functions (since $\mathbf{R}^\sharp$ does) and is, in fact, a decidable theory of arithmetic. Alas, the conditions in Q4 are not met, since Contraction is valid in **RM3**, which we will complain about in §5.1.

Contraction aside, all this sounds really nice, given the guiding questions we have raised. For the relevant logician, though, the $\mathbf{RM3}^i$ come with some bad news as well: Relevance is lost. Indeed, some irrelevant implications, like Safety, $(A \wedge \neg A) \rightarrow (B \vee \neg B)$, are valid in **RM3**. Moreover, being a finitely valued logic, Dugundji formulas [2, p. 426] hold. For a three valued logic, for any four A_0, A_1, A_2, A_3,

$$\vdash_{\mathbf{RM3}} \bigvee_{0 \leqslant j < k \leqslant 3} A_j \leftrightarrow A_k$$

This shows that desirable properties of paraconsistent arithmetics, as displayed throughout Q1–Q4, are partially attainable but logical relevance might be lost in the process of securing these.

There is a worry, too, that paraconsistency itself is at risk in such a setup. The strength of the **RM3** conditional, plus the expressive resources of arithmetic, return something very close to classical negation. If one defines $\perp$ as $0 = 1$, then ${\sim}A \overset{df}{=} A \rightarrow \perp$ is a definable *explosive* negation, $A, {\sim}A \vdash_{\mathbf{RM3}^i} B$ and, by the instance $A \vee (A \rightarrow \perp)$ of the characteristic **RM3** axiom $A \vee (A \rightarrow B)$, this negation is also *exhaustive*, i.e., $A \vee {\sim}A$ for all A. This is not classical negation, since ${\sim}{\sim}A \rightarrow A$ fails when $I(A) = \mathsf{b}$, but exclusion and exhaustion properties certainly give this negation a classical flavor, and would be sufficient for a strengthened liar $L \leftrightarrow {\sim}L$ to do damage. See §4.4.

And regarding computability theory, the function problem is not avoided by $\mathbf{RM3}^i$, since any characteristic function $\chi_A(x)$ of some set A, defined through cases of the form $x = c$ and $x \neq c$ (for some constant c in the domain of χ_A), will lead to $0 = 1$ given that $c = c$ and $c \neq c$ both hold in $\mathbf{RM3}^i$. What this shows, as indicated before, is that a paraconsistent computability theory defined over an inconsistent arithmetic needs to depart from standard methods and definitions of classical computability theory on pain of triviality. We return to this issue in §6. So arithmetic in **RM3** has significant drawbacks.

4.3 Arithmetic Based on PAC

Avron's **PAC** [5], [6, §3.2–§3.2.2] has exactly the same matrix as **RM3**, except when it comes to its conditional:

$\to_{\mathbf{PAC}}$	t	b	f
t	t	b	f
b	t	b	f
f	t	t	t

Whereas $\to_{\mathbf{RM3}}$ contraposes and does not weaken, $\to_{\mathbf{PAC}}$ weakens but does not contrapose. The $\to_{\mathbf{PAC}}$ conditional has the nice property for the logic **LP** [47] that $\Gamma, A \vDash_{\mathbf{LP}} B$ iff $\Gamma \vDash_{\mathbf{PAC}} A \to_{\mathbf{PAC}} B$. Interestingly, $\to_{\mathbf{RM3}}$ is definable in **PAC** as

$$A \to_{\mathbf{RM3}} B \stackrel{df}{=} (A \to_{\mathbf{PAC}} B) \wedge (\neg B \to_{\mathbf{PAC}} \neg A)$$

and $\to_{\mathbf{PAC}}$ is definable in **RM3** as

$$A \to_{\mathbf{PAC}} B \stackrel{df}{=} (A \to_{\mathbf{RM3}} B) \vee B$$

Hence, **PAC** and **RM3** are definitionally equivalent systems.

In [62], Tedder axiomatizes Peano arithmetic in **PAC**—he calls this logic **A3** and the arithmetic based on it is dubbed there $\mathbf{A3}^\sharp$. Just like $\mathbf{R}^\sharp$, $\mathbf{A3}^\sharp$ is properly contained in $\mathbf{P}^\sharp$ [62, p. 534]. Unsurprisingly, given the definitional equivalence of **PAC** and **RM3**, $\mathbf{A3}^\sharp$ has finite models which are non-trivial, inconsistent and decidable; moreover, these properties hold good when one uses either cyclic or heap models along with either **PAC** or **RM3** [63, Prop. 2.11].

Tedder [63] also studies two versions of Robinson arithmetic in **PAC**, with a focus on decidability. There are many details, e.g., about which sort of models (cyclic or heap) are under consideration. Let us just give a flavor of the results.

1. Using **PAC**, the characteristic axiom of Robinson arithmetic, $\forall x\,(x \neq 0 \to \exists y\,(x = y'))$, i.e., that every number is either zero or a successor, is satisfiable in cycliс models. However,

   $$(*) \qquad \forall x\,(x \leqslant n \to (x = 0 \vee \cdots \vee x = n))$$

   is *not*. Tedder shows that **Q** in **PAC** + $(*)$, *plus* $(*)$'s contrapositive, is undecidable. The contrapositive must be added by hand, so to speak, since $\to_{\mathbf{PAC}}$ does not contrapose.

2. Analogously, **Q** in **RM3** is undecidable if we add special instances of the weakening axiom

   $$(**) \qquad A(n) \to (n \leqslant m \to A(n))$$

   where A is any Δ_0 formula. Here we are adding infinitely many axioms, since A is schematic.

The undecidability of such theories might actually be an attractive feature. If some arithmetics in **RM3**/**PAC** are undecidable (or close to it), that might be a good thing since the 'arithmetic' described is more likely to be close to the conventional picture. However, the choice between cyclic or heap models might come with its own problems. Consider the successor axiom

$$(***) \qquad x' = y' \to x = y$$

This states that the successor function is injective, but we might regard it as saying that the successor function is an injection from $\mathbb{N}$ to a proper subset of itself, which is Dedekind's definition of being infinite. This looks problematic for heap models. Let $\mathfrak{n}$ be the least number such that $\mathfrak{n} = \mathfrak{n}'$. Then for the $m < \mathfrak{n}$ such that $m' = \mathfrak{n}$, we have $m' = \mathfrak{n}'$, so by $(***)$ $m = \mathfrak{n}$. This means the initial segment of the model that is supposed to be before the collapse nevertheless gets 'sucked in', so to speak, leading to the inconsistent identification of all numbers. This seems bad. And it does not even help to rephrase and say that the axiom only applies to numbers before $\mathfrak{n}$, i.e.,

$$(x < \mathfrak{n} \wedge y < \mathfrak{n}) \to (x' = y' \to x = y)$$

because $\mathfrak{n} < \mathfrak{n}$, so the antecedent is satisfied by $\mathfrak{n}$! This is an echo of the function problem in the order relation, in one of the simplest recursive notions.

4.4 Consistency Operators and the LFIs

Logics of formal inconsistency, like **LFI1** [16], can be thought of as **LP** plus a primitive connective $\circ$, a consistency operator, with the truth table

A	$\circ A$
t	t
b	f
f	t

With $\neg$ as **LP** negation, one can then define a second operator, $\bullet$, which is a dual 'non-classicality' or *inconsistency* marker, $\bullet A \overset{df}{=} \neg\circ A$. Indeed, adding $\circ$ to the language is very powerful; one may further define classical negation as $\sim A \overset{df}{=} \neg A \wedge \circ A$, a bottom constant $\bot \overset{df}{=} A \wedge \sim A$, and detachable conditional $A \rightarrow_{\mathbf{LFI1}} B \overset{df}{=} \sim A \vee B$. Crucially, the $\circ$ and $\bullet$ operators are mutually exclusive ($\circ A, \bullet A \vDash B$) and cannot themselves be subject to non-classicality [45].

Having (effectively) full classical logic, especially a consistency operator, might help with the function problem. Coming back to the $\mathfrak{n}$ conundrum, if a consistency operator is at hand, we may have

$$(x < \mathfrak{n} \wedge y < \mathfrak{n} \wedge \circ(x < \mathfrak{n}) \wedge \circ(y < \mathfrak{n})) \rightarrow (x' = y' \rightarrow x = y)$$

In general, if $\mathbb{N}^\circ$ is the 'consistent' fragment of arithmetic, maybe we can work with functions in it. Problematic collapses may be averted by restricting to consistent cases, as in:

$$f_A(n) = \begin{cases} 1 & \text{if } A(n) \wedge \circ(A(n)) \\ 0 & \text{if } \neg A(n) \wedge \circ(\neg A(n)) \end{cases}$$

This is the intuition Priest suggests at various points: for heap models, consistent reasoning is legitimate in the consistent initial segment. So there is evidence that from the strong logic standpoint, the function problem may be dealt with (if not solved, since there is no decision procedure for consistency $\circ$). This is enough reason to peer a bit farther down the **LFI1** path.

4.4.1 Paraconsistent Turing Machines in LFI1

There is an inchoate sense that a paraconsistent machine should be able to 'do two things at the same time'. Less fancifully, and quite conventionally, one might conjecture that a *non-deterministic* Turing machine has the potential to out-compute a deterministic Turing machine. This turns out not to be so. For finite automata, we have:

Theorem (Rabin and Scott 1959)**.** *Anything computable by a non-deterministic finite automaton is also computable by a deterministic finite automaton [52].*

More generally (though through a different proof strategy), we have:

Theorem. *For every non-deterministic Turing machine there is an equivalent deterministic Turing machine [57, p. 178].*

Could the paraconsistent case be different? Perhaps, but not if the paraconsistent logic is **LFI1** (or any of the equivalent logics we've been studying at the top of the strength ranking). Agudelo and Carnielli show that paraconsistent Turing Machines axiomatized in **LFI1** (ParTMs) do not out-compute their classical counterparts.

Theorem (Agudelo and Carnielli 2010)**.** *Any ParTM can be simulated by a (classical) Turing Machine [1].*

Perhaps, this is *good news*. Agudelo and Carlielli explicitly state that they are not looking to deviate from classical theory too much. For example, if a ParTM could compute, say, the halting function, then this would imply by Church's theorem that first-order logic is decidable, and for Agudelo and Carnielli such a conclusion would be far too radical. Perhaps, a logic that stays as close to classicality as possible, with concomitant results that are as close to classical computability as possible, is desirable, insofar as it remains within the bounds of accepted science.

4.4.2 The Downsides of Consistency Operators

As with the Rabin–Scott theorem, the strong approach has all the costs and benefits of standard, conservative arithmetic and computability. Alternatively, if one is motivated to work on paraconsistent computability theory because of a genuine expectation of inconsistency, or at least a genuine uncertainty about inconsistency, then there are real problems here.

Most saliently, the suggestion that we can solve the function problem by assuming consistency in consistent situations is subject to the ever-present fact that we do not in fact have any guarantee of consistency in almost any situation. Put more technically, a consistency operator $\circ$ must itself be consistent for this to work:

$$(\circ A \wedge \neg \circ A) \to \bot$$

But why suddenly presume the consistency of consistency? Any qualms about classicality that were meant to be solved by going paraconsistent would be very likely recur one level up. For example, revenge liar-type sentences of the form

$$L : \neg L \wedge \circ(L)$$

(saying ‘this sentence is false and consistently so’) are prone to explode. This means the prospects are dim for dealing any better with diagonal arguments to incompleteness in arithmetic based in **LFI1** (or in an equivalent theory). A Gödel sentence that says ‘this sentence is not provable, and consistently so’ will evade decision. From a new-seeking non-classical direction, all this looks too classical; see [45].

5 Arithmetic in Deep Relevant Logics

Having looked at the strongest possible logics for paraconsistent computability theory, we now turn to logics weaker than **RQ** which may also accommodate (inconsistent) theories of arithmetic.

5.1 The Need for Contraction-free Logics

The alert reader will also have already long-noted that strong logics like **R**, **RM** and **RM3** support Contraction, $\vdash A \to (A \to B) \therefore \vdash A \to B$. In the case of arithmetic, this is not a catastrophe, since Curry-paradox-inducing axioms like a naïve truth schema are not assumed to be around. But contraction is still a cost, since it will support the (pernicious) reasoning behind Löb's theorem: If we have $\Box$ representing provability in arithmetical theories, and provability minimally satisfies

- $\Box(A \to B) \to (\Box A \to \Box B)$
- $\Box A \to \Box\Box A$
- $\vdash A \therefore \vdash \Box A$

then one can show that $\vdash \Box A \to A$ implies $\vdash A$, which is Löb's theorem, and which seems wrong. Questionable steps include Permutation, $\vdash A \to (B \to C) \therefore \vdash B \to (A \to C)$, and Contraction [58, p. 230]. A logic that validates such steps will be forced to say that the provability predicate representable in arithmetic is not the real, informal proof relation that we really use over arithmetic. It is not provable (in the second sense) that all sentences provable (in the first sense) are true, even though it *is* true that all provable sentences (in the second sense) *are* true [49, Ch. 17]. Incompleteness!

Again, this makes the relevant arithmetician no *worse* off than their classical counterparts. Löb's theorem says that arithmetic cannot express its own soundness, which the world has been coming to terms with since Gödel. It does close off, however, the possibility of finding a new way around.

5.2 Ultralogic to the Rescue?

When it comes to relevant, Contraction-free logics, the family of *deep* relevant logics contains the more interesting systems—though of course one has Contraction-free systems which are not depth relevant, like **RW**. Among these, Routley's ultralogic, **DK** [54, p. 48], is a quite strong system—yet weaker than the most well-known systems containing **TW**. Moreover, Routley advanced **DK** as a candidate for a universal logic, i.e., one with which we may reason safely (paradox-free) in any domain of science and philosophy. As such, **DK** was already considered by Routley as a possible base for arithmetic.

Routley's Arithmetic **DKA** [54, §9] is not just the result of adding the Peano axioms to **DKQ**. He has some qualms with the fact that Meyer's $\mathbf{R}^\sharp$ proves theorems like $\forall x\, \forall y\, (x = x \leftrightarrow y = y)$ or the $\forall x\, \forall y\, \forall z\, (x = y \rightarrow z = z)$ we noted when reporting on relevant Robinson arithmetic. His view is that such theorems fail to be relevant implications, for how could $3 = 5$ be relevant (sufficient) to $9 = 9$? He blames $x = y \rightarrow (y = z \rightarrow x = z)$, axiom (A2) from $\mathbf{R}^\sharp$, for the validity of such irrelevant-looking principles, whence he replaces it with $(x = y \wedge y = z) \rightarrow x = z$ and $x = y \rightarrow y = x$. Moreover, he defines $1 \overset{df}{=} 0'$ and $\tau \overset{df}{=} (1 = 1)$ and also modifies the successor axioms thus: $(x = y \wedge \tau) \rightarrow x' = y'$ and $(x' = y' \wedge \tau) \rightarrow x = y$.

Mortensen [42] disputes Routley mainly on the grounds that a true implication need not be a relevant implication. He considers that the relevance of a logic is a matter of its logical theorems satisfying criteria like variable sharing, but these conditions should not be demanded from non-logical theorems of arithmetical theories. Whether Mortensen is right in this point is a matter we do not discuss here but which we recommend to fellow relevant logicians, since it is quite important for any intended applications of relevant logics.

Nevertheless, if one wishes to take Routley's criticism of $x = x \leftrightarrow y = y$ and $x = y \rightarrow z = z$ very seriously, there is a relevantist way to respond to him. As suggested in [25], the implication relation in those principles is not entirely irrelevant even though the Variable Sharing Principle is violated. Estrada-González and Tapia-Navarro advance the notion of weak q-relevance [25, p. 515]: if $A \rightarrow B$ is a theorem, then A and B share weak q-content—where the weak q-content of a formula is defined as the set of its terms and (relevant) predicates. Thus, the identity predicate is shared q-content in the principles criticized by Routley.[10]

[10] Besides being a tool to respond to Routley on arithmetical principles with dubious validity as entailments, weak q-relevance is also an interesting response to the objection against relevant mathematics in general, to wit, that some forms of weakening, ruled out in relevant logics, are needed to develop sufficiently strong mathematical theories worth of consideration. See [25, p. 516] for a restricted formulation of weakening which a relevant logician might endorse.

Not enough attention—almost no attention—has been given to **DKA** or other arithmetics based on **DKQ**, owing perhaps in part to weak logics' historical performance against GUIDING QUESTION Q5: they tend to be hard to use.[11] As Routley points out in [54, p. 68], he does not know whether **DKA** can represent all primitive recursive functions.[12] Insofar as **DKA** admits inconsistent models, interesting results might be found. We invite readers to follow this path.

5.3 The Logic of Meaning Containment

A much more developed Contraction-free system is that of Brady's $\mathbf{MC}^\sharp$ [13], based on his logic $\mathbf{MCQ}^-$, a weaker formulation of his logic **MC** of meaning containment [14]. This logic is also weaker than Routley's **DK**, since **MC** lacks LEM, $A \vee \neg A$, and Distribution, $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$—though the latter holds in rule form. Moreover, $\mathbf{MCQ}^-$ further removes the quantifier principles $\forall x\,(A \to B) \to (A \to \forall x\, B)$ and $\forall x\,(A \vee B) \to (A \vee \forall x\, B)$, and their corresponding rule forms as well, and instead adds the weaker $A \to \forall x\, A$. The removal of LEM and these quantifier principles gives $\mathbf{MCQ}^-$ an intuitionistic flavor, which may be considered a desirable feature given its intended application in mathematical theories.

Subtle but important modifications to the arithmetical axioms and rules included in $\mathbf{MCQ}^-$ are also made. For instance, identity principles are similar to those of **DKA**, and the successor principles are given in rule form as $\vdash x' = y' \;\therefore\; \vdash x = y$ and $\vdash x = y \;\therefore\; \vdash x' = y'$ and their contrapositives. Most notably, a classicality axiom for equations, $x = y \vee x \neq y$, is needed given the lack of LEM.

Brady is then able to prove through metavaluations that $\mathbf{MC}^\sharp$ is negation consistent and can also represent all primitive recursive functions [13, §8] [12, §6]. Of course, then, by Gödel's first incompleteness theorem, $\mathbf{MC}^\sharp$ is incomplete. What is quite outstanding of $\mathbf{MC}^\sharp$ is that general recursion can also be accommodated in it [13, §8]. To introduce expressions of the form $\mu x A x$, i.e., the least x such that Ax holds, Brady adds the following least number principles for a formula A:

- $Aa \vee \neg Aa$
- $\vdash \exists x\, Ax \;\therefore\; \vdash A\mu x A x$

[11] "Several logicians (including Brady, Meyer, Mortensen, Priest, and Routley) have attempted to reconstruct various fragments of classical reasoning in this way. While the results are not definitive, they are not terribly encouraging" [49, p. 221]. Cf. §5.4.

[12] Given Brady's results for $\mathbf{MC}^\sharp$, which we report in the next subsection, one may suspect that arithmetics based on **DKQ** (perhaps, not exactly **DKA** due to its particular axiomatization) can represent not only all primitive recursive functions but also general recursive functions given that Brady's $\mathbf{MCQ}^-$ is contained in **DKQ**.

- $\vdash \exists x\, Ax, \vdash m < \mu x Ax \;\therefore\; \vdash \neg Am$

where the expression $A\mu x Ax$ means that A is true of the least x such that Ax.

Finally, regarding Gödel's second incompleteness theorem, Brady [13, p. 469] remarks that

> [...] arithmetic, set up using the above metavaluational proof process and incorporating primitive and general recursion, is simply consistent. This proof is finitary, but that does not mean that Gödel's Second Theorem is contradicted, as the classical component of the logic is restricted by not including the rule $\forall x\,(A \vee B) \Rightarrow A \vee \forall x\, B$, which then prevents the LEM from automatically extending to the two quantifiers.

The careful technical work of Brady provides a nice example of a relevant arithmetic that can carry out classical computability theory, which is a remarkable result. Alas, due to its negation consistency, any hopes of sidestepping the classical limitations of arithmetic and computability theory are futile. But this, of course, was not Brady's goal; but it keeps us looking elsewhere.

5.4 Second-order Dialectical Logic

This brings us to the deepest circle, where a weak logic is used to build foundations. Logan and Boccuni [31] consider how to resuscitate arithmetic from set theory, based on the logic **DL**.[13] They use second-order **DL**, plus Frege's Basic Law V:

$$\forall F\, \forall G\, ((\{F\} = \{G\} \wedge t) \leftrightarrow \forall x\, (Fx \leftrightarrow Gx))$$

(with $\{\}$ a functional abstraction operator). Membership is definable,

$$x \in y \overset{df}{=} \exists F\, (y = \{F\} \wedge Fx)$$

so first-order relations are available. They show that this package can be used to derive the Peano axioms, as we will briefly observe.

In reviewing this impressive achievement, we also note that several of our complaints in the previous sections were about strong logics committing fallacies of relevance. How does a depth-relevant logic like **DL** fare? This is a place where guiding question Q5 arises; deep relevant logics are designed to be weak, and they are. How is this dealt with? Well, constant t plays an essential role, with axiom

$$\vdash t$$

[13] Cf. [55]. Routley studied both set theory and arithmetic in the closely related logic **DKQ** [54], but without attempting to derive the axioms of arithmetic from set theory.

and rule

$$\vdash A \therefore \vdash t \to A.$$

Define $0 \stackrel{df}{=} \{x\colon \neg(x = x)\}$, define successor zermelodically, $s(x) \stackrel{df}{=} \{x\}$, and define the natural numbers, in the spirit of Frege but with much more attention to the *Sinn* than the *Bedeutung*:

$$\mathbb{N}(x) \stackrel{df}{=} \forall F\,(\forall y\,(Fy \to (Fs(y) \wedge t)) \to (F0 \to Fx)).$$

This is a clever permutation of a more usual statement (that the numbers are the intersection of all inductive structures) that turns out to be more usable in a somewhat rigid logic. For example, the axiom

$$\forall x\, s(x) \neq 0$$

follows using $A \to A$, $(A \to \neg A) \to \neg A$, and, crucially, t. Other axioms, like the previously problematic successor injection, $s(x) = s(y) \to x = y$, reduce to the straightforward application of the definition $\{x\} \stackrel{df}{=} \{z\colon z = x\}$.

From a non-contractive standpoint, the challenging axiom is

$$(\mathbb{N}(x) \wedge t) \to \mathbb{N}(s(x))$$

For example, using light affine set theory, Terui [65] needs an extra modality to get this through; otherwise all one can show in his setting is that if x is a number *and* x is a number, then its successor is, too. Logan and Boccuni solve this with judicious use of t, and lattice conjunction.

Obtaining the axioms, recall, tells us not much more than that the axioms are obtained. We do not know what arithmetic follows from the Peano Axioms in this logic. So far, "an analogue of a piece of arithmetic is available" [31, p. 19]. In this direction lies slow, hard, honest toil. A very intriguing possibility here, meanwhile, is a solution to—or avoidance of—the function problem. The only function symbol in the language is the variable binding term forming operator. Addition is done using a three place relation:

$$S(x, y, z) \stackrel{df}{=} \forall F\,(\forall a\,\forall b\,(F(a, b) \to (F(s(a), s(b)) \wedge t)) \to (F(0, x) \to F(y, z)))$$

From this it can be shown that the sum of any two numbers exists—up to contraction, at least.

Foundational work in depth-relevant logic promises to be significantly different than standard classical theory, free of many problems observed above, and with the potential for entirely new results. It is also significantly more difficult to work on, because the deeper the logic, the farther we need to climb. This remains a tantalising speculative exercise.

6 A Lesson from the Function Problem

Having now amassed a good deal of conflicting knowledge about the options for relevant-ish logics for computability, let us look ahead at how any such logics will fare in mathematical practice. We have seen strong logics that will underwrite incomplete arithmetics, and weaker logics that may yet do better but have not yet. In all cases, though, the discussion assumes that key notions like *recursive* and *representation* are fixed, more or less as classically. It is becoming clear that if a non-classical result (like a decidable arithmetic) is what one wants, then a presupposed classical starting point is not what one needs. What might a genuinely paraconsistent *computability theory* look like?

We'll consider a simple example—something you find on the first pages of a computability theory textbook (like [24]). We will go through some simple informal reasoning in mathematical English, based on new definitions intended for paraconsistency, remaining neutral about which of our logics might best support it. The idea being illustrated is that apparent barriers to progress in this area can be taken as markers for where a redesign, rather than recapture, is needed.

Kleene's theorem says that decidable sets are exactly the semi-decidable sets with semi-decidable complements:

Theorem (Kleene's theorem)**.** *S is decidable iff S and $S^{\complement}$ are semi-decidable.*

What here? The idea is to return to the function problem, and use it as a basis for a certain kind of reductio proof. The strategy is to *decide* functionality, or not, as a special property case by case.

Let's presume we have some very basic set theory, and recall that a relation $R \subseteq X \times Y$ is

- *total* iff $\forall x(x \in X$ implies $\exists y(y \in Y \wedge \langle x, y\rangle \in R))$;
- *partial* iff $\exists x(x \in X \wedge \neg\exists y(y \in Y \wedge \langle x, y\rangle \in R))$;
- *overcomplete* iff $\exists x \in X \exists y, z \in Y(y \neq z \wedge \langle x, y\rangle \in R \wedge \langle x, z\rangle \in R)$;
- a *function* iff $\forall x \in X \forall y, z \in Y(\langle x, y\rangle \in R \wedge \langle x, z\rangle \in R$ implies $y = z)$.

A relation can be total *and* partial. An overcomplete relation may be total or partial. If an overcomplete relation is a function, then $z \neq z$ for some $z \in Y$.

In search of some properly paraconsistent relations, consider:

Definition 1. *A set $X \subseteq \mathbb{N}$ is*

- pre-decidable *iff there is a computable relation* $f \subseteq \mathbb{N} \times \{0,1\}$ *such that*

$$\textit{if} \;\; ((x \in X \wedge y = 1) \vee (x \notin X \wedge y = 0)) \;\; \textit{then} \;\; \langle x, y \rangle \in f;$$

- decidable *iff it is pre-decidable and* f *is a function;*

- pre-semi-decidable *iff there is a computable relation* $f \subseteq \mathbb{N} \times \{0,1\}$ *such that*

$$\textit{if} \;\; (x \in X \wedge y = 1) \;\; \textit{then} \;\; \langle x, y \rangle \in f;$$

- semi-decidable *iff pre-semi-decidable and* f *is a function.*

Lemma 1. *If a set is decidable it is semi-decidable. If a set is pre-decidable it is pre-semi-decidable.*

Proof. Let $S \subseteq \mathbb{N}$ be a decidable set. Then S is pre-decidable and there is a computable function $f \colon \mathbb{N} \to \{0,1\}$ such that if $(x \in S \wedge y = 1) \vee (x \notin S \wedge y = 0)$ then $\langle x, y \rangle \in f$. Since any logic we consider will agree that $(A \vee B) \to C$ implies $(A \to C) \wedge (B \to C)$, it follows, by $\wedge$-elimination, that if $(x \in S \wedge y = 1)$ then $\langle x, y \rangle \in f$, so S is pre-semi-decidable, and since f is a function, S is semi-decidable. The same argument shows that if S is pre-decidable then it is pre-semi-decidable, with the only difference that this time f is a computable relation only. □

Theorem 2 (*Weak Kleene's theorem*)**.** *A set (or relation)* S *is pre-decidable iff both it and its complement* $S^{\complement}$ *are semi-decidable. But any inconsistent sets that are semi-decidable and have semi-decidable complements, are not decidable.*

Proof. If a set is decidable, then both it and its complement $S^{\complement}$ are semi-decidable, by the previous Lemma. Suppose both S and $S^{\complement}$ are semi-decidable. Then there are computable relations f_S and $f_{S^{\complement}}$. Putting these together $f_S \cup f_{S^{\complement}} = f$ shows that S is pre-decidable. Now suppose S is decidable—that is, suppose f is a function.

$$f(n) = \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \notin S \end{cases}$$

If there is an inconsistent non-empty subset of $\mathbb{N}$, i.e., some S such that some $n \in S$ and $n \notin S$, then it would be that $f(n) = 0$ and $f(n) = 1$, which is impossible. So for any such S, f is not a function. Inconsistent computable sets are pre-decidable, but not decidable. □

This suggests that the function problem can be *useful* in reductios. What we mean by this is that in some reductios ending in $0 = f(x) = 1$, instead of concluding that some substantial assumption was wrong (like the decidability of some set or the existence of some Turing machine), all we may have determined is that f is not a function. Perhaps the assumptions made are still compatible with f being a computable relation instead. Functionality is now one of the moving parts, a variable parameter, in the overall picture of computability. This does not tell us what a computable relation *is*, if not a relation with a recursive characteristic function. But it suggests that recasting familiar definitions to rethink results is a worthwhile strategy trying to answer our GUIDING QUESTIONS. This is a start.

7 Conclusion: Which Logic(s) for Paraconsistent Computable Functions?

Starting from **R** as a mid-point for a spectrum of logics for arithmetic, we see a choice—turning to the stronger logics like **RM3** and maximally paraconsistent systems, or turning to logics significantly weaker, like **DL** and other depth-relevant systems. How do these choices fare in light of our GUIDING QUESTIONS? We summarize our findings:

	$\mathbf{R}^\sharp$	$\mathbf{R}^{\sharp\sharp}$	$\mathbf{Q_R}$	$\mathbf{RM3}^i$	$\mathbf{Q_{PAC}}$	**DKA**	$\mathbf{MC}^\sharp$	$\mathbf{DL2Q}^{t,fc}$
Non-trivial	✓	✓	✓	✓	✓	✓	✓	?
Negation consistent	✓	✓	✓	×	?	?	✓	?
Negation complete	×	×	×	✓	?	?	×	?
Decidable	×	×	×	✓	?	?	×	?
Contractive	✓	✓	✓	✓	✓	×	×	×

The question marks indicate that there is a good deal still to uncover. As elsewhere in the relevant programme, there is no clear 'winner'.

Here are some concluding observations.

- If one works with the stronger, or even maximal, systems, there are many benefits—the most apparent being staying close to classical arithmetic and computability theory. The class of computable functions is unchanged from the classical case. This benefit, though, is also the most significant cost, in that the resulting computability theory will not significantly advance on previous attempts. While in some cases decidability is regained (*Entscheidung!*), proving completeness via 'internal' proofs of soundness are as out of reach as they have been since the mid-twentieth century. *Ignoramus et ignorabimus.*

- If one works with weaker systems, there are many costs—from the nigh-intractability of the function problem, to more mundane difficulties in deriving even the simplest of results (our guiding Q5). These costs, though, are only costs if one is holding out for a non-classical approach that somehow is not different from the classical approach. A complete, decidable system that represents its own proof relation and expresses its own soundness and consistency is going to be unlike classical arithmetic, which does none of these things.

The choice, as we noted from the start, may turn on the answer to the question: *paraconsistent* or *inconsistent*?

An inconsistent approach anticipates inconsistent subsets of $\mathbb{N}$. If these exist, either because of naïve set comprehension, or some inconsistency from Gödel's theorem (cf. [56]), or something else, they go hand in hand with weaker logics. On this approach, $\circ$ and equivalent operators should give us profound pause. They cannot reckon with Gödel's or Turing's limits any better than classical logic.

A simply paraconsistent approach not concerned with the ghosts of departed inconsistent sets can work much more flexibly with logics like **R** and its extensions. Strong logics close to classical—some might say *extremely* close to classical—are easy to work with and think about. They support familiar (but not exactly the same) arithmetic, as well as models of inconsistent theories with nice properties.[14]

Insofar as Löb's theorem is an incompleteness, an inability to express something manifestly true, then stronger logics are terminally incomplete. They are easy to work with and have some good-making features, but ultimately do not—will not—answer Hilbert's call. On that count, if the aim is to find a genuine solution to the function problem and with it a class of *properly* non-classical computable operations that cross the Church–Turing barrier, something more radical, only yet half-imagined, is still needed.

Appendix 1: Axiomatizations for R and Other Systems

RQ is given by the following axiom schemata and rules.

Axioms:

1. $A \to A$
2. $(A \wedge B) \to A$
3. $(A \wedge B) \to B$

[14] And they can describe, and can be represented by, hardware you can literally build this afternoon if you wanted to [15].

4. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
5. $A \to (A \vee B)$
6. $B \to (A \vee B)$
7. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C))$
8. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
9. $\neg\neg A \to A$
10. $(A \to \neg B) \to (B \to \neg A)$
11. $(A \to B) \to ((B \to C) \to (A \to C))$
12. $A \to ((A \to B) \to B)$
13. $(A \to (A \to B)) \to (A \to B)$
14. $\forall x\, A \to A[y/x]$ (where y is free for x in A)
15. $\forall x\, (A \to B) \to (A \to \forall x\, B)$ (where x is not free in A)
16. $\forall x\, (A \vee B) \to (A \vee \forall x\, B)$ (where x is not free in A)

Rules:

17. $\vdash A, \vdash A \to B \;\therefore\; \vdash B$
18. $\vdash A, \vdash B \;\therefore\; \vdash A \wedge B$
19. $\vdash A \;\therefore\; \vdash \forall x\, A$

Other logics discussed in this paper may be obtained from **RQ** as follows:

RMQ = **RQ** $+ A \to (A \to A)$.
RM3Q = **RMQ** $+\ A \vee (A \to B)$.
DKQ = **RQ** $-$ 11. $-$ 12. $-$ 13. $+ A \vee \neg A\ +\ ((A \to B) \wedge (B \to C)) \to (A \to C)\ +$ $\vdash A \to B, \vdash C \to D \;\therefore\; \vdash (B \to C) \to (A \to D)$.
MCQ = **DKQ** $-$ 8. $- A \vee \neg A -$ 16. + the following metarules:

- If $\vdash A, \vdash B \;\therefore\; \vdash C$ then $\vdash D \vee A, \vdash D \vee B \;\therefore\; \vdash D \vee C$.
- If $\vdash A, \vdash B[y/x] \;\therefore\; \vdash C[y/x]$ then $\vdash A, \vdash \exists x\, B \;\therefore\; \vdash \exists x\, C$ where y does not occur in A, B or C, and where 19. does not generalize on any free variable in the premises A and $B[y/x]$ in the derivation $\vdash A, \vdash B[y/x] \;\therefore\; \vdash C[y/x]$.

MCQ$^-$ = **MCQ** $-$ 15. $+ A \to \forall x\, A$ (where x does not occur in A).
DLQ = **DKQ** $- A \vee \neg A\ +\ (A \to \neg A) \to \neg A$.
DLQ^t = **DLQ** $+\ t\ +\ \vdash A \;\therefore\; \vdash t \to A$.

$\mathbf{DL2Q}^{t,fc} = \mathbf{DLQ}^t + \exists F^n \forall y_1 \ldots \forall y_n (A \leftrightarrow A(F^n/B(y_1, \ldots, y_n)))$ where F does not occur free in B and B is free for F in A. Regarding 14.–16. and 19., quantification can be either first- or second-order.

See the appendix in [63] for an axiomatization of **PAC** (dubbed **A3** therein).

Appendix 2: Some Features of RM

An **RM**-frame $\langle W, w_0, \leqslant \rangle$ has an ordering that is reflexive, transitive, and connected. That is, **RM** has a linearly ordered infinite characteristic matrix, making it particularly easy to think about. Following Dunn, **RM**-models are obtained by adding valuations $v\colon \texttt{prop} \times W \longrightarrow \{\{1\}, \{0\}, \{1, 0\}\}$ that obey heredity (if $w_x \leqslant w_y$ then $v_{w_x}(p) \subseteq v_{w_y}(p)$) and the semantic clauses for $\neg$ and $\wedge$,

$$\begin{aligned} 1 \in v_w(\neg A) &\quad \text{iff} \quad 0 \in v_w(A) \\ 0 \in v_w(\neg A) &\quad \text{iff} \quad 1 \in v_w(A) \\ 1 \in v_w(A \wedge B) &\quad \text{iff} \quad 1 \in v_w(A) \text{ and } 1 \in v_w(B) \\ 0 \in v_w(A \wedge B) &\quad \text{iff} \quad 0 \in v_w(A) \text{ or } 0 \in v_w(B) \end{aligned}$$

The conditional has truth condition

$$\begin{aligned} 1 \in v_{w_x}(A \to B) \quad \text{iff} \quad & \forall y \geqslant x((1 \in v_{w_y}(A) \text{ only if } 1 \in v_{w_y}(B)) \\ & \wedge (0 \in v_{w_y}(B) \text{ only if } 0 \in v_{w_y}(A)) \end{aligned}$$

This is nice, but unlike Routley–Meyer's $Rxyz$, it is hard for $A \to A$ to fail "honestly" [20, p. 172]. The falsity condition is disjunctive:

$$0 \in v_{w_x}(A \to B) \quad \text{iff} \quad 1 \notin v_{w_x}(A \to B) \vee (1 \in v_{w_x}(A) \wedge 0 \in v_x(B))$$

Dunn calls the first disjunct the "escape clause".[15]

Dunn [20] finds a countable family of weakenings of **RM**. Let $A \circ B = \neg(A \to \neg B)$. Then the **RM** axiom is $(A \circ A) \to A$.[16]
This in hand, we can recursively define

$$A^1 = A \qquad A^{n+1} = A^n \circ A$$

[15] See [8, p. 29] and [10, p. 126] for this and other very useful background. If we wrote the falsity condition as a conditional, it says:

$$0 \in v(A \to B) \;\leftrightarrow\; (1 \in v(A \to B) \to (1 \in v(A) \wedge 0 \in v(B)))$$

which we think is funny, and can be permuted/contraposed to say all sorts of things.

[16] By the following reasoning: $\neg(A \to \neg A) \to A$ implies $\neg A \to (A \to \neg A)$ by contraposition, which implies $A \to (\neg A \to \neg A)$ by permutation, which implies $A \to (A \to A)$ by contraposition and transitivity.

Then the **RM**s are successive weakenings of $A^{n+1} \to A^n$ or idempotence $(a \circ a)^{n+1} = (a \circ a)^n$ in the algebra.

Proof of Linearity from Mingle (reconstructed from [2, §29.5], "Why we don't like mingle"):

1. $\neg A \to (\neg A \to \neg A)$ (mingle)
2. $\neg A \to (A \to A)$ (1, contrapose)
3. $A \to (\neg A \to A)$ (2, permute)
4. $((A \to A) \wedge (B \to B)) \to (\neg((A \to A) \wedge (B \to B)) \to ((A \to A) \wedge (B \to B)))$ (instance of 3)
5. $(\neg(A \to A) \vee \neg(B \to B)) \to ((A \to A) \wedge (B \to B))$ (from 4, fiddling)
6. $\neg(A \to A) \to (B \to B)$ (5, axioms 5 and 3, transitivity)
7. $B \to (\neg(A \to A) \to B)$ (6, permute)
8. $B \to (A \circ \neg A \to B)$ (7, definition)
9. $B \to (\neg A \circ A \to B)$ (8, commute)
10. $B \to (\neg A \to (A \to B))$ (9, export)
11. $B \to (\neg(A \to B) \to A)$ (10, contrapose)
12. $\neg(A \to B) \to (B \to A)$ (11, permute)
13. $(A \to B) \vee (B \to A)$ (12, implication to disjunction)

Now, the above derivation does use steps that are disputable. If you are used to working in weaker systems, it feels very dirty! As Dunn says,

> One not used to relevance logic might be surprised at the author's delicious sense of forbidden pleasure derived from the use of the material conditional here and throughout the metalanguage [19, footnote 4].

So too with such casual permutation, among other moves, above.[17]

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